Geometric Nonlinear Analysis using the Two-cycle Method in Ftool

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Abstract. With recent advances in design and material technology, increasingly slender structures are being conceived, which makes nonlinear analysis an important task for efficient and safe projects. Geometrically nonlinear (second-order) problems are usually solved using the Finite Element Method (FEM) along with iterative numerical schemes, in which the structure response is directly influenced by the level of discretization and the nonlinear solution algorithm used. Thus, nonlinear analysis demands some experience from the analyst in terms of the parameters involved in the solution algorithms and structural behavior in general. To reduce the discretization dependence, exact solutions based on the deformed infinitesimal element equilibrium can be used as interpolation functions. To circumvent the difficulties in dealing with parameters related to the numerical methods, the two-cycle method can be used, since it is not dependent on load or displacement steps. In this work, the educational software Ftool, widely used by Civil Engineers and students, was adapted to perform two-cycle analyses employing frame elements based on solutions of the differential equations obtained from the deformed configuration. The results in terms of displacements and rotations for the studies examples are identical to the analytical solutions, showing that the combination of the two-cycle method with the exact element formulation is promising and can diminish the need for discretization and the use of complicated nonlinear solution algorithms.

Keywords: Ftool, structural analysis, geometric nonlinearity, second-order effects, educational software.

1 Introduction

Unlike first-order linear analysis, a geometric nonlinear analysis using the Finite Element Method (FEM) usually requires discretization (subdivision) of the structural members. Among other reasons, discretization is paramount since the traditional cubic interpolation functions (Hermitian polynomials) are not the homogeneous solution of the problem differential equation.

Some works use the differential equation of the problem to develop an “exact” element, for example, using the equilibrium of infinitesimal element as in Goto and Chen [1], Chan and Gu [2], and Rodrigues et al. [3]. Most of these researchers consider shear deformation using Timoshenko beam theory. The importance of shear deformation is greater in structures with small slenderness or composite materials (sandwich beams, for instance). This work will use the Euler-Bernoulli (EB) beam theory for the sake of simplicity in implementation.

Geometric nonlinearity is especially relevant in slender structures, which present lateral displacements that are large enough to significantly change the way loads act upon them. The usual way to capture this behavior is to subdivide the elements. Since discretization can sometimes be undesirable, especially when dealing with undergraduate students who are unaware of this concept, a solution to overcome it is interesting from a didactic point of view. In this context, Ftool (Two-dimensional Frame Analysis Tool, [4]), a widely used software for structural analysis, stands out as the obvious choice for implementing the findings of this work since it is renowned for its simplicity and educational purposes.

This work proposes the use of interpolation functions coming from the exact solution of the differential equation of an axially loaded beam to obtain tangent stiffness coefficients, which will be used in a two-cycle scheme. These functions are obtained following the work of Burgos and Martha [5]. All examples were implemented in an improved version of Ftool. This improved version performs the traditional first-order analysis and applies the two-cycle iterative method [6] without the need to subdivide the domain to obtain reliable results for second-order analyses. Some examples of single columns and simple frames were analyzed, and comparisons were made against analytical solutions when available.

2 Solution of the differential equation of the deformed beam element

In this section, the equilibrium conditions of an infinitesimal element considering the deformed condition will be developed. Figure 1 represents the free-body diagram of an element subjected to transverse and axial loads, which is used for the development of the equations.

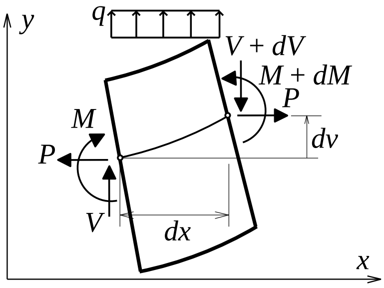

Figure 1. Equilibrium of the infinitesimal element

2.1 Equilibrium equations

Figure 1 shows a deformed infinitesimal element subjected to a distributed transverse load q and an axial constant load P. Vertical force and bending moment are allowed to vary along the x-axis. Equilibrium equations are obtained according to Eq. (1), in which $v(x)$ is transverse displacement, $V(x)$ is vertical force, and $M(x)$ is the bending moment acting upon the cross-section.

$$\sum F_y \rightarrow -dV + q\left(x\right)dx = 0 \rightarrow \frac{dV}{dx} = q(x)$$
$$\sum M \rightarrow dM - \left(V + dV\right)dx - Pdv + q(x)\frac{dx^2}{2} = 0 \rightarrow \frac{dM}{dx} - P\frac{dv}{dx} = V \quad (1)$$

It is interesting to notice that one of the consequences of using the deformed configuration is that the transversal load $V(x)$ is no longer the derivative of the bending moment. This is because, in the deformed configuration, the shear force Q and the vertical force V are not the same. Figure 2 shows this difference.

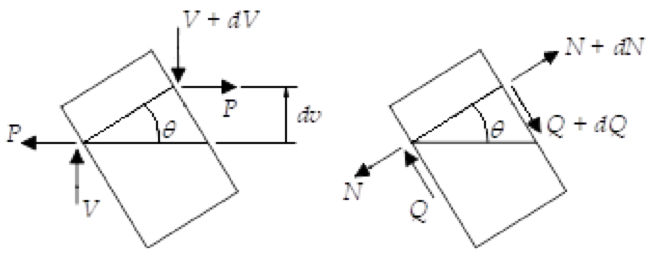

Figure 2. Normal and shear forces in the deformed configuration

Using the relation between bending moment and curvature, $M(x) = EI\, d\theta/dx$, in which $\theta(x)$ is the cross-section rotation, then taking the derivative and substituting $dV/dx = q(x)$ from equation (1), the general equation for a deformed beam element is obtained:

$$EI\frac{d^3\theta}{dx^3} - P\frac{d^2v}{dx^2} = q(x) \tag{2}$$

For Euler-Bernoulli beam theory (EBBT), the cross-section rotation is the derivative of the lateral displacements, so Eq. (2) can be written only in terms of the displacement $v(x)$. This is not true when Timoshenko beam theory is used, and, in this case, there is the need for an additional equation that relates v and θ [5].

$$\frac{d^4v}{dx^4} - \frac{P}{EI}\frac{d^2v}{dx^2} = \frac{q(x)}{EI} \tag{3}$$

The homogenous solution of eq. (4) depends on the sign of the load P. If P is positive (tension), a parameter μ^2 given by P/EI is introduced, and the homogeneous equation solution is given in terms of hyperbolic functions:

$$\frac{d^4v}{dx^4} - \mu^2\frac{d^2v}{dx^2} = 0, \quad \mu^2 = \frac{P}{EI} \tag{4}$$

$$\begin{aligned} v_h(x) &= A\sinh(\mu x) + B\cosh(\mu x) + Cx + D \\ \theta_h(x) &= A\mu\cosh(\mu x) + B\mu\sinh(\mu x) + C \end{aligned} \tag{5}$$

If P is negative (compression), $\mu^2 = -P/EI$, and Eq. (4) is rewritten in a way such that the solution is given by trigonometric functions:

$$\frac{d^4v}{dx^4} + \mu^2\frac{d^2v}{dx^2} = 0, \quad \mu^2 = -\frac{P}{EI} \tag{6}$$

$$\begin{aligned} v_h(x) &= A\sin(\mu x) + B\cos(\mu x) + Cx + D \\ \theta_h(x) &= A\mu\cos(\mu x) - B\mu\sin(\mu x) + C \end{aligned} \tag{7}$$

2.2 Interpolation functions

In the context of the direct stiffness method, the analytical behavior of a frame element can be approximated by a discrete behavior. The discrete solution is represented by nodal displacements, while the continuous solution is obtained by interpolating the nodal displacements using shape functions. Figure 3 shows the deformed configuration of an element obtained from interpolation of nodal values. Axial displacement $u(x)$ uses nodal values d_1 and d_4, while transversal displacement $v(x)$ uses d_2, d_3, d_5 and d_6.

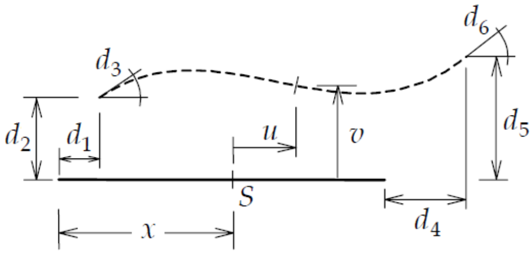

Figure 3. Deformed configuration of a frame element

The deformed shape of an element in terms of lateral displacement and rotations can be written based on the nodal values using interpolation functions, as shown in Eq. (8). These interpolation functions ($N_j(x)$) are calculated directly using the homogenous solution of the problem differential equation, i.e., from the equilibrium of a deformed infinitesimal element subjected to a compressive force, as in Eqs. (5) or (7), depending on the sign of the axial force. Due to space limitations, the expressions of interpolation functions will not be presented here. They can be found in [5]. These functions are then used in a scheme based on the Virtual Work Principle to generate the tangent stiffness matrix [7].

$$\begin{aligned} u(x) &= N_1^u(x)d_1 + N_4^u(x)d_4 \\ v(x) &= N_2^v(x)d_2 + N_3^v(x)d_3 + N_5^v(x)d_5 + N_6^v(x)d_6 \\ \theta(x) &= N_2^\theta(x)d_2 + N_3^\theta(x)d_3 + N_5^\theta(x)d_5 + N_6^\theta(x)d_6 \end{aligned} \rightarrow \begin{Bmatrix} u(x) \\ v(x) \\ \theta(x) \end{Bmatrix} = [N]\{d\} \tag{8}$$

3 Geometric nonlinear analysis

In the context of a geometric nonlinear analysis, there's a need to consider the axial force effect in the stiffness matrix using the deformed configuration of the element. By applying the usual procedure to calculate the stiffness matrix in the context of a Finite Element Analysis, based on the Virtual Work Principle, the following expressions are obtained:

$$[k] = \int_0^L [B]^{\mathrm{T}}[E][B]dx$$

$$[B] = \begin{bmatrix} \frac{d(N_1^u)}{dx} & 0 & 0 & \frac{d(N_4^u)}{dx} & 0 & 0 \\ 0 & \frac{d(N_2^\theta)}{dx} & \frac{d(N_3^\theta)}{dx} & 0 & \frac{d(N_5^\theta)}{dx} & \frac{d(N_6^\theta)}{dx} \\ 0 & \frac{d(N_2^v)}{dx} & \frac{d(N_3^v)}{dx} & 0 & \frac{d(N_5^v)}{dx} & \frac{d(N_6^v)}{dx} \end{bmatrix}, \quad [E] = \begin{bmatrix} EA+P & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & P \end{bmatrix} \tag{9}$$

The unique values for stiffness coefficients (only for the bending part) are given in Eqs. (10) and (11) for the case of positive and negative values of *P*, respectively.

$$\begin{gathered} k_{22} = EI\,\frac{\mu^3 \sinh(\mu L)}{D}, \quad k_{23} = EI\,\frac{\mu^2\left[\cosh(\mu L) - 1\right]}{D}, \\ k_{33} = EI\,\frac{\mu\left[\mu L\cosh(\mu L) - \sinh(\mu L)\right]}{D}, \quad k_{36} = EI\,\frac{\mu\left[\sinh(\mu L) - \mu L\right]}{D} \\ D = 2 - 2\cosh(\mu L) + \mu L\sinh(\mu L), \quad \mu = \sqrt{\frac{P}{EI}} \end{gathered} \tag{10}$$

$$\begin{gathered} k_{22} = EI\,\frac{\mu^3 \sin(\mu L)}{D}, \quad k_{23} = EI\,\frac{\mu^2\left[1 - \cos(\mu L)\right]}{D} \\ k_{33} = EI\,\frac{\mu\left[\sin(\mu L) - \mu L\cos(\mu L)\right]}{D}, \quad k_{36} = EI\,\frac{\mu\left[\mu L - \sin(\mu L)\right]}{D} \\ D = 2 - 2\cos(\mu L) - \mu L\sin(\mu L), \quad \mu = \sqrt{-\frac{P}{EI}} \end{gathered} \tag{11}$$

In Eqs. (10) and (11), *P* is the axial force, *EI* is the bending stiffness, and *L* is the element's length. If Hermitian polynomials are used as interpolation functions, Eq. (10) leads to a tangent matrix which can be separated into the traditional elastic and geometric stiffness matrices. In fact, if a first-order Taylor Series expansion is performed on the coefficients in Eq. (11), the same matrices are obtained [5].

The expressions in Eqs. (10) and (11) are meant to be used in a two-cycle framework [8], without the need for iterative procedures like Newton-Raphson. For a given value of the axial load *P*, the solution in terms of displacements and rotations will be equal to the one obtained when solving the differential equation analytically. Of course, this is true only if the problem remains within the limit of small rotations and pre-critical behavior. The tangent stiffness matrix of the frame element to be used in the second cycle is as follows:

$$K_t = \begin{bmatrix} \frac{EA+P}{L} & 0 & 0 & \frac{-EA-P}{L} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & -k_{22} & k_{23} \\ 0 & k_{23} & k_{33} & 0 & -k_{23} & k_{36} \\ \frac{-EA-P}{L} & 0 & 0 & \frac{EA+P}{L} & 0 & 0 \\ 0 & -k_{22} & -k_{23} & 0 & k_{22} & -k_{23} \\ 0 & k_{23} & k_{36} & 0 & -k_{23} & k_{33} \end{bmatrix} \quad (12)$$

3.1 Two-cycle Iterative Method

This method, developed by Chen & Lui [6], uses the following equilibrium equation in two stages, where $\{F\}$ and $\{U\}$ are the force and displacement vectors, respectively; $\{K\}$ is the stiffness matrix:

$$\{F\} = [K]\{U\} \quad (16)$$

According to Silva *et al.* [8], the method begins with the first cycle, which involves a linear analysis of the structure using the elastic (first-order) stiffness matrix K. This first cycle is used to capture the values of the axial loads in each element. In the second cycle, the consideration of the geometric nonlinearity of the structure is added, and another analysis is started. Finally, nodal forces are recalculated in the elements using the last obtained matrix. This process is iterative, and the forces are applied only once in each cycle.

3.2 Implementation in Ftool

The Two-cycle iterative method was implemented in Ftool, and a context menu was added to allow this analysis. A tolerance was introduced to avoid division by zero when the axial load is null in an element. Figure 4 shows how this menu appears in Ftool.

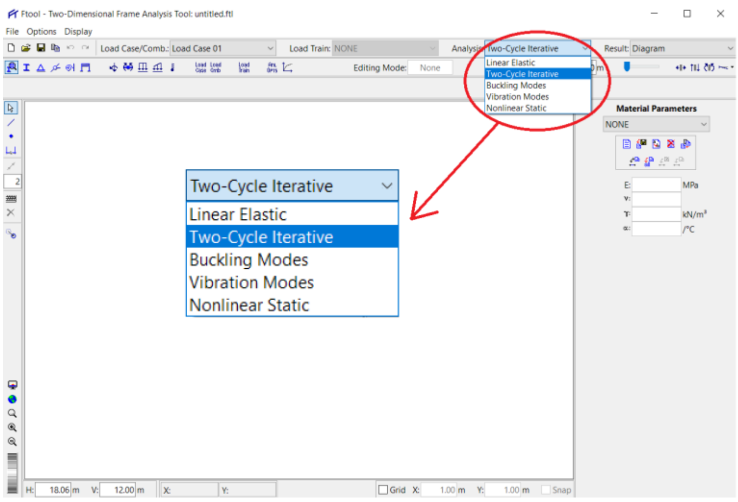

Figure 4. Two-cycle in Ftool

4 Examples

In this section, the updated version of Ftool was used to obtain the solutions for the models shown in Figure 5. A second-order analysis using the two-cycle iterative method in Ftool was compared with the second-order analysis performed by the nonlinear analysis software *Mastan2* [9], which uses the predictor-corrector method and cubic (Hermitian) polynomials. The models in *Mastan2* were discretized using 1 and 5 elements per bar for comparison. Results are also compared with analytical solutions, where available. For the two-cycle solution, each load value results in a displacement vector that is independent of the previous steps. In all examples, the proposed formulation without discretization obtained results that are equivalent to the analytical or to the over discretized solutions.

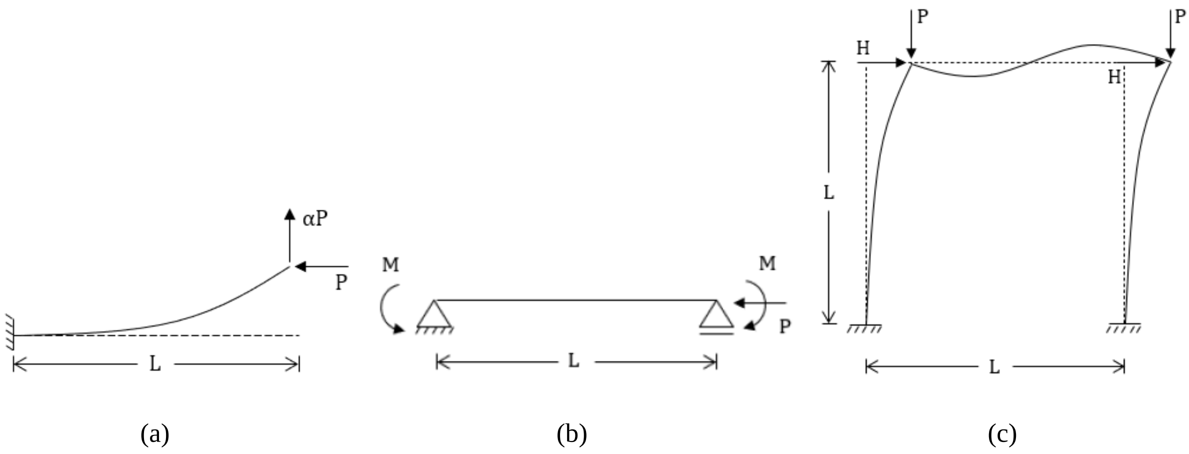

Figure 5. Proposed examples: a) Cantilever; b) Simply-supported; c) Three bar frame.

4.1 Cantilever column

According to Figure 5a, the first model analyzed is a column fixed at one end and free at the other, subjected to a compressive load P and a transverse load αP. This model has an analytical solution that will be used for comparison. The parameters for this example are L = 6 m; E = 10^8 kN/m²; A = 10^{-2} m^2; I = 10^{-5} m⁴; α = 0.01. Results in terms of displacement at the free end are shown in Figure 6.

4.2 Simply-supported column

The second example is a simply supported column, according to Figure 5b, subjected to a compressive load P and two moments with opposite directions at the ends, given by M = 0.001PL. All the other parameters are the same as in the previous example. Results in terms of rotation at the right end are shown in Figure 7.

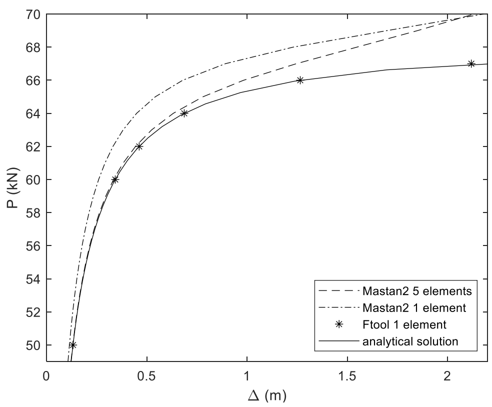

Figure 6. Results from example 1 (cantilever)

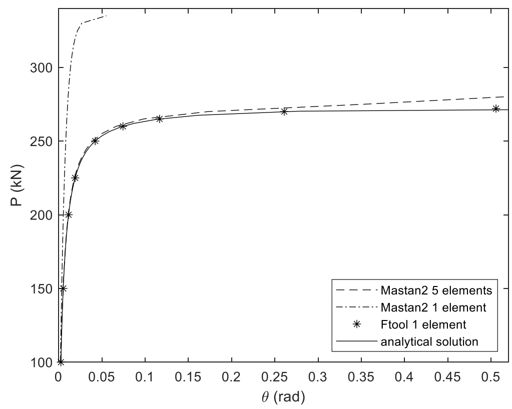

Figure 7. Results from example 2 (simply-supported)

4.3 Three-bar frame with sway allowed

Figure 5c shows an unbraced (sway) frame subjected to a vertical load P and horizontal loads H =0.001P. All the other parameters are the same as in previous examples. Results in terms of horizontal displacement at the top are shown in Figure 8.

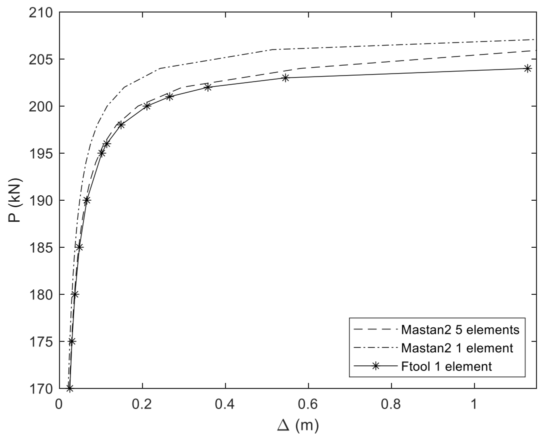

Figure 8. Results from example 3 (sway frame)

5 Conclusions

In this paper, to reduce the need for discretization in a second-order analysis, exact solutions based on the deformed infinitesimal element equilibrium were used as interpolation functions. The methodology was implemented in Ftool, with excellent results. Considering that Ftool is an educational software, and many students and even young professionals lack the necessary experience in element subdivision and iterative nonlinear schemes, the proposed solution is elegant and convenient.

In the case of single elements, analytical solutions are available, and the comparison showed that the numerical results obtained are identical to the exact ones. In the case of frames, the comparison was made against discretized solutions obtained in Mastan2. The results were also excellent.

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Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work and that all material that has been herein included as part of the present paper is the authorship of the authors.

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