Airfoil lift finite element computations using $\mathbf{H}^1(\Omega)$ and $\mathbf{H}(\mathbf{div};\Omega)$ approximation spaces

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Abstract. Computing the lift generated by an airfoil is crucial in aircraft design. The problem of incompressible flow around an airfoil is herein studied as a weakly irrotational flow, resulting a div-curl problem in a 2D double-connected domain. In traditional potential flow, $H^1(\Omega)$ finite elements are adopted and a cut in the original domain with additional constrain for the velocity potential and velocity potential gradient at the cut are required for the numerical solution of the problem. The lift coefficient can be finally computed from the circulation of the numerical solution. In the present work, the lift of an airfoil is computed from finite element solutions using different approximation spaces: The conventional $H^1(\Omega)$ potential-field space and a special class of $\mathbf{H}(div;\Omega)$ velocity-field space, i.e., the divergence-free space. To ensure accuracy, an a posteriori error estimator for the problem is derived, which is based on the difference in the velocity solution obtained using $H^1(\Omega)$ and $\mathbf{H}(div;\Omega)$ spaces. The convergence of the error estimator, and by consequence, the convergence of the lift coefficient is verified by means of uniform h-refinement strategies.

Keywords: Lift of airfoils, Finite element solutions, H(div; Ω) approximation spaces, A posteriori error estimator

1 Introduction

The problem of flow around airfoils has been subject of experimental and numerical research for a long time. The solution of the problem allows the computation of the lift generated by the airfoil, which is crucial in aircraft design. Depending on the Reynolds number, the incompressible Navier-Stokes equations can be simplified and, for suitable boundary conditions, the problem can be studied as a weakly irrotational flow model. Regardless of the flow governing equations, finite element methods are widely employed for the approximation of the problem [1]. In the context of weakly irrotational flow models, the most common finite element approach is to obtain a weak formulation in terms of velocity potential and use $H^1(\Omega)$ finite element approximation spaces to solve for the potential. However, this approach do not ensure local mass conservation. The appropriate selection of $\mathbf{H}(div;\Omega)$-$L^2(\Omega)$ spaces for the velocity and pressure fields has been demonstrated to ensure local mass conservation of the approximation [2]. The divergence-free space is a $\mathbf{H}(div;\Omega)$-type space which results null divergence and, therefore, can be employed to obtain more accurate solutions in incompressive flow problems [3]. The divergence-free space, denoted $\mathbf{H}(div;\Omega)$-Kernel or $\mathbf{H}(div_k;\Omega)$ space, has a reduced number of shape functions when compared to $\mathbf{H}(div;\Omega)$-Standard space [3]. In the flow around an airfoil problem, the accuracy of the numerical approximation is affected by the local singularity that arise at the trailing edge. The challenge is to identify these regions and refine the elements such that the accuracy is optimal as a function of the number of degrees of freedom. A posteriori error estimators extract a computable quantity as a function of the numerical approximation and they can be used as a guide in adaptive procedures [4]. Extensive research has been done in the context of a posteriori error estimators [5–7]. The main objective of the present work is to accurately compute numerical flow solutions around airfoils to ensure accurate lift computations. To ensure accuracy, an a posteriori error estimator for the the weakly irrotational flow around an airfoil problem is derived based on $H^1(\Omega)$ velocity potential and $\mathbf{H}(div_k;\Omega)$ velocity finite element solutions. The study was conducted with the finite element library NeoPZ [8, 9].

2 Problem statement

The accurate solution of the flow around an airfoil problem allows the computation of the circulation, which leads to the computation of the lift force generated by the airfoil. According to classical fluid mechanics literature [1], the problem of flow around an airfoil can be studied using the governing equations of weakly irrotational flows.

2.1 Governing equations of irrotational flows

If the density ρ is constant, the viscosity and thermal diffusion are negligible, and the flow does not depend on time, then the Navier-Stokes equations simplifies and the velocity $\mathbf{u}(\mathbf{x})$ and pressure $p(\mathbf{x})$ are given for all points $\mathbf{x} \in \Omega$ of the fluid by

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \tag{1}$$

in which $\mathbf{f}$ are the domain forces. For suitable boundary conditions, there exist solutions of equations (1) satisfying $\nabla \times \mathbf{u} = \mathbf{0}$. These solutions are called irrotational [1]. For $\mathbf{f} = \mathbf{0}$, we say that (1) have such irrotational solutions if the velocity $\mathbf{u}(\mathbf{x})$ is given for all points $\mathbf{x} \in \Omega$ of the fluid by

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \times \mathbf{u} = \mathbf{0}, \tag{2}$$

and the pressure can be post-processed from the incompressible Bernoulli's equation as $p = k - \frac{1}{2}\rho \mathbf{u} \cdot \mathbf{u}$, in which k is a constant that can be computed from undisturbed flow conditions. The governing equations can also be presented in terms of a velocity potential $\varphi(\mathbf{x})$, defined as $\mathbf{u} = \nabla\varphi$. Thus, both governing equations in (2) are simultaneous satisfied by

$$\Delta\varphi = 0, \qquad \mathbf{u} = \nabla\varphi \tag{3}$$

where $\Delta\,() = \nabla \cdot \nabla\,()$ is the Laplace operator.

2.2 Flow around an airfoil profile

The flow around an airfoil profile Γ_0 corresponds, in principle, to flow in an unbounded exterior domain, but we approximate infinity numerically by a boundary Γ_∞ at a finite distance; so Ω is a two dimensional domain with boundary $\Gamma = \Gamma_0 \cup \Gamma_\infty$, such as the one illustrated in Figure 1 (disregarding the line Σ for now).

One often takes $\mathbf{u}_\infty$ constant and the normal velocity boundary conditions $g = \mathbf{u} \cdot \mathbf{n}$ reads

$$g|_{\Gamma_\infty} = \mathbf{u}_\infty \cdot \mathbf{n}, \quad g|_{\Gamma_0} = 0. \tag{4}$$

Unfortunately, the numerical results show that with these boundary conditions, the flow generally goes around the trailing edge P. As P is a singular point of Γ, $|\mathbf{u}(\mathbf{x})|$ tends to infinity when $\mathbf{x} \to P$ and the viscosity effects (η and ζ) are no longer negligible in the neighborhood of P (see Fig. 1). The modeling of the flow as an incompressible, inviscid and irrotational flow by (2) is not valid. The irrotational flow condition has to be replaced by $\nabla \times \mathbf{u} = \mathbf{c}\delta_\Sigma$, where $\mathbf{c}$ is a constant and δ_Σ is the Dirac function on an arbitrary line Σ which connects the trailing edge P with the external boundary Γ_∞ (see [1]). The problem can thus be stated as: Consider the doubly-connected domain $\Omega \subset \mathbf{R}^2$, illustrated in Fig. 1, with boundary $\Gamma := \partial\Omega$, and a vector field $\mathbf{u} : \bar{\Omega} \to \mathbf{R}^2$ satisfying the div-curl problem

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \times \mathbf{u} = \mathbf{c}\delta_\Sigma. \tag{5}$$

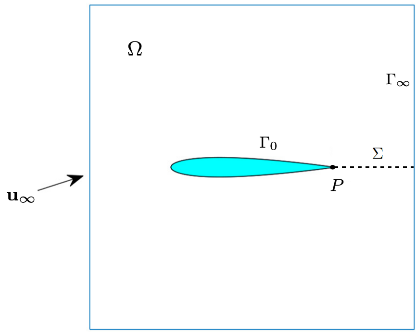


Figure 1. Flow around an airfoil profile [10].

Once obtained the solution $\mathbf{u}$, the circulation κ can be computed as

$$\int_{\Gamma_0} u_t d\Gamma = \kappa, \tag{6}$$

in which $u_t = \mathbf{u}_0 \cdot \mathbf{t}$ is the tangential velocity on the profile. The lift coefficient C_ℓ of the airfoil can be computed from the circulation as: $C_\ell = \kappa\rho\,|\mathbf{u}_\infty|$.

3 $H^1(\Omega)$ and $\mathbf{H}(div_k;\Omega)$ approximation spaces

For problems defined in $\mathbf{R}^2$, a finite element shape function $\phi(x,y) \in H^1(\Omega)$ is firstly converted into an $\mathbf{H}(curl;\Omega)$ function by multiplying it by the orthonormal basis component $\mathbf{e}_z$: $\phi(x,y)\mathbf{e}_z \in \mathbf{H}(curl;\Omega)$. Then, a divergence-free finite element shape function $\mathbf{\Phi}(x,y) \in \mathbf{H}(div_k;\Omega)$ can be defined as

$$\mathbf{\Phi}(x,y) = \nabla \times (\phi \mathbf{e}_z) = -\frac{\partial \phi}{\partial \mathrm{y}}\mathbf{e}_x + \frac{\partial \phi}{\partial \mathrm{x}}\mathbf{e}_y, \tag{7}$$

with the property

$$\nabla \cdot \mathbf{\Phi} \equiv 0 \Longrightarrow \mathbf{\Phi} \in \mathbf{H}(div_k;\Omega). \tag{8}$$

More details details on the velocity discretization at the element level with $\mathbf{H}(div_k;\Omega)$ approximation spaces can be found in [3].

4 Solution strategies by variational methods

The defined problem can be approximated by finite elements either in terms of velocity potential φ, using $H^1(\Omega)$ approximation spaces into a variational weak formulation, or in terms of velocity $\mathbf{u}$, using $\mathbf{H}(div_k;\Omega)$ approximation spaces into another variational weak formulation.

4.1 Solution for the velocity potential field φ: $H^1(\Omega)$ variational approach

To avoid the ill-posed div-curl problem (5) defined over the doubly-connected domain (see [1, 11]), a cut Σ in the original domain (see Fig. 1), and additional constraints for the velocity potential and velocity potential gradient at the cut, are required. The cut Σ includes two separate lines Σ^+ and Σ^- joining the upper and lower surfaces (respectively) of the airfoil boundary Γ_0 with the far field boundary Γ_∞. Therefore, a new, simply-connected domain of computation is defined: $\Omega - \Sigma$. It is necessary to add a boundary condition on Σ^+ and Σ^-. Since $\mathbf{u}$ must be continuous along Σ, we have:

$$\left.\frac{\partial \varphi}{\partial n}\right|_{\Sigma^+} = \left.\frac{\partial \varphi}{\partial n}\right|_{\Sigma^-}, \qquad \left.\frac{\partial \varphi}{\partial t}\right|_{\Sigma^+} = \left.\frac{\partial \varphi}{\partial t}\right|_{\Sigma^-}. \tag{9}$$

Integrating one of these equations over Σ one obtains: $\frac{\partial}{\partial t}\left(\varphi|_{\Sigma^+} - \varphi|_{\Sigma^-}\right) = 0$ or $\frac{\partial}{\partial n}\left(\varphi|_{\Sigma^+} - \varphi|_{\Sigma^-}\right) = 0$. For some constant β, one obtains from the previous result: $\varphi|_{\Sigma^+} - \varphi|_{\Sigma^-} = \beta$, where β is a constant to be determined by imposing the flux continuity at the trailing edge: $|\nabla\varphi(P^+)|^2 = |\nabla\varphi(P^-)|^2$, in which

P^+ and P^- are the trailing edge points on Σ^+ and Σ^-, respectively. It can be proven using conformal mappings that the solution φ does not depend on the position of Σ [1], even though the numerical result is influenced by it. Choosing an arbitrary path Σ, the solution for the velocity potential φ can be obtained by solving: find $\varphi \in H^1(\Omega)$ such that

$$\Delta\varphi = 0 \quad on \quad \Omega - \Sigma, \qquad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = g \quad on \quad \Omega - \Sigma, \tag{10}$$

with the conditions

$$\varphi|_{\Sigma^+} - \varphi|_{\Sigma^-} = \beta, \qquad \left|\nabla\varphi\left(P^+\right)\right|^2 = \left|\nabla\varphi\left(P^-\right)\right|^2 \tag{11}$$

on the new boundaries [1]. To solve (10)-(11), a simple method is to note that the solution is linear in β:

$$\varphi(\mathbf{x}) = \varphi^0(\mathbf{x}) + \beta\varphi^1(\mathbf{x}), \tag{12}$$

where φ^0 is the solution of (10), with g defined in (4), and the first condition in (11) with $\beta = 0$:

$$\Delta\varphi^0 = 0, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = g, \quad \varphi \quad \text{continuous} \quad \text{across} \quad \Sigma, \tag{13}$$

and φ^1 is the solution of (10), with $g = 0$, and the first condition in (11) with $\beta = 1$:

$$\Delta\varphi^1 = 0, \quad \left.\frac{\partial\varphi^1}{\partial n}\right|_{\Gamma} = 0, \quad \left.\varphi^1\right|_{\Sigma^+} - \left.\varphi^1\right|_{\Sigma^-} = 1, \quad \left.\nabla\varphi^1\right|_{\Sigma^+} = \left.\nabla\varphi^1\right|_{\Sigma^-}. \tag{14}$$

The variational $H^1(\Omega)$ weak form of the first and second problems can be stated as: find $\varphi^0 \in W_0$ such that

$$\int_\Omega \nabla\varphi^0 \cdot \nabla w d\Omega = \int_{\partial\Omega} g d\Gamma \quad \forall w \in H^1(\Omega), \tag{15}$$

with $W_0 = \left\{\varphi^0 \in H^1(\Omega): \quad \left.\varphi^0\right|_{\Sigma^+} - \left.\varphi^0\right|_{\Sigma^-} = 0, \quad \left.\nabla\varphi^0\right|_{\Sigma^+} = \left.\nabla\varphi^0\right|_{\Sigma^-}\right\}$, and find $\varphi^1 \in W_1$ such that

$$\int_\Omega \nabla\varphi^1 \cdot \nabla w = 0 \quad \forall w \in H^1(\Omega), \tag{16}$$

with $W_1 = \left\{\varphi^1 \in H^1(\Omega): \quad \left.\varphi^1\right|_{\Sigma^+} - \left.\varphi^1\right|_{\Sigma^-} = 1, \quad \left.\nabla\varphi^1\right|_{\Sigma^+} = \left.\nabla\varphi^1\right|_{\Sigma^-}\right\}$.

The discretization of the computational domain into finite elements with piecewise $H^1(\Omega)$ approximations for φ^0, φ^1 and w allows to obtain numerical solutions for φ^0 and φ^1. Finally, with these solutions is possible to find β by solving the second condition in (11) with (12): it is an equation in one variable β. In this approach, it is possible to show (see [1]) that β corresponds to the circulation and, therefore, the lift coefficient C_ℓ is proportional to β: $C_\ell = \beta\rho\left|\mathbf{u}_\infty\right|$, where ρ is the density of the fluid.

4.2 Solution for the velocity field u: $\mathbf{H}(div_k;\Omega)$ variational approach

To solve the problem with the $\mathbf{H}(div_k;\Omega)$ variational formulation, another approach to avoid the ill-posed div-curl problem is adopted. For the problem double-connected domain, illustrated in Fig. 1, the problem becomes well-posed with a unique solution, derived from potentials, when a certain line integral is further prescribed [10]. In aerodynamics this is known as "circulation condition": we need to prescribe the circulation along a closed curve that encloses the airfoil. One possible choice is to prescribe the line integral around the boundary of the airfoil defined in (6), for some constant κ. The solution for the velocity field $\mathbf{u}$, satisfying $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} = \nabla\varphi$, can be obtained by solving the weighted integral form: find $\mathbf{u} \in W^{div}$ such that

$$\int_\Omega \nabla \cdot \mathbf{u} w d\Omega = 0, \quad \forall w \in L^2(\Omega) \tag{17}$$

$$\int_\Omega \mathbf{u} \cdot \mathbf{w} d\Omega - \int_\Omega \nabla\varphi \cdot \mathbf{w} d\Omega = 0, \quad \forall \mathbf{w} \in \mathbf{H}(div_k;\Omega) \tag{18}$$

with $W^{div} = \left\{\mathbf{u} \in \mathbf{H}(div_k;\Omega) : \mathbf{u}\cdot\mathbf{n} = u_n|_\Gamma = g, \quad \int_{\Gamma_0} u_t d\Gamma = \kappa\right\}$. Since $\mathbf{u} \in \mathbf{H}(div_k;\Omega)$, equation (17) is automatically satisfied. Integration by parts of the second integral in equation (18) allows to rewrite the problem as: find $\mathbf{u} \in W^{div}$ such that

$$\int_\Omega \mathbf{u}\cdot\mathbf{w}d\Omega - \int_\Omega \nabla\cdot\mathbf{w}\varphi d\Omega + \int_{\partial\Omega}\varphi w_n d\Gamma = 0, \quad \forall \mathbf{w}\in\mathbf{H}(div_k;\Omega) \tag{19}$$

in which $w_n = \mathbf{w}\cdot\mathbf{n}$. Knowing that $\mathbf{w}\in\mathbf{H}(div_k;\Omega)$ and that, for the flow around an airfoil problem with normal velocity u_n prescribed in the whole boundary $\partial\Omega$, the problem reduces to: find $\mathbf{u}\in W^{div}$ such that

$$\int_\Omega \mathbf{u}\cdot\mathbf{w}d\Omega = 0, \quad \forall\mathbf{w}\in\mathbf{H}(div_k;\Omega). \tag{20}$$

The velocity potential decomposition presented in (12), which came from the Helmholtz decomposition of vector fields, can also be presented in terms of velocity as:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \beta\mathbf{u}^1(\mathbf{x}). \tag{21}$$

The variational $\mathbf{H}(div_k;\Omega)$ weak form of the problems to find $\mathbf{u}^0$ and $\mathbf{u}^1$ can be stated as: find $\mathbf{u}^0\in W_0^{div}$ such that

$$\int_\Omega \mathbf{u}^0\cdot\mathbf{w}d\Omega = 0, \quad \forall\mathbf{w}\in\mathbf{H}(div_k;\Omega):\ \mathbf{w}\cdot\mathbf{n} = w_n|_\Gamma = 0, \tag{22}$$

with $W_0^{div} = \left\{\mathbf{u}^0\in\mathbf{H}(div_k;\Omega):\ \mathbf{u}^0\cdot\mathbf{n} = u_n^0\big|_\Gamma = g, \quad \int_{\Gamma_0} u_t^0 d\Gamma = 0\right\}$ and find $\mathbf{u}^1\in W_1^{div}$ such that

$$\int_\Omega \mathbf{u}^1\cdot\mathbf{w}d\Omega = 0, \quad \forall\mathbf{w}\in\mathbf{H}(div_k;\Omega):\ \mathbf{w}\cdot\mathbf{n} = w_n|_\Gamma = 0, \tag{23}$$

with $W_1^{div} = \left\{\mathbf{u}^1\in\mathbf{H}(div_k;\Omega):\ \mathbf{u}^1\cdot\mathbf{n} = u_n^1\big|_\Gamma = 0, \quad \int_{\Gamma_0} u_t^1 d\Gamma = 1\right\}$.

The discretization of the computational domain into finite elements with piecewise $\mathbf{H}(div_k;\Omega)$ approximations for $\mathbf{u}^0$, $\mathbf{u}^1$ and $\mathbf{w}$ allows to obtain numerical solutions for $\mathbf{u}^0$ and $\mathbf{u}^1$. Finally, with these solutions is possible to find β by ensuring the continuity of the solution $\mathbf{u}$, given by (11), at the trailing edge. However, β does not have the physical meaning of circulation anymore.

5 Prager–Synge theorem

The Prager–Synge theorem is the basis of the proposed error estimator [5, 7]. The theorem states: Let $\varphi\in H^1(\Omega)$ be the solution of the variational problem, like those presented in (15)-(16), and let $\varphi_h\in H^1(\Omega)$ and $\mathbf{u}_h\in\mathbf{H}(\text{div},\Omega)$ with $\nabla\cdot\mathbf{u}_h = f$, be arbitrary (f is not defined and zero in our problem). Then

$$\|\nabla(\varphi-\varphi_h)\|^2 + \|\nabla\varphi+\mathbf{u}_h\|^2 = \|\nabla\varphi_h+\mathbf{u}_h\|^2. \tag{24}$$

The proof can be demonstrated by adding and subtracting $\nabla\varphi$, resulting

$$\begin{aligned}\|\nabla\varphi_h+\mathbf{u}_h\|^2 &= \|\nabla(\varphi_h-\varphi)+\nabla\varphi+\mathbf{u}_h\|^2\\ &= \|\nabla(\varphi_h-\varphi)\|^2 + \|\nabla\varphi+\mathbf{u}_h\|^2 + 2\left(\nabla(\varphi_h-\varphi),\nabla\varphi+\mathbf{u}_h\right)\end{aligned} \tag{25}$$

Note (from $\mathbf{u} = -\nabla\varphi$ and $\nabla\cdot\mathbf{u} = 0$) that $\nabla\varphi\in\mathbf{H}(\text{div},\Omega)$ with $\nabla\cdot(\nabla\varphi) = 0$. Thus $(\nabla\varphi+\mathbf{u}_h)\in\mathbf{H}(\text{div},\Omega)$ and in particular $\nabla\cdot(\nabla\varphi+\mathbf{u}_h) = 0$. Thus, using that $\varphi_h-\varphi\in H^1(\Omega)$, the Green theorem holds

$$\left(\nabla(\varphi_h-\varphi),\nabla\varphi+\mathbf{u}_h\right) = -\left(\nabla\cdot(\nabla\varphi+\mathbf{u}_h),\varphi_h-\varphi\right) = 0. \tag{26}$$

Hence, under the assumptions of the theorem, it follows from (24) that

$$\|\nabla(\varphi-\varphi_h)\| \le \|\nabla\varphi_h+\mathbf{u}_h\|. \tag{27}$$

This is an estimate for the error $\|\nabla(\varphi-\varphi_h)\|$ and has been brought to our attention for a posteriori analysis by Vohralik in [6].

6 Error Estimator

Knowing the global finite element solutions $\varphi_h \in H^1(\Omega)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}_\mathrm{k}, \Omega) \in \mathbf{H}(\mathrm{div}, \Omega)$, an elemental a posteriori error estimator $\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)}$ can be defined as:

$$\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)} = \int_{\Omega_e} \|\nabla\varphi_h + \mathbf{u}_h\| \, d\Omega_e. \tag{28}$$

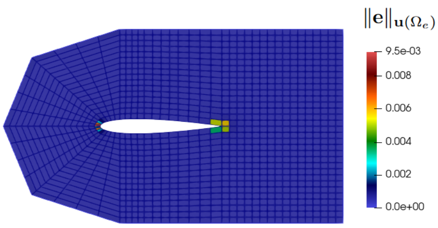
The a posteriori error estimator $\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)}$ can be employed to decide whether the numerical solutions φ_h and $\mathbf{u}_h$ are accurate enough, or even to guide hp-adaptive refinements.

7 Results

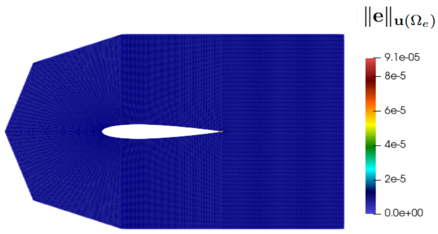
In this section, the error estimator convergence and the accuracy of the lift computation is analyzed for an incompressible inviscid flow around an NACA 0012 airfoil problem. The analytic equation describing the NACA 0012 airfoil is given in [12]. The flow density is considered to be unitary, i.e. $\rho = 1.0\ kg/m^3$, and the velocity at infinity is set as $\mathbf{u}_\infty = (1.0, 0.5)\ m/s$.

7.1 Error estimator convergence

The initial mesh adopted for the convergence study is composed of 72 quadrilateral elements of polynomial order $p = 2$ for the velocity potential in the case of $H^1(\Omega)$ analysis, and $p = 1$ for the velocity in the case of $\mathbf{H}(div_k; \Omega)$ analysis. Blended elements are employed to maintain the exact geometry of the NACA profile. The a posteriori error estimator $\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)}$ is then computed for uniform h-refinement levels: h=1,2,3,4 and 6. The refinement level h=6 resulted in very accurate $H^1(\Omega)$ and $\mathbf{H}(div_k; \Omega)$ velocity solutions. Figure 2 illustrates the error results for the refinement levels h=2 and h=4, while Figure 3 illustrates the convergence of a global error, defined as the sun of the elemental errors.



(a) velocity norm error $\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)}$ for h-refinement h=2



(b) velocity norm error $\|\mathbf{e}\|_{\mathbf{u}(\Omega_e)}$ for h-refinement h=4

Figure 2. Error estimator results for h-refinement analysis.

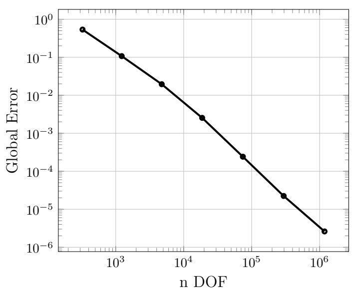


Figure 3. Global error convergence

7.2 Accurate lift computations

Table 1 presents the lift coefficient results for the refinement levels h = 0, 2, 4 and 6, as well as the respective generated Number of Equations (NE) and relative error between the $H^1(\Omega)$ and the $\mathbf{H}(div_k; \Omega)$ C_ℓ solutions. It is worth mentioning that static condensation can be applied to eliminate the equations related to the internal shape functions of the $\mathbf{H}(div_k; \Omega)$ approximations. The rates of convergence of the lift coefficient C_ℓ resulted $q \approx 1.06$ for the $H^1(\Omega)$ solutions and $q \approx 1.02$ for the $\mathbf{H}(div_k; \Omega)$ solutions. Even thought the problems solution is singular at the trailing edge, the convergence rates of the lift coefficient were not sub-optimal.

Table 1. Lift coefficient results obtained from the $H^1(\Omega)$ and $\mathbf{H}(div; \Omega)$ velocity solutions

h-refinement	NE	C_ℓ $H^1(\Omega)$	C_ℓ $\mathbf{H}(div_k; \Omega)$	relative error (%)
h=0	325	6.65	5.43	18.34
h=2	4753	18.40	18.09	1.66
h=4	74305	22.21	22.11	0.44
h=6	1181953	23.19	23.16	0.12

8 Concluding Remarks

This work presented a new a posteriori error estimator applied to weakly irrotational flows around airfoils and lift computations, which is based on the difference in the velocity solution obtained using $H^1(\Omega)$ and $\mathbf{H}(div_k; \Omega)$ spaces. The convergence of the error estimator, and by consequence, the convergence of the lift coefficient is verified by means of uniform h-refinements. Further investigation may include to apply the a posteriori error estimator to guide hp-adaptive refinement strategies in more complex flow around airfoil problems.

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