Educational interactive graphics tool for teaching Mohr's circle

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Abstract. Mohr's circle is a two-dimensional graphical representation of the transformation law for the Cauchy stress tensor. The components of the Cauchy stress tensor at a specific material point are known with respect to a coordinate system. Mohr's circle graphically determines the stress components acting in a rotated coordinate system, i.e., acting on a differently oriented plane passing through that point. The e-Mohr program is an interactive graphics tool that explains the Mohr circle to Solid Mechanics (Strength of Materials) students in engineering courses. The program demonstrates how Mohr's circle works for plane stress states, allowing users to interactively manipulate the plane's orientation where the stress tensor components are calculated. Furthermore, e-Mohr involves determining a point in the Mohr's circle called the pole or origin of planes. Any straight line drawn from the pole will intersect Mohr's circle at a point representing the state of stress on an inclined plane in the same orientation (parallel) in space as that line.

Keywords: Mohr's circle; plane stress state; interactive educational software.

1 Introduction

Mohr's circle is a two-dimensional graphical representation of the transformation law for the Cauchy stress tensor. It is widely used in mechanical engineering for the strength of materials, geotechnical engineering for the strength of soils, and structural engineering for the strength of built structures. This graphical method allows the calculation of stresses in different planes, reducing them to vertical and horizontal components, known as principal planes, where the principal stresses are determined.

After performing a stress analysis on a material body, the components of the Cauchy stress tensor at a specific point are known concerning a coordinate system. Mohr's circle (Fig. 1) is then used to graphically visualize the stress components acting in a rotated coordinate system, which differs from the original system.

On Mohr's circle, the abscissa and ordinate (σ, τ) of the points represent the magnitudes of the normal stress and shear stress components acting on the rotated coordinate system. In essence, Mohr's circle is the locus of points representing the state of stress in the individual planes in all their orientations, where the axes correspond to the normal and shear stress components.

According to Parry [1], German engineer Karl Culmann was the first to devise a graphical representation for stresses in the 19th century, inspiring Christian Otto Mohr to extend this concept to two and three-dimensional stresses and develop a failure criterion based on the stress circle. Mohr's circle can also be applied to other 2x2 symmetric tensor matrices, such as the strain and moment of inertia tensors.

In structural, mechanical, or geotechnical engineering, for example, the distribution of stresses within an object, such as the stresses in a mass of rock around a tunnel, in the wings of an airplane, or in the columns of a building, is determined using stress analysis. This calculation involves determining the stresses at each object's point (or material particle). The stress at any point on an object (considered a continuum) is entirely defined by stress components of the tensor known as the Cauchy stress tensor. In two dimensions, the Cauchy stress tensor at a material point is completely defined by just three stress components with respect to any two perpendicular directions (see Fig. 2-a). These components are the stresses σ_x and σ_y normal to the specific coordinate system, and the shear stress τ_{xy}. The symmetry of the Cauchy stress tensor can be demonstrated from the angular momentum balance, implying that the shear stress $\tau_{yx} = \tau_{xy}$.

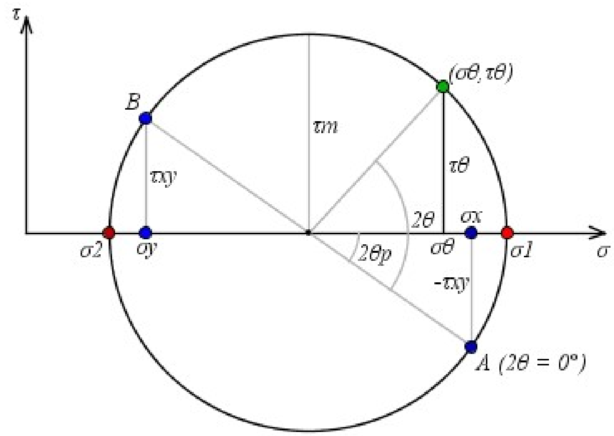

Figure 1. Mohr's circle of a plane stress state at a material point

After the stress distribution within the object has been determined with respect to a coordinate system, it may be necessary to calculate the components of the stress tensor (σ_θ, τ_θ) at a particular material point with respect to a rotated coordinate system, i.e., the stresses acting on a plane with a different orientation passing through that point of interest forming an angle θ with the coordinate system (Fig. 2-b).

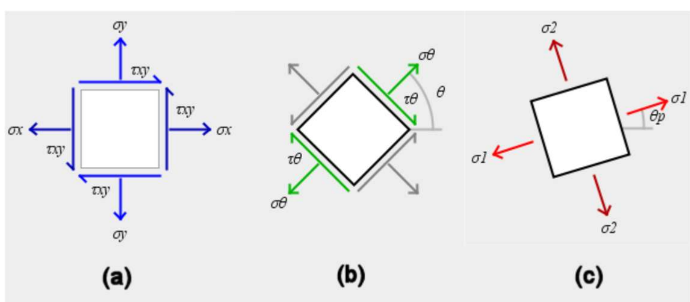

Figure 2. (a) Cauchy stress tensor; (b) stresses acting on a plane with a generic orientation; (c) plane orientation with the principal (maximum and minimal) stresses

One plane orientation of interest (θ_p) is the one in which the maximum normal stresses act (see Fig. 2-c). Another is the plane orientation with the maximum shear stress. To determine these special plane orientations, performing a tensor transformation under a rotation of the coordinate system is necessary. From the tensor definition, the Cauchy stress tensor obeys the tensor transformation law. Mohr's circle for stress is a graphical representation of this transformation law for the Cauchy stress tensor.

This article describes a graphical tool called e-Mohr that demonstrates how Mohr's circle works for plane stress states, allowing users to interactively manipulate the plane's orientation where the stress tensor components are calculated.

The e-Mohr program has been successfully used in the Solid Mechanics (Strength of Materials) disciplines of the Civil Engineering and Mechanical Engineering courses at PUC-Rio for over two decades. The program was originally created as a semester project of one discipline of Computer Graphics in the post-graduate program of Civil Engineering at PUC-Rio in 2004. However, this experience was not published before this article.

The main advantages of the e-Mohr program come from the programming language used: Java. This language was chosen because it allows the program to be executed online via the Internet or offline after downloading the program and the JRE (Java Runtime Environment). The program can be obtained via the URL: https://www.tecgraf.puc-rio.br/etools/mohr. Another advantage of the Java language is that it is based on the object-oriented programming paradigm, allowing code to be reused naturally. The source code of e-Mohr is open, and it is available on its website.

Figure 3 shows the sign convention for normal and shear stresses adopted in the work, which is based on the book *Mechanics of Materials* by Gere and Timoshenko [2].

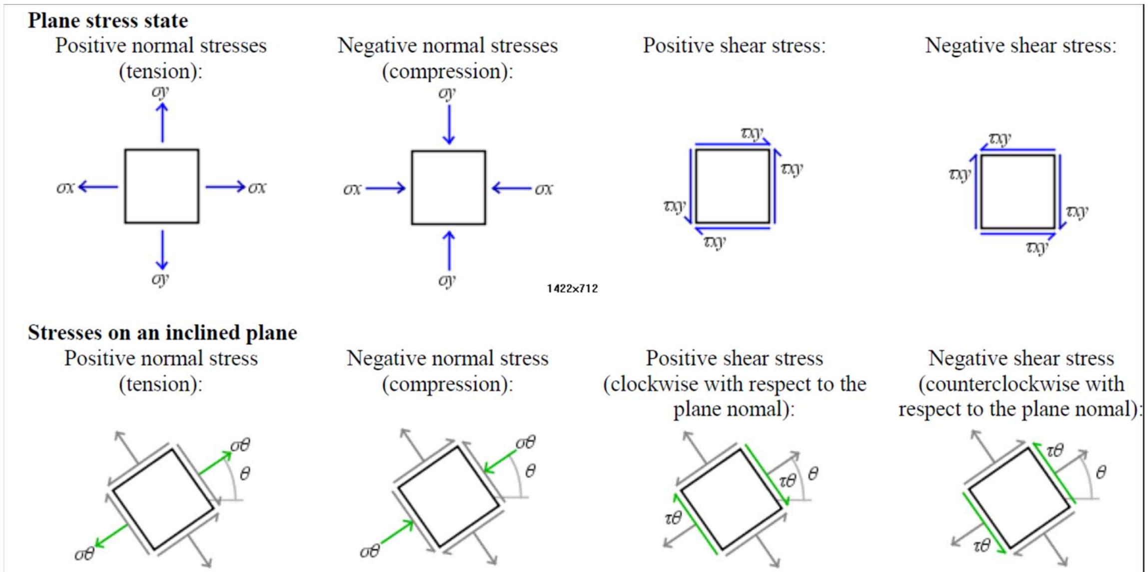

Figure 3. Sign convention for normal and shear stresses [2]

One may observe the consequence of this sign convention: on the plane for which $\theta = 0°$ (the face of σ_x), $\tau_\theta = -\tau_{xy}$; and, on the plane $\theta = 90°$ (the face of σ_y), $\tau_\theta = +\tau_{xy}$.

2 Graphical representation of the tensor transformation law: the Mohr's circle

The equilibrium of an infinitesimal element is deduced with the aid of Fig. 4.

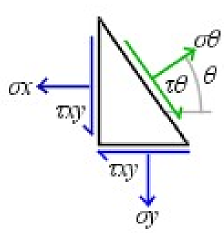

Figure 4. Equilibrium of an infinitesimal element at a material point with Cauchy stress components and stress components acting on a plane with a generic orientation

Equilibrium of forces in the direction of σ_θ, Gere and Goodno [3]:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \cos 2\theta + \tau_{xy} \cdot \text{sen} 2\theta \quad (1)$$

Equilibrium of forces in the direction of τ_θ, Gere and Goodno [3]:

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2} \cdot \text{sen}2\theta - \tau_{xy} \cdot \cos 2\theta \tag{2}$$

Equations (1) and (2) can be represented graphically by means of a circle, the Mohr's circle – Gere and Goodno [3], in a two-dimensional coordinate system (σ, τ), as depicted in Fig. 1. Stresses σ_θ and τ_θ can be obtained graphically by laying off an angle 2θ, counterclockwise, from line AB that connects points $(\sigma_x, -\tau_{xy})$ and (σ_y, τ_{xy}).

From the Mohr's circle in Fig. 1, it is possible to identify the maximum principal stress σ_1 – eq. (3), the minimum principal stress σ_2 – eq. (4), the angle θ_p that defines the maximum principal stress – eq. (5), and the maximum shear stress τ_m – eq. (6)

$$\sigma_1 = \frac{\left(\sigma_x + \sigma_y\right)}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \tag{3}$$

$$\sigma_2 = \frac{\left(\sigma_x + \sigma_y\right)}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \tag{4}$$

$$\theta_p = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \tag{5}$$

$$\tau_m = \frac{\sigma_1 - \sigma_2}{2} \tag{6}$$

As illustrated in Fig. 1, to determine the stress components (σ, τ) acting on a plane inclined at an angle θ counterclockwise in relation to plane A where σ_x acts, an angle 2θ is launched in the same counterclockwise direction around the Mohr's circle from the known stress point A $(\sigma_x, -\tau_{xy})$ to the point (σ, τ).

The double-angle approach is based on the fact that the θ angle between the normal vectors of any two physical planes passing through a material point (Fig. 4) is half the angle between two lines connecting their respective stress points (σ, τ) on Mohr's circle to the center of the circle.

This double-angle relationship arises from the parametric equations of Mohr's circle, which are functions of 2θ. Furthermore, it is observed that planes A $(\sigma_x, -\tau_{xy})$ and B (σ_y, τ_{xy}) in the material element of Fig. 4 are separated by an angle $\theta = 90°$, while in Mohr's circle, this angle is represented by 180° (twice the angle).

3 The pole

The procedure of launching an angle 2θ from the inclined line joining points A and B on Mohr's circle is relatively confusing. Fortunately, it is possible to define a point on the circumference from which the θ angle of the inclined plane may be launched. This point is defined as the *pole* or the *origin of planes* – Brandenberg [4].

The pole definition is based on one important property of a circle: the relation between a central angle and its inscribed angle (see Fig. 5): the central angle is twice the inscribed angle for a given subtended arc.

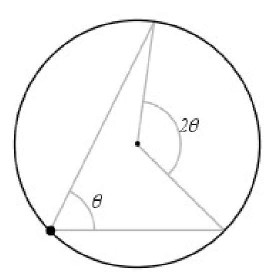

Figure 5. A central angle of a circle and its inscribed angle

There are numerous alternatives for the pole on the circumference of Mohr's circle. Figure 6 demonstrates that a point with coordinates (σ_y,$-\tau_{xy}$) in the two-dimensional space of Mohr's circle is a convenient choice for the pole. In this way, the angle θ is always measured starting from a horizontal line in the counterclockwise direction. This horizontal line represents the vertical plane where the σ_x stress acts (see Fig. 6). From the pole emanate normal directions (defined by angle θ) of inclined planes with stresses corresponding to the coordinates of the point (σ_θ, τ_θ) on the circumference.

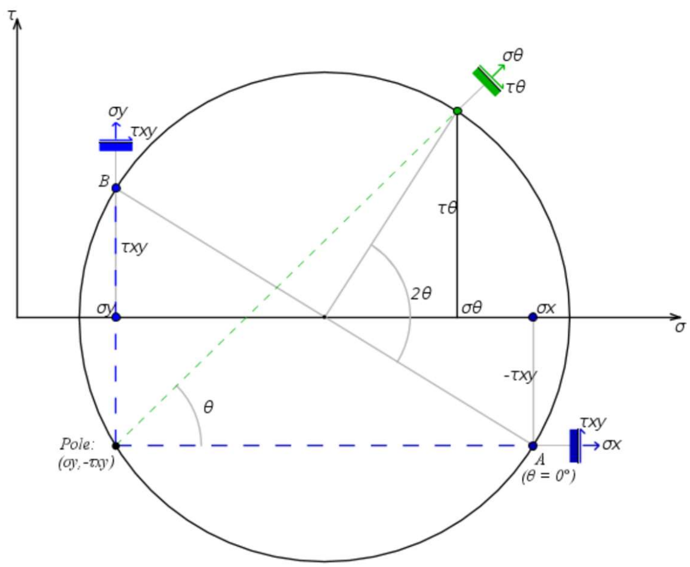

Figure 6. A convenient choice for the pole: point with coordinates (σ_y,$-\tau_{xy}$) in the stress space

4 The e-Mohr program

Figure 7 shows an image of the graphics interface of the e-Mohr program. There are four canvases in the interface of the e-Mohr program. The Mohr's circle is displayed on the biggest canvas, and the other three smallest canvases display, respectively, the infinitesimal element at a material point with the input stress state (Cauchy stress components), the stress components acting on a plane with a generic orientation θ, and the infinitesimal element oriented in the principal stress directions.

In the horizontal control bar at the top of the interface, there are the following options:

- To switch language between Portuguese ( ) and English ( );
- To always show the stress axes in the image of Mohr's circle in the main canvas ( );
- To frame the circle on the canvas ( );
- To turn on or off the display of the input stress state ( );
- To turn on or off the display of the stress components acting on a plane with a generic orientation θ ( );
- To turn on or off the display the principal stresses ( );
- To give information about the program ( ); and
- To launch a window with an explanation of the theoretical background of the program and its manipulation procedures ( ).

The input parameters of the program are the three components of the stress state (σ_x, σ_y, τ_{xy}) and the orientation θ of the generic plane where the stress components (σ, τ) act. These parameters may be input by entering their values in the correspondent canvases on the right of the interface. A potentiometer is also available to set the value of angle θ.

Alternatively, these parameters may be interactively adjusted by manipulating some control points in the Mohr's circle of the main canvas (see Fig. 8). Normal stresses σ_x e σ_y are adjusted by dragging, with the mouse, the corresponding points of Mohr's circle in the horizontal direction (along the normal stress axis). Shear stress τ_{xy} is adjusted dragging point (σ_x,$-\tau_{xy}$) or point (σ_y, τ_{xy}) of the Mohr's circle in the vertical direction (in the direction of the shear stress axis). Finally, the angle that defines the direction of the inclined plane normal is adjusted dragging point ($\sigma_\theta, \tau_\theta$) along the circumference of the Mohr's circle, around the pole. In the case in which point ($\sigma_\theta, \tau_\theta$) coincides with the pole, the direction θ of the inclined plane corresponds to the circle tangent direction at the pole.

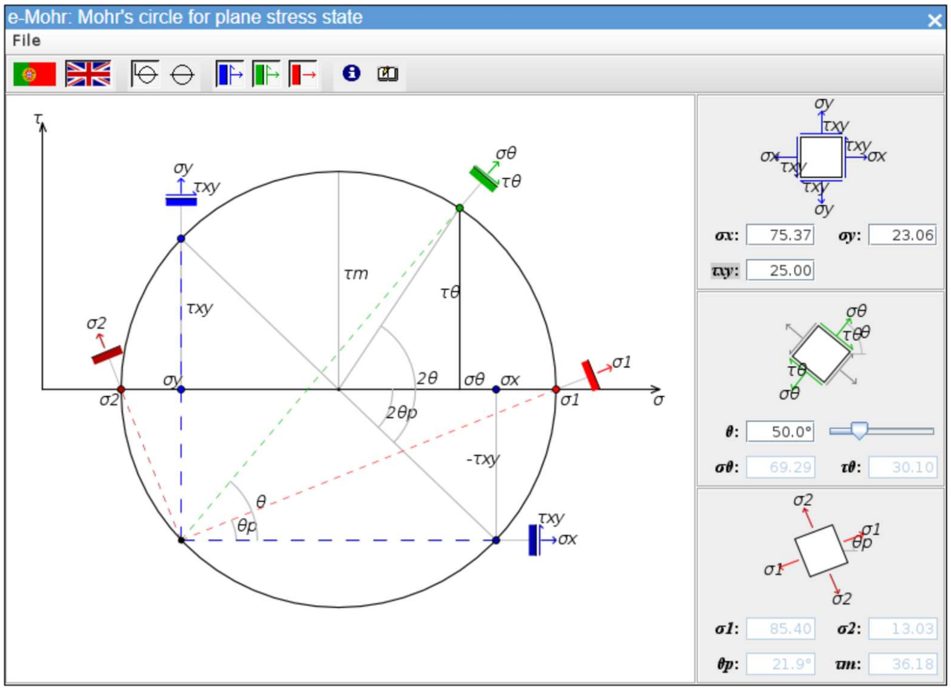

Figure 7. Graphics interface of the e-Mohr program

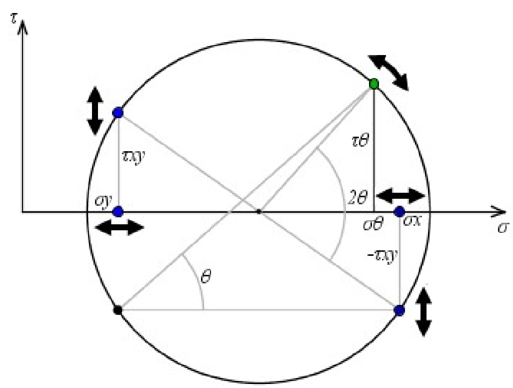

Figure 8. Control points for adjusting the plain stress state and inclined plane angle

5 Conclusions

The e-Mohr program is an educational tool used during Solid Mechanics (Strength of Materials) classes in any Civil Engineering or Mechanical Engineering course when teaching the Mohr's circle. Students can also access the program from their homes, making learning the method much more accessible.

The program demonstrates how Mohr's circle works for plane stress states, allowing users to interactively manipulate the plane's orientation where the stress tensor components are calculated. Furthermore, e-Mohr involves determining a point in the Mohr's circle called the pole or origin of planes. Any straight line drawn from the pole will intersect the circle at a point representing the state of stress on an inclined plane in the same orientation (parallel) in space as that line.

The software requires the installation of JRE - Java Runtime Environment. This is available at the following URL address: https://www.java.com/download/manual.jsp. The online (applet) version requires the installation of CheerpJ Applet Runner (https://cheerpj.com/cheerpj-applet-runner) in the browser.

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Authorship statement. The author hereby confirms that he is the sole liable person responsible for the authorship of this work and that all material that has been herein included as part of the present paper is either the property (and authorship) of the author or has the permission of the owner PUC-Rio.

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