Extension of an object-oriented framework to simulate reticulated structures considering the Reddy beam model

M. A. C. Rodrigues[1], R. B. Burgos[2], R. L. Rangel[3], E. Ruocco[4], L. F. Martha[5]

[1] *Dept. of Civil Engineering, University of Espírito Santo (UFES)*
Avenida Fernando Ferrari, 514, 29075-910, Vitória/ES, Brazil
marcos.a.rodrigues@ufes.br
[2] *Dept. of Structures and Foundations, State University of Rio de Janeiro (UERJ)*
Rua São Francisco Xavier, 524, 20550-900, Rio de Janeiro/RJ, Brazil
rburgos@eng.uerj.br
[3] *International Center for Numerical Methods in Engineering (CIMNE)*
C. Gran Capità S/N, 08034 Barcelona, Spain
rrangel@cimne.upc.edu
[4] *Dept. of Engineering, University of Campania "L. Vanvitelli" (Unicampania)*
Via Roma, 29, 81031, Aversa/CE, Italy
eugenio.ruocco@unicampania.it
[5] *Dept. of Civil and Environmental Engineering, Pontifical Catholic University of Rio de Janeiro (PUC-Rio)*
Rua Marquês de São Vicente, 225, 22453-900, Rio de Janeiro, RJ, Brazil
lfm@tecgraf.puc-rio.br

Abstract. Classical structural analysis software based on the Finite Element Method (FEM) typically considers the Euler-Bernoulli and Timoshenko beam theories. However, in some cases, these theories may not provide stress results with sufficient accuracy. Therefore, this study presents the extension of an open-source object-oriented framework for structural analysis of frame models to incorporate the Reddy beam model, as well as a proposed modified Reddy element. The latter enables discretizing the structural members with just one element. The computational implementation evidences the importance of developing object-oriented codes. The presented examples clearly show the improvements in shear stress prediction when high-order beams formulations are considered for the frame elements. Furthermore, the results also clarify some differences between the most usual beam formulations.

Keywords: Structural analysis, Reddy beam model, Shear stress, Object-oriented programming.

1 Introduction

Most software for structural analysis of frame models, such as Ftool [1], LESM [2], and Mastan [3], usually considers classical beam theories that follow Euler-Bernoulli (EBBT) and Timoshenko (TBT) element formulations. The Euler-Bernoulli beam theory is generally used to model framed structures. It assumes that the beam cross-section remains plane and orthogonal to the longitudinal axis after deformation, thereby disregarding shear deformation. Therefore, shear stresses are undetermined when using EBBT elements. On the other hand, the Timoshenko theory is usually employed to simulate structures with moderate slenderness ratio or small shear-to-bending ratio [4,5], as it considers shear deformation in an approximated manner in its formulation. It assumes that the beam cross-section remains plane, but not necessarily orthogonal to the deformed longitudinal axis, thus introducing a shear deformation due to the additional rotation of the cross-section. This leads to a constant shear stress distribution dependent on a shear correction factor [6] that may help to predict the global structural response in terms of deflection and vibration frequency [7]. However, it is well known that the shear stress distribution of beams with symmetric cross-section is usually parabolic, with the upper and lower surfaces being stress-free. Therefore, the Timoshenko theory fails to predict shear stress distributions over the cross-section of structures like composite beams, even when a shear correction factor is considered. Consequently, both EBBT and TBT elements, while often predicting comparable and reliable displacement fields, may not provide results of shear stress distribution with sufficient accuracy.

On the other hand, high-order beam theories can accurately predict shear stress distributions and are suited for simulating steel-concrete composite elements, laminated composite and sandwich beams, and functionally graded beams. For instance, the Reddy beam theory (RBT) [8-10] is a high-order beam model widely used to predict the shear stress distribution on plates and beams. It assumes that the displacement field follows a cubic function along the height of the cross-section, which is capable of warping. However, despite their accuracy in predicting shear stress distribution, high-order formulations may require a refined discretization of the beams to achieve accurate results. To deal with this, Rodrigues et al. [11] proposed an improved Reddy element able to simulate framed structures with reduced model discretizations.

The main objectives of this work is to briefly describe the theory of the standard and improved Reddy beam elements for 2D analyses, present their implementations in a structural analysis program dedicated to frame models, and show the advantages of considering high-order beam theories in structural modeling. The implementation was done in the program NUMA-TF (Numerical Analysis of Trusses and Frames) [12], an open-source MATLAB program.

2 Reddy beam model

2.1 Governing equations

Figure 1 illustrates the differences between Euler-Bernoulli, Timoshenko, and Reddy models for the bending behavior of a beam element. In the Reddy theory, warping of a cross-section is considered with the condition that the warped section's profile intersects the upper and lower surfaces of the element orthogonally.

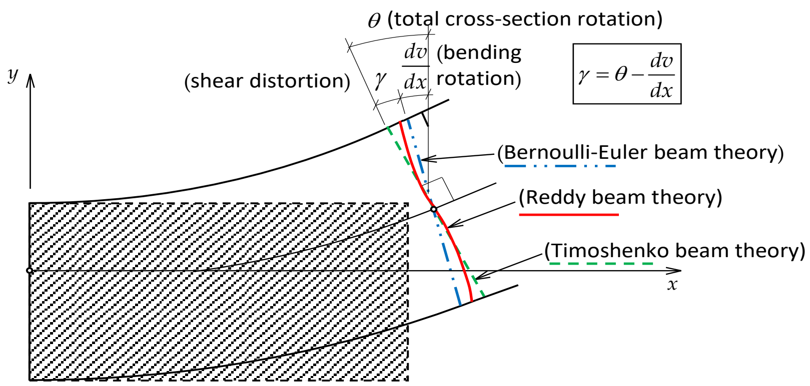

Figure 1. Euler-Bernoulli, Timoshenko, and Reddy bending theories [11].

The displacement field along the height of the cross-section is assumed as a cubic function in Reddy theory:

$$u(x,y) = u_0(x) - y\theta(x) + \alpha y^3\left(\theta(x) - \frac{dv_0(x)}{dx}\right) \qquad v(x,y) = v_0(x), \tag{1}$$

where $\alpha = 4/3h^2$ is the Reddy constant, h is the cross-section height, $u_0(x)$ and $v_0(x)$ are the cross-section axial and transverse displacement at the centroid, respectively, and $\theta(x)$ is the cross-section rotation. As in the Timoshenko theory, the cross-section rotation and transversal displacement are not associated, and both are independent variables.

The Green-Lagrange strain tensor is then expressed as:

$$\varepsilon_x = \frac{du_0}{dx} - y\frac{d\theta}{dx} + \alpha y^3\left(\frac{d\theta(x)}{dx} - \frac{d^2v_0(x)}{dx^2}\right), \qquad \gamma_{xy} = \left(\theta - \frac{dv_0}{dx}\right)(3\alpha y^2 - 1) \tag{2}$$

Under the assumption of linear-elastic behavior of a homogenous and isotropic material, the stresses are $\sigma_x = E\varepsilon_x$ and $\tau = G\gamma$. Thus, the virtual work principle gives:

$$\int_V \sigma_x \delta\left[-y\frac{d\theta}{dx}+\alpha y^3\left(\frac{d\theta}{dx}-\frac{d^2v_0}{dx^2}\right)\right]dV+\int_V \tau\delta\left[\left(\theta-\frac{dv_0}{dx}\right)(3\alpha y^2-1)\right]dV-\int_0^L(q\delta v_0+m\delta\theta)dx=0, \quad (3)$$

where q is the transversal load and m is the distributed moment along the x (longitudinal) axis of the element.

The usual internal forces (Q^R, M^R) and the high-order internal forces (V, P), due to the new contributions from deformations, are defined as:

$$Q=-\int_A \tau dA\,; \quad M=-\int_A y\sigma_x dA\,; \quad V=-\int_A y^2\tau dA\,; \quad P=-\int_A y^3\sigma_x dA\,, \quad (4)$$

which leads to:

$$\int_0^L(M-\alpha P)\,\delta\frac{d\theta}{dx}dx+\int_0^L\alpha P\,\delta\frac{d^2v_0}{dx^2}dx+\int_0^L(Q-3\alpha V)\left(\delta\theta-\delta\frac{dv_0}{dx}\right)dx-\int_0^L(q\delta v_0+m\delta\theta)dx=0. \quad (5)$$

Finally, the system of governing differential equations of the Reddy model is obtained integrating by parts:

$$\begin{cases}\delta v\colon \dfrac{d(Q-3\alpha V)}{dx}+\dfrac{d^2(\alpha P)}{dx^2}+q=0\rightarrow\dfrac{dT}{dx}+\dfrac{d^2(\alpha P)}{dx^2}+q=0\\ \delta\theta\colon(Q-3\alpha V)-\dfrac{d(M-\alpha P)}{dx}+m=0\rightarrow T-\dfrac{dM^R}{dx}+m=0\end{cases}, \quad (6)$$

The three boundary conditions are:

$$\left(v, T+\frac{d(\alpha P)}{dx}\right),\left(\frac{dv}{dx},\alpha P\right),(\theta,M^R), \quad (7)$$

where $Q^R = T + d(\alpha P)/dx$ is the effective shear force and $M^R = M-\alpha P$ is the bending moment associated to the Reddy theory. Therefore, the primary variables are the geometric boundary condition v, dv/dx, and θ, and the related secondary variables are the force boundary condition Q^R, P, and (M^R), respectively. One should note that this beam theory requires the specification of both dv/dx, and θ [13].

2.2 Finite element model

The finite element model aims to discretize the continuous behavior into a discretized problem based on the nodal displacements using interpolation functions. In the RBT element, each node has four degrees of freedom: the two translations (u, v) and the two rotations (dv/dx, θ). The nodal displacements are depicted in Fig. 2.

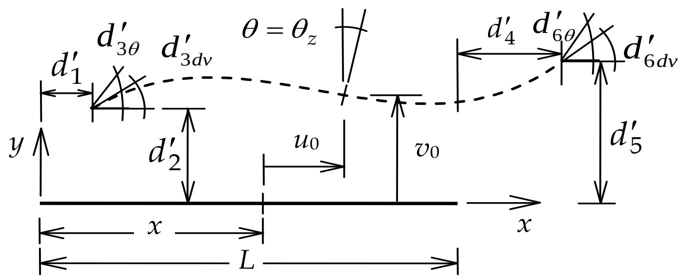

Figure 2. Deformed configuration of an isolated Reddy element [11].

The displacement field along the element is written as follows:

$$u_0(x)=N_1^u(x)d_1'+N_4^u(x)d_4'\rightarrow u_0(x)=\{N_u(x)\}\{u\}, \quad (8)$$

$$v_0(x)=N_2^v(x)d_2'+N_3^v(x)d_{3dv}'+N_5^v(x)d_5'+N_6^v(x)d_{6v}'\rightarrow v_0(x)=\{N_v(x)\}\{v\}, \quad (9)$$

$$\frac{dv_0(x)}{dx}=\frac{N_2^v(x)}{dx}d_2'+\frac{N_3^v(x)}{dx}d_{3dv}'+\frac{N_5^v(x)}{dx}d_5'+\frac{N_6^v(x)}{dx}d_{6dv}'\rightarrow\frac{dv_0(x)}{dx}=\left\{\frac{dN_v(x)}{dx}\right\}\{v\}, \quad (10)$$

$$\theta(x)=N_1^\theta(x)d_{3\theta}'+N_4^\theta(x)d_{6\theta}'\rightarrow\ \theta(x)=\{N_\theta(x)\}\{v\}, \quad (11)$$

where $N_v(x)$ corresponds to the Hermitian interpolation functions for the out-of-plane component $v_0(x)$, and $N_u(x)$ and $N_\theta(x)$ are linear interpolation functions for the axial displacement $u_0(x)$ and rotation $\theta(x)$, respectively.

The stiffness matrix is then calculated as:

$$\begin{aligned}
K^{RBT} &= \{\delta u\}^T \int_0^L EA\{N_u{}'\}\{N_u{}'\}^T dx\{u\} + \{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_\theta{}'\}^T dx\{\theta\} - \frac{1}{5}\{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_\theta{}'\}^T dx\{\theta\} \\
&+ \frac{1}{5}\{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_v{}''\}^T dx\{v\} - \frac{1}{5}\{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_\theta{}'\}^T dx\{\theta\} + \frac{1}{21}\{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_\theta{}'\}^T dx\{\theta\} \\
&- \frac{1}{21}\{\delta\theta\}^T \int_0^L EI_z\{N_\theta{}'\}\{N_v{}''\}^T dx\{v\} + \frac{1}{5}\{\delta v\}^T \int_0^L EI_z\{N_v{}''\}\{N_\theta{}'\}^T dx\{\theta\} - \frac{1}{21}\{\delta v\}^T \int_0^L EI_z\{N_v{}''\}\{N_\theta{}'\}^T dx\{\theta\} \\
&+ \frac{1}{21}\{\delta v\}^T \int_0^L EI_z\{N_v{}''\}\{N_v{}''\}^T dx\{v\} + \frac{8}{15}GA\left(\{\delta v\}^T \int_0^L \{N_v{}'\}\{N_v{}'\}^T dx\{v\} + \{\delta\theta\}^T \int_0^L \{N_\theta{}'\}\{N_\theta{}'\}^T dx\{\theta\}\right),
\end{aligned} \tag{12}$$

which leads to an 8x8 matrix for K^{RBT}:

$$\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 \\
0 & \left(\frac{4EI}{7L^3}+\frac{16GA}{25L}\right) & \left(\frac{2EI}{7L^2}+\frac{4GA}{75}\right) & \frac{4GA}{15} & 0 & \left(-\frac{4EI}{7L^3}-\frac{16GA}{25L}\right) & \left(\frac{2EI}{7L^2}+\frac{4GA}{75}\right) & \frac{4GA}{15} \\
0 & \left(\frac{2EI}{7L^2}+\frac{4GA}{75}\right) & \left(\frac{4EI}{21L}+\frac{16GAL}{225}\right) & \left(\frac{16EI}{105L}-\frac{2GAL}{45}\right) & 0 & \left(-\frac{2EI}{7L^2}-\frac{4GA}{75}\right) & \left(\frac{2EI}{21L}-\frac{4GAL}{225}\right) & \left(-\frac{16EI}{105L}+\frac{2GAL}{45}\right) \\
0 & \frac{4GA}{15} & \left(\frac{16EI}{105L}-\frac{2GAL}{45}\right) & \left(\frac{68EI}{105L}+\frac{8GAL}{45}\right) & 0 & -\frac{4GA}{15} & \left(-\frac{16EI}{105L}+\frac{2GAL}{45}\right) & \left(-\frac{68EI}{105L}+\frac{4GAL}{45}\right) \\
-\frac{EA}{L} & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 \\
0 & \left(-\frac{4EI}{7L^3}-\frac{16GA}{25L}\right) & \left(-\frac{2EI}{7L^2}-\frac{4GA}{75}\right) & -\frac{4GA}{15} & 0 & \left(\frac{4EI}{7L^3}+\frac{16GA}{25L}\right) & \left(-\frac{2EI}{7L^2}-\frac{4GA}{75}\right) & -\frac{4GA}{15} \\
0 & \left(\frac{2EI}{7L^2}+\frac{4GA}{75}\right) & \left(\frac{2EI}{21L}-\frac{4GAL}{225}\right) & \left(-\frac{16EI}{105L}+\frac{2GAL}{45}\right) & 0 & \left(-\frac{2EI}{7L^2}-\frac{4GA}{75}\right) & \left(\frac{4EI}{21L}+\frac{16GAL}{225}\right) & \left(\frac{16EI}{105L}-\frac{2GAL}{45}\right) \\
0 & \frac{4GA}{15} & \left(-\frac{16EI}{105L}+\frac{2GAL}{45}\right) & \left(-\frac{68EI}{105L}+\frac{4GAL}{45}\right) & 0 & -\frac{4GA}{15} & \left(\frac{16EI}{105L}-\frac{2GAL}{45}\right) & \left(\frac{68EI}{105L}+\frac{8GAL}{45}\right)
\end{bmatrix} \tag{13}$$

However, the employed interpolation functions do not correspond to the solution of the differential equations. Therefore, a refined discretization of the structure is necessary to solve simple problems with this formulation. Meanwhile, in linear analyses with both classical theories (EBBT and RBT), just one element is required to reach the exact solution using cubic interpolation functions. To overcome this limitation of the Reddy beam model, Rodrigues et al. [11] proposed new interpolation functions for the cross-section rotation directly derived from the differential equation solution of Eq. (6), being expressed as:

$$\theta(x) = N_2^\theta(x)d_2' + N_3^\theta(x)d_{3dv}' + N_1^\theta(x)d_{3\theta}' + N_5^\theta(x)d_5' + N_6^\theta(x)d_{6v}' + N_4^\theta(x)d_{6\theta}', \tag{14}$$

where,

$$N_2^\theta(x) = N_5^\theta(x) = 6\frac{x^2}{L^3} - 6\frac{x}{L^2} \quad N_3^\theta(x) = N_6^\theta(x) = 3\frac{x^2}{L^2} - 3\frac{x}{L} \quad N_1^\theta(x) = 1 - \frac{x}{L} \quad N_4^\theta(x) = \frac{x}{L}, \tag{15}$$

These interpolation functions enable linear structural analyses using just one element per member, as it is done with EBBT and TBT models. The stiffness matrix of this modified Reddy beam theory (MRBT), K^{MRBT}, can be derived based on Eq. (12), and the result is:

$$\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
0 & \frac{6EI}{L^2} & \left(\frac{64EI}{21L}+\frac{8GAL}{45}\right) & \left(\frac{16EI}{105L}-\frac{8GAL}{45}\right) & 0 & -\frac{6EI}{L^2} & \left(\frac{62EI}{21L}+\frac{4GAL}{45}\right) & \left(-\frac{16EI}{105L}-\frac{4GAL}{45}\right) \\
0 & 0 & \left(\frac{16EI}{105L}-\frac{8GAL}{45}\right) & \left(\frac{68EI}{105L}+\frac{8GAL}{45}\right) & 0 & 0 & \left(-\frac{16EI}{105L}-\frac{4GAL}{45}\right) & \left(-\frac{68EI}{105L}+\frac{4GAL}{45}\right) \\
-\frac{EA}{L} & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\
0 & \frac{6EI}{L^2} & \left(\frac{62EI}{21L}+\frac{4GAL}{45}\right) & \left(-\frac{16EI}{105L}-\frac{4GAL}{45}\right) & 0 & -\frac{6EI}{L^2} & \left(\frac{64EI}{21L}+\frac{8GAL}{45}\right) & \left(\frac{16EI}{105L}-\frac{8GAL}{45}\right) \\
0 & 0 & \left(-\frac{16EI}{105L}-\frac{4GAL}{45}\right) & \left(-\frac{68EI}{105L}+\frac{4GAL}{45}\right) & 0 & 0 & \left(\frac{16EI}{105L}-\frac{8GAL}{45}\right) & \left(\frac{68EI}{105L}+\frac{8GAL}{45}\right)
\end{bmatrix}. \tag{16}$$

Finally, the stresses are calculated directly from the displacements as:

$$\sigma_x = E\varepsilon_x = E\left[\frac{du_0}{dx} - y\frac{d\theta}{dx} + \alpha y^3\left(\frac{d\theta(x)}{dx} - \frac{d^2v_0(x)}{dx^2}\right)\right], \quad (17)$$

$$\tau = G\gamma_{xy} = G\left(\theta - \frac{dv_0}{dx}\right)(3\alpha y^2 - 1). \quad (18)$$

Notice that the shear stress has a quadratic term, and the normal stress a cubic one, that does not exist in the TBT beam element.

3 Computational implementation

The Reddy beam model was implemented in the NUMA-TF program [14] following the OOP paradigm. The implementation required some classes to be modified or added. Figure 3 provides a general overview of the new code architecture with a Class Diagram, developed according to the UML (Unified Modeling Language) format [15] and adapted from Cavalcanti et al. [16].

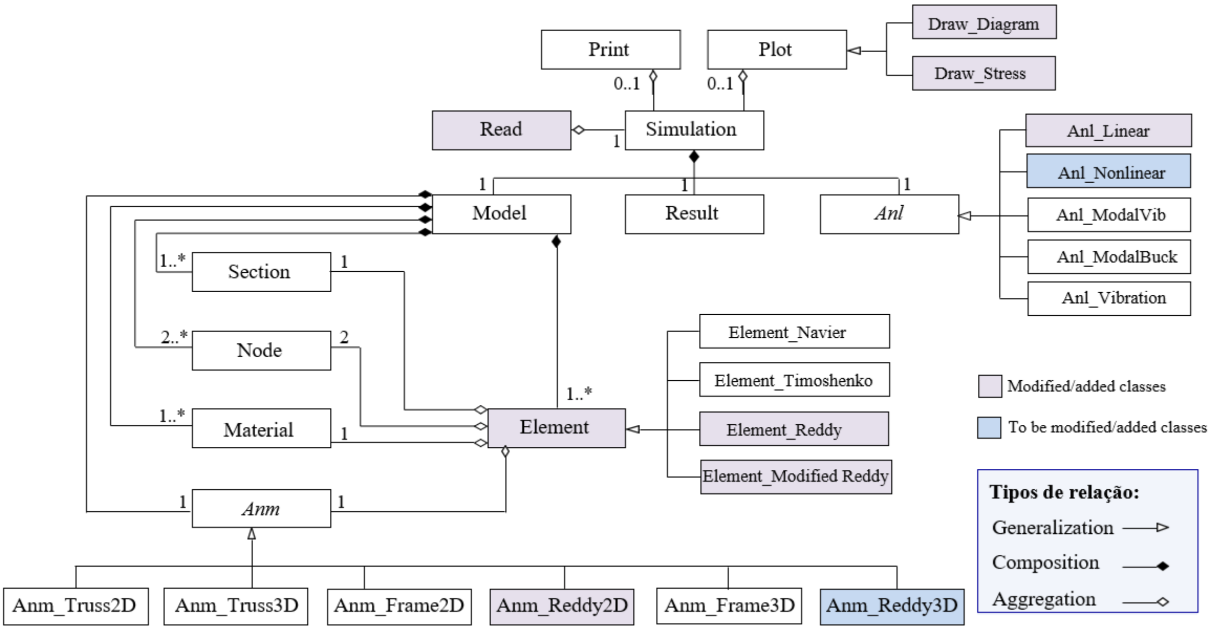

Figure 3. UML class diagram of NUMA-TF after the implementation of Reddy beam model.

The Reddy element (*Anm_Reddy*) is a subclass of *Anm*, as the new model has different degrees of freedom from the previously existing subclasses (*Anm_Truss, Anm_Frame*). Also, the generic beam class *Element* is now specialized into an RBT element (*Element_Reddy*) or an MRBT element (*Element_Modified Reddy*), besides EBBT (*Element_Navier*) and TBT (*Element_Timoshenko*) elements. A class to draw the stress results (*Draw_Stress*) was also implemented.

4 Examples

Examples are presented here in order to show a briefly comparison between the distinct beam theories. A clamped beam and a frame presented in Rodrigues et al. [11] are analyzed, as shown in Fig. 4. Both structural models have a length L = 1 m and a Young's modulus of E = 10^7 kN/m^2. The clamped beam is subjected to a concentrated load P = 1030 kN and a bending moment M = 1.03 kNm, while the frame has a distributed load q =10000 kN/m. The elements have a slenderness ratio $\lambda = L/h = 10$. The cross-section has a form factor of $\chi = 5/6$ for the TBT element.

Figure 6 shows the solutions for the transversal displacement v and cross-section rotation θ of the clamped beam model, when discretized by a single RBT and MRBT element, compared against the analytical solution given by Ruocco and Reddy [17]. Due to the high slenderness, the transversal displacement and rotation are similar in all formulations using only one element for discretizing the members, except for the RBT element that requires a refine discretization to provide the same result. The shear force and bending moment diagrams are also the same. However, the shear stress distributions along the height of cross-sections are different, as it can be seen in Fig. 7 to Fig. 10. Figures 7 presents the shear stress distributions in a cross-section of the clamped beam model using different discretization levels, while Fig. 8 shows the results along the beam obtained with the RBT (or MRBT) and TBT formulations and an 80-element discretization. Figure 9 and 10 provide the same results for the upper horizontal beam of the frame model. It should be remarked that the shear stress for the EBBT element is always null.

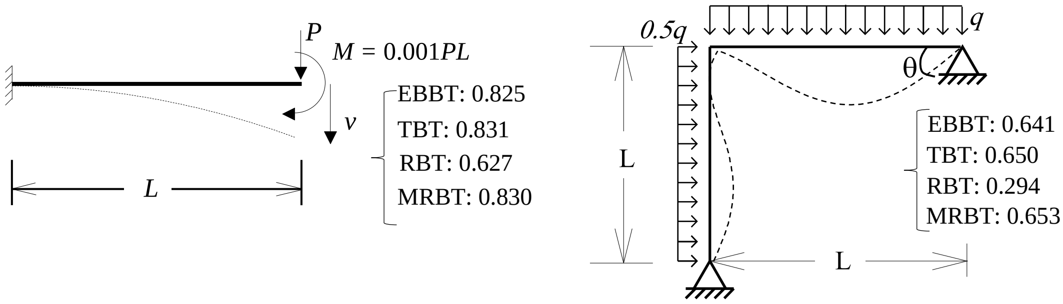

Figure 5. Example models [11].

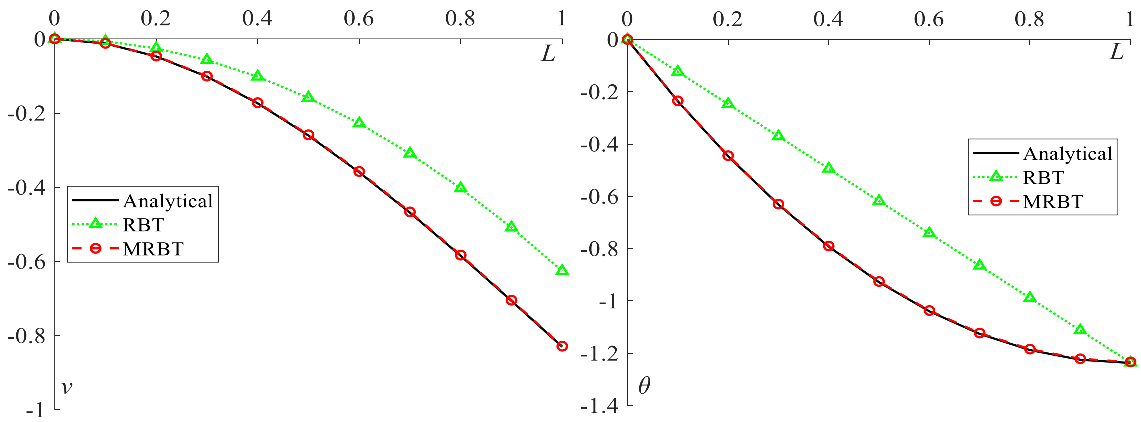

Figure 6. Solutions for transversal displacement and rotation with a one-element discretization of the clamped beam model: Analytical, RBT and MRBT [11].

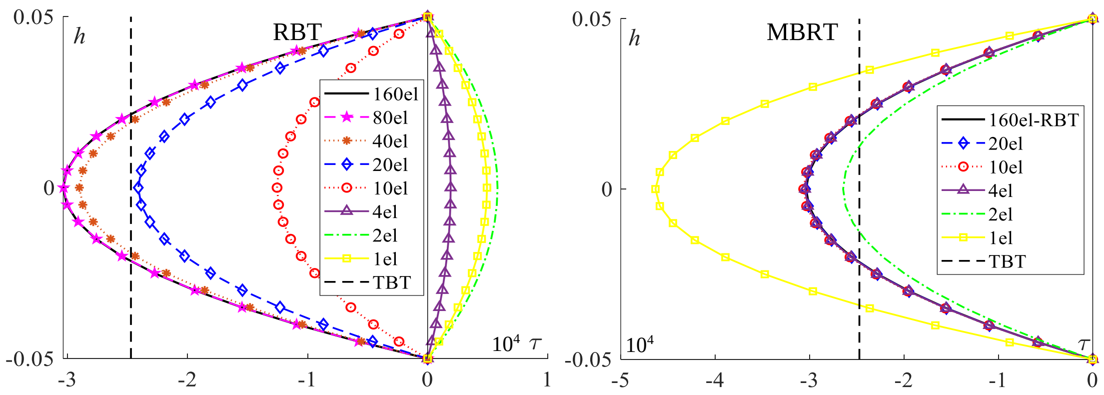

Figure 7. Cross-section shear stress distributions of the clamped beam model with different discretizations.

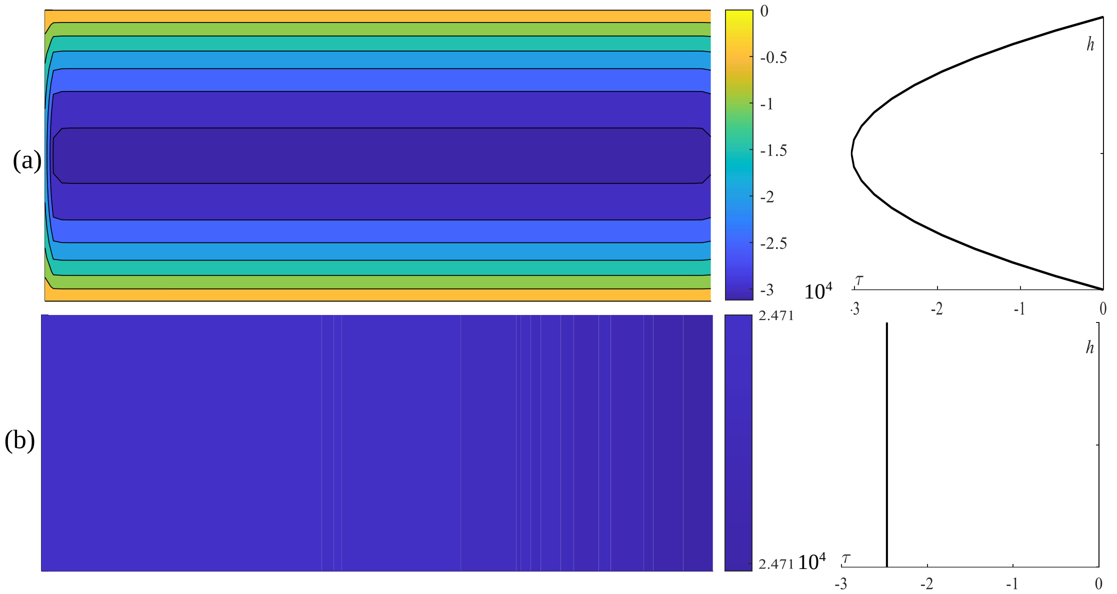

Figure 8. Shear stress distributions along the clamped beam: (a) RBT/MRBT element and (b) TBT element.

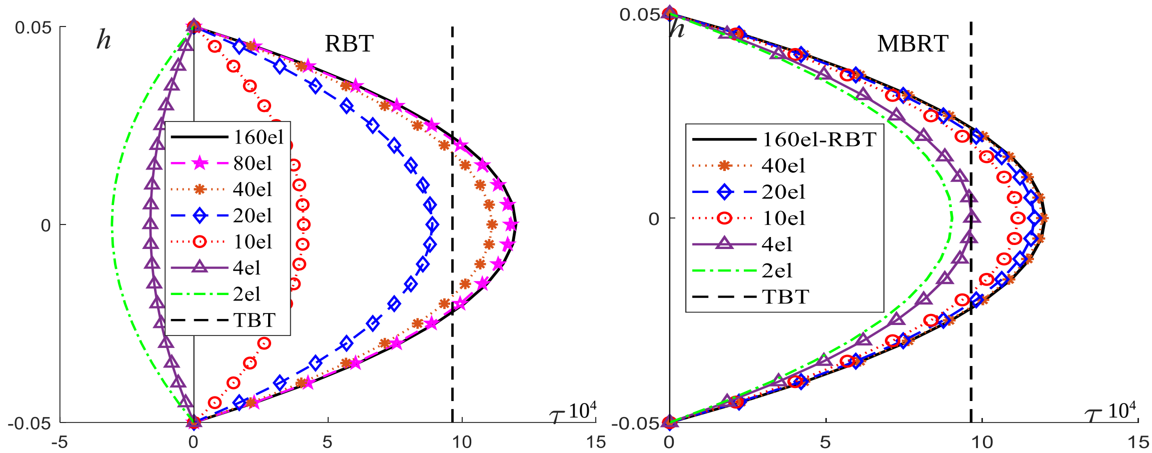

Figure 9. Cross-section shear stress distributions of the frame model (horizontal beam) with different model discretizations.

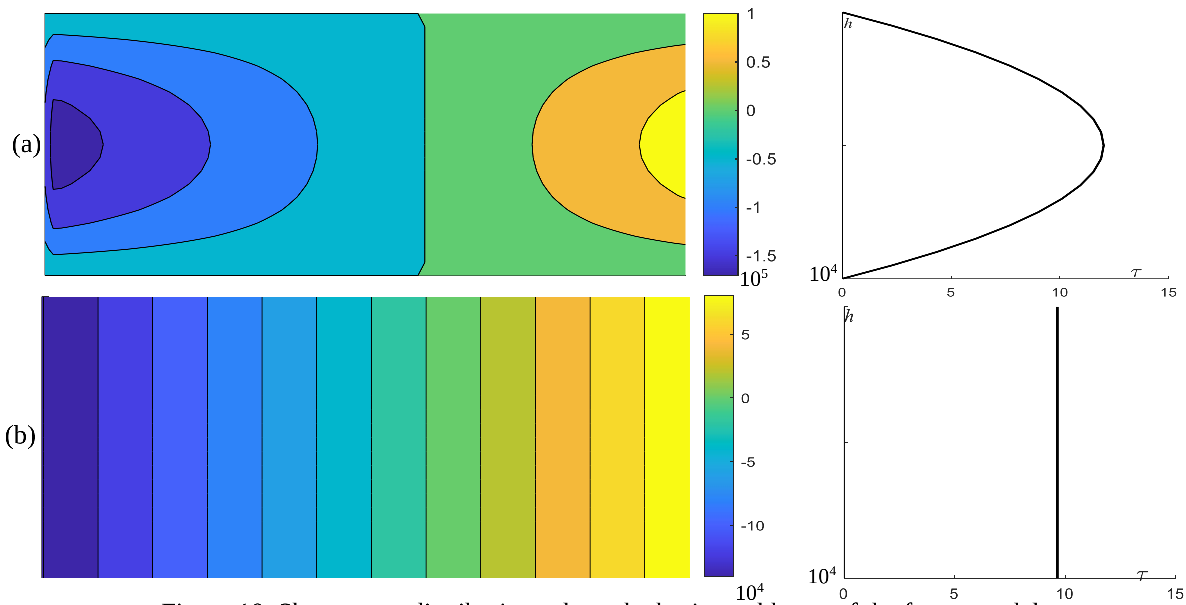

Figure 10. Shear stress distributions along the horizontal beam of the frame model: (a) RBT/MRBT element and (b) TBT element.

5 Conclusions

This work discussed the development and implementation of standard and improved Reddy beam theories for 2D structural analysis of frame models.

The examples evidenced that the use of the presented high-order beam theories provides the same results as the classical beam theories for the displacements and internal force diagrams. However, the Euler-Bernoulli theory is not able to calculate such stresses and, if the Timoshenko beam theory is used, the resulting shear distribution is constant in the cross-section. Accurate results of shear stress distribution are only achieved when Reddy models are considered. Therefore, in cases where the shear stress distribution is relevant for structural design, a high-order theory, such as the presented Reddy models, becomes necessary. In particular, if a single element discretization is applied, it is fundamental to employ the modified RBT element.

It was also demonstrated that the Reddy beam models can be easily implemented in a finite element code, despite the increased degrees of freedom. In fact, the computational implementation took advantage of the generality and modularity provided by an OOP framework.

Finally, the 3D formulation and implementation of improved Reddy elements, as well as their use for nonlinear structural analyses, are under development and will be discussed in a future work.

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