Nonlinear Dynamics of a Cantilever Beam under a Transversal Harmonic Follower Force

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Abstract. The geometrically nonlinear analysis of cantilever beams is well disseminated in current literature since many structures can be modelled as a cantilever beam from high towers and long wind turbine blades to microbeams used in atomic microscopy. In many applications, particularly those with fluid forces the loads do not maintain its direction throughout the analysis. This work addresses the case of large displacements for beams considering a harmonic follower transversal load. This type of situation generates interesting nonlinearities and has several applications, for example, wind turbine blades where the wind loads remain perpendicular to beam axis. In this case, the nonlinear partial differential equation of motion has time varying coefficients, leading to a type of Mathieu-Hill equations. This influences the nonlinear dynamic behavior of the structural system as well as its stability. According results for large response it's possible to observe the great influence of load level and initial boundary condition, for high level of load or high initial displacement an instability its observed.

Keywords: stability, nonlinear dynamic, parametric excitation.

1 Introduction

The stability of structures subjected to follower compressive loads has been addressed by many researchers in the past [1], [2]. Flutter-type instability or divergence of beams under follower compressive forces depends on several factors and parameters. A non-prismatic free-attached beam subjected to an end force that remains tangent to the beam axis during deformation (Beck's problem) is a classic example of follower force instability problems [1]. However, the investigation of dynamic transversal follower forces is not as widespread in the technical literature. This type of situation can be observed, for example, in wind turbine blades. The wind pressure on the blade is always perpendicular to the surface of the blade and the change in direction during the deformation process becomes important in slender structures subjected to large displacements. Deformations and stability arising from static transversal loads were addressed by several researchers adopting analytical [3], [4] and numerical solution [5], [6]. When dynamic analysis is included in transversal follower load problems, one eventually falls back on a parametric excitation analysis, since the dynamic load also depends on the rotation of the beam at its point of application.

The study of harmonic, forced, damped oscillations and even coupled oscillators is frequently addressed in the nonlinear dynamics literature. A particular problem is the analysis of structures under parametric excitation where at least one of the coefficients in the differential equation of motion varies with time [7]. When this parameter variation induces an oscillatory movement in the system, triggering an internal build-up of energy, we have the phenomenon known as parametric instability or parametric resonance. [8], [9]. This has been for many decades an important problem in structural dynamics. Parametric excitation leads to a system of homogeneous differential equations with time-varying coefficient. The linearized equations result in a system of Mathieu–Hill equations, where the unstable solutions grow exponentially. The main instability region occurs when the driving frequency is twice the system's natural frequency.

In the present paper, the behavior of a cantilever beam subjected to a follower transversal dynamic load is considered. This investigation aims to understand the structural behavior of typical slander elements subjected to dynamic wind load, in which it always acts perpendicular to the surface. In mathematical terms, the problem is analytically modeled for large displacement and subsequently a modal reduction is made using the Galerkin method to obtain the nonlinear equations of motion.

2 Mathematical Formulation

2.1 Galerkin Method Reduction

Figure 1a shows the undeformed cantilever beam with a Young modulus E, span L, moment of inertia I, cross section area A and concentrated dynamic load $P(t)$ applied at the free end. Figure 1b shows the deformed cantilever beam where $\theta(s)$ is the slope, $y = y(s,t)$, the transversal displacement, $x = x(s,t)$ the longitudinal displacement, s, the local coordinate in deformed axis and t is time. Also, $x = x(L,t) = x_L$, $y = y(L,t) = y_L$ and $\theta = \theta(L,t) = \theta_L$

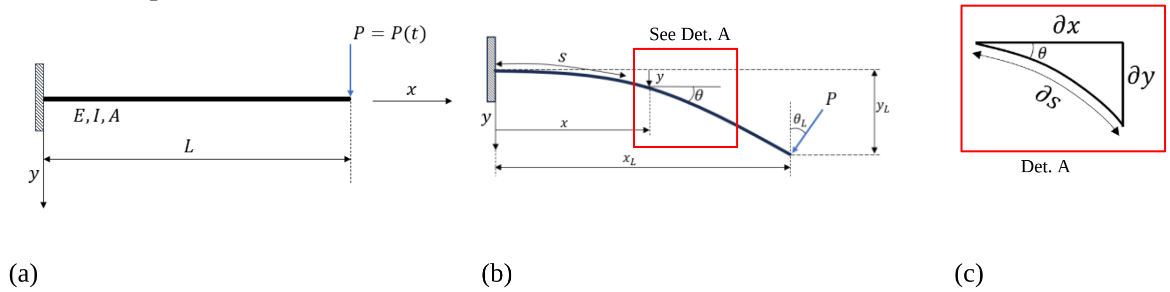


Figure 1. Cantilever beam with end dynamic follower load in (a) undeformed state, (b) deformed state with load P following end slope θ_L and (c) Infinitesimal element of deformed beam in a generic position (x, y)

Knowing that bending moment is proportional to the change of curvature and stiffness EI; and the curvature is a first derivative of slope [1], the flexural moment at a position (x, y) due to P is given by:

$$EI\frac{\partial\theta}{\partial s} = P_y(x_L - x) + P_x(y_L - y) \tag{1}$$

P_x and P_y are the components of the follower force in the x and y directions, respectively, and depend on the slope at the free end, θ_L:

$$P_x = P_x(t) = P(t)\cdot\sin(\theta_L) \qquad P_y = P_y(t) = P(t)\cdot\cos(\theta_L) \tag{2}$$

Taking the second derivative of equation (1) with respect to s, one obtains:

$$EI\frac{\partial^2}{\partial s^2}\left[\frac{\partial\theta}{\partial s}\right] = -P_y\frac{\partial}{\partial s}\left[\frac{\partial x}{\partial s}\right] - P_x\frac{\partial^2 y}{\partial s^2} \tag{3}$$

By geometry, as illustrated in Figure 1c, the following expressions for the derivatives of x and y with respect to s are obtained:

$$\frac{\partial x}{\partial s} = \cos(\theta) \qquad \frac{\partial y}{\partial s} = \sin(\theta) \tag{4}$$

The derivative of Eq. (4) with respect to s results in:

$$\frac{\partial^2 y}{\partial s^2} = \cos(\theta)\frac{\partial\theta}{\partial s} \quad \rightarrow \quad \frac{\partial\theta}{\partial s} = \frac{1}{\cos(\theta)}\frac{\partial^2 y}{\partial s^2} \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3), leads to:

$$EI\frac{\partial^2}{\partial s^2}\left[\frac{1}{\cos(\theta)}\frac{\partial^2 y}{\partial s^2}\right]=-P_y\frac{\partial}{\partial s}\left[\cos(\theta)\right]-P_x\frac{\partial^2 y}{\partial s^2} \tag{6}$$

Considering Taylor series expansions of the trigonometric functions in (5), the following approximations are obtained:

$$\frac{1}{\cos(\theta)}\approx 1+\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2 \qquad \cos(\theta)\approx 1-\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2 \tag{7}$$

Substituting Eqs. (7) into Eq. (6), the following static nonlinear differential equation is obtained:

$$EI\frac{\partial^2}{\partial s^2}\left[\left(1+\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2\right)\frac{\partial^2 y}{\partial s^2}\right]=-P_y\frac{\partial}{\partial s}\left[1-\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2\right]-P_x\frac{\partial^2 y}{\partial s^2} \tag{8}$$

Adding the inertial and damping forces, results in the following nonlinear partial differential equation of motion:

$$m\frac{\partial^2 y}{\partial t^2}+C\frac{\partial y}{\partial t}+EI\frac{\partial^2}{\partial s^2}\left[\left(1+\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2\right)\frac{\partial^2 y}{\partial s^2}\right]=-P_y\frac{\partial}{\partial s}\left[1-\frac{1}{2}\left(\frac{\partial y}{\partial s}\right)^2\right]-P_x\frac{\partial^2 y}{\partial s^2} \tag{9}$$

and adopting $\frac{\partial y}{\partial s}=y_{,s}$ and $\frac{\partial y}{\partial t}=\dot{y}$ as the notation, results in:

$$m\ddot{y}+C\dot{y}+EI\left[\left(1-0.5y_{,s}^2\right)y_{,ssss}+3\cdot y_{,s}\cdot y_{,ss}\cdot y_{,sss}+y_{,ss}^3\right]=P_y\cdot y_{,s}\cdot y_{,ss}-P_x\cdot y_{,ss} \tag{10}$$

Taking into account Eqs. (4) and (5) and substituting Eq. (7) into Eq. (2), one obtains:

$$P_x=P\cdot y_{,s=L} \qquad P_y=P\cdot\left(1-0.5\cdot y_{,s=L}^2\right) \tag{11}$$

Finally, substituting Eq. (11) into the Eq. (10), the nonlinear equation of motion up to cubic nonlinearities takes the form

$$m\ddot{y}+C\dot{y}+EI\left[\left(1-0.5y_{,s}^2\right)y_{,ssss}+3\cdot y_{,s}\cdot y_{,ss}\cdot y_{,sss}+y_{,ss}^3\right]=P\cdot\left(1-0.5\cdot y_{,s=L}^2\right)\cdot y_{,s}\cdot y_{,ss}-P\cdot y_{,s=L}\cdot y_{,ss} \tag{12}$$

Using separation of variables, the transversal displacement can be written as $y=y(s,t)=d(t)\Phi(s)$. Thus, Eq. (12) takes the form:

$$\begin{aligned}&m\ddot{d}\Phi+\bar{C}\dot{d}\Phi+EI\left[\left(1-0.5d^2\Phi_{,s}^2\right)d\Phi_{,ssss}+3\cdot d^3\Phi_{,s}\cdot\Phi_{,ss}\cdot\Phi_{,sss}+d^3\Phi_{,ss}^3\right]=\\&P\cdot\left(1-0.5\cdot d^2\Phi_{,s=L}^2\right)\cdot d^2\Phi_{,s}\cdot\Phi_{,ss}-P\cdot d^2\Phi_{,s=L}\cdot\Phi_{,ss}\end{aligned} \tag{13}$$

Applying Galerkin to Eq. (13), and disregarding (for now) damping for simplification purposes, we have:

$$M\ddot{d}+K_1d+K_2d^3=P(t)\cdot K_3d^2+P(t)\cdot K_4d^4 \tag{14}$$

Considering $P=P(t)=P_0\cos(\Omega t)$ where P_0 and Ω are module for dynamic force and the excitation frequency, respectively, the following equation is obtained after some algebraic manipulations:

$$\ddot{d}+\left[\omega^2+\frac{K_2}{M}d^2-\frac{P_0}{M}d\cdot\left(K_3+K_4d^2\right)\cdot\cos(\Omega t)\right]d=0 \tag{15}$$

where $\omega=\sqrt{\frac{K_1}{M}}$ is the natural frequency of structure and

$$M=\int_0^L \Phi m\Phi ds \tag{16}$$

$$K_1=\int_0^L EI\cdot\Phi_{,ssss}\Phi\cdot ds \tag{17}$$

$$K_2 = \int_0^L \left[EI\left(-0.5\Phi_{,s}^2 \cdot \Phi_{,ssss} + 3\cdot\Phi_{,s}\cdot\Phi_{,ss}\cdot\Phi_{,sss} + \Phi_{,ss}^3\right)\Phi \right] ds \quad (18)$$

$$K_3 = \int_0^L \Phi_{,s}\cdot\Phi_{,ss}\Phi\cdot ds - \int_0^L \Phi_{,s=L}\cdot\Phi_{,ss}\Phi\cdot ds \quad (19) \qquad K_4 = \int_0^L -0.5\cdot\Phi_{,s=L}^2\Phi_{,s}\cdot\Phi_{,ss}\cdot\Phi\cdot ds \quad (20)$$

The adopted interpolating functions are the free vibration modes of a clamped-free beam given by

$$\Phi(s) = \frac{1}{2}\left[\cosh\left(\frac{\lambda s}{L}\right) - \cos\left(\frac{\lambda s}{L}\right) - \vartheta\left[sinh\left(\frac{\lambda s}{L}\right) - sin\left(\frac{\lambda s}{L}\right)\right]\right] \quad (21)$$

where parameters λ and ϑ are given by Blevins [10] for each vibration mode.

2.2 Approximation by Galerkin-Urabe Method

The Galerkin-Urabe Method [11], [12] has been susecfully applied to abtain an approximate responses of nonlinear dynamic systems under harmonic loads. Assuming the the steady state of Eq. (15) as $d = A\cos(\Omega t)$, multiplying it by the weighting function $\cos(\Omega t)$ and integrating over one period (from 0 to $2\pi/\Omega$) results in:

$$\int_0^{2\pi/\Omega}\left[-A\Omega^2\cos(\Omega t) + \frac{\left[K_1 + K_2A^2\cos^2(\Omega t) - P_0A\cos^2(\Omega t)\cdot(K_3 + K_4A^2\cos^2(\Omega t))\right]A\cos(\Omega t)}{M}\right]\cos(\Omega t)dt = 0$$

(22)

Solving Eq. (22) we have the following analytical expression correlating the load magnitude P_0, the forcing frequency Ω and the ampitude of the steady-state response A

$$P_0 = \frac{6K_2A^2 - 8\Omega^2M + 8K_1}{5A^3K_4 + 6AK_3} \quad (23)$$

Figure 2 shows the correlation between the normalized beam displacement A/L with the normalized dynamic load P_0L^2/EI for selected values of Ω/ω, based on Eq. (23).

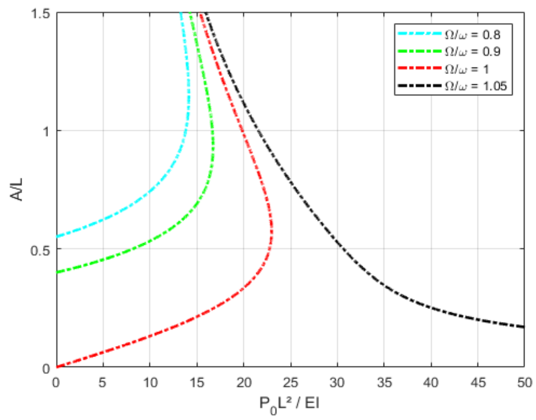


Figure 2. Correlation between the normalized beam displacement A/L with the normalized dynamic load parameter P_0L^2/EI for selected values of Ω/ω

3 Numerical Results and Discussions

To understand the dynamic response and stability of the dynamical system, Eq. (15) is numerically integrated using the Runge-Kutta algorithm implemented in Matlab [13]. Table 1 presents the geometric and material parameters for the beam used in the numerical analysis. Considering only the first vibration mode in the Galerkin process, $\lambda = 1.875$ and $\vartheta = 0.734$ in Eq. (24).

Table 1. Beam Geometrical and Material Properties

Parameter	Nomenclature	Value
Beam span	L	$5.0\ m$
Young Modulus	E	$2.0 \cdot 10^{11}$ N/m²
Distributed mass	m	$1256\ kg/m$
Moment of inertia	I	$0.0021\ m^4$

The curve for $\Omega/\omega = 1$ (main resonance region) in Figure 2 is compared with the numerically obtained results in Figure 3 for selected values of the normalized transverse load P_0L^2/EI. The highlighted values in Figure 3 corresponds to the approximate response of A/L for the following load parameters: $P_0L^2/EI = 5$, $P_0L^2/EI = 15$, $P_0L^2/EI = 20$ e $P_0L^2/EI = 22.3$. For these same points, the response in time domain was obtained using the Runge-Kutta method considering an initial displacement $d_0/L = 0.1$. Figure 4 shows the steady state displacement are in agreement with the value estimated by the approximate response in Figure 3 for the load levels $P_0L^2/EI = 5$, $P_0L^2/EI = 15$ and $P_0L^2/EI = 20$.

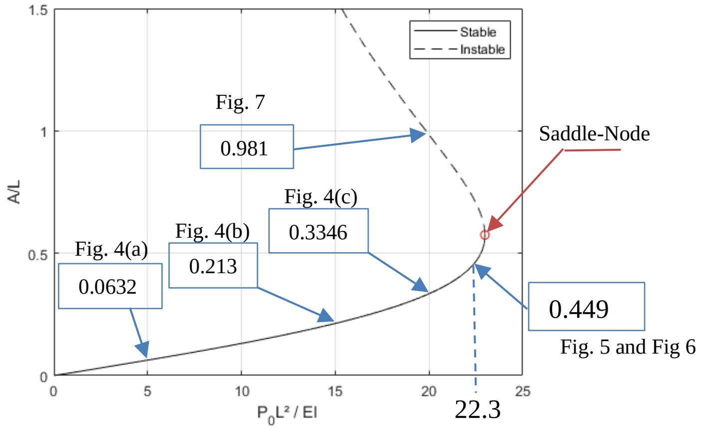
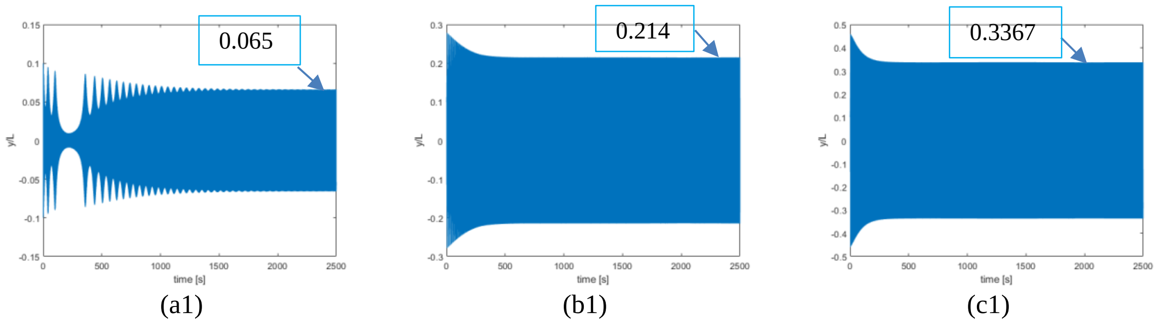
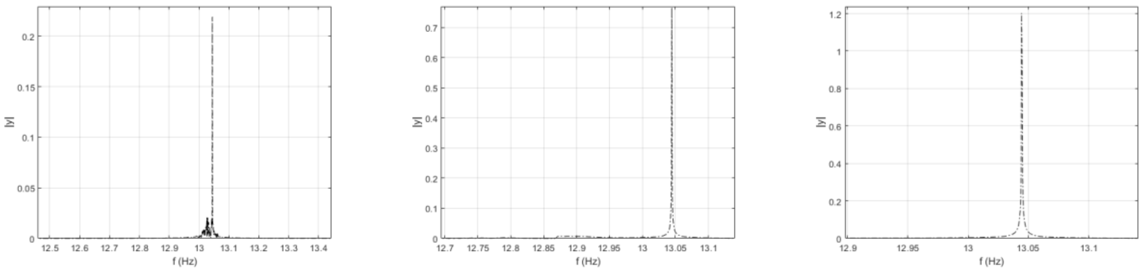


Figure 3. Approximate response for dynamic module displacement normalized by beam length A/L for several normalized dynamic load module P_0L^2/EI and $\Omega/\omega = 1$

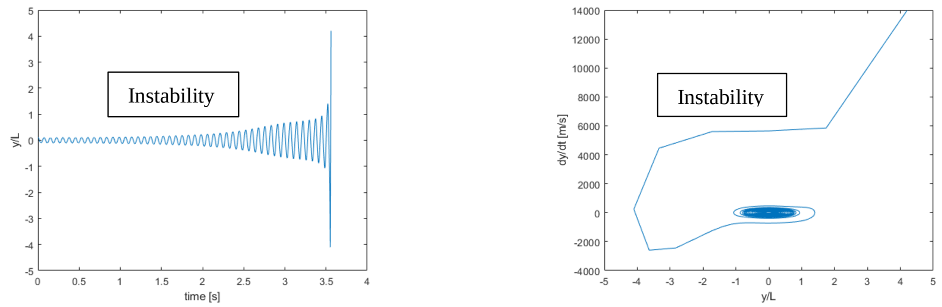
For $P_0L^2/EI = 22.3$ (see Figure 5) adopting an initial displacement $d_0/L = 0.1$ the reponse is clearly unstable, with the time response growing exponentially up to a point where it tends to infinity. However, changing the initial displacement to a value closer to the expected response amplitude, $d_0/L = 0.2$ (see Figure 6) the response converges to the steady state magnitude $A/L \approx d/L = 0.453$ near the predicted value ($A/L = 0.449$, see Figure 3). For $P_0L^2/EI = 20$ a large amplitude steady state response of $A/L = 0.981$ (see Figure 3) is predicted. Using in the numerical integration the initial displacement $d_0/L = 0.9$, close to expected amplitude, it is observed in Figure 7 that, after a long transient, the displacement decreases to that previously obtained in Figure 4(c1) for the same load level. This demonstrates that the lower branch of the response is stable up to the limit point in Figure 3 becoming after this point unstable. Also the results show that, as one approaches the limit points, smaller perturbations are necessary to achive convergence, indicating a decrease of the basin of attraction of the stable solution.





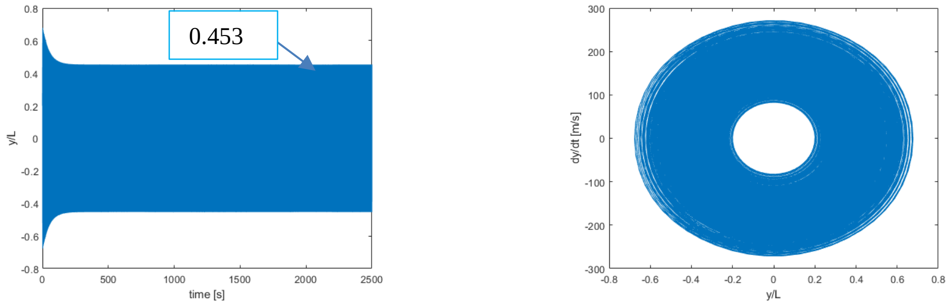
(a2) (b2) (c2)

Figure 4. Transversal displacement time history (a1 to c1), and FFTs (a2 to c2) considering $d_0 / L = 0.1$, $\Omega = \omega$ and for several values of dynamic module load P_0 : (a) $P_0L^2 / EI = 5$, (b) $P_0L^2 / EI = 15$, (c) $P_0L^2 / EI = 20$



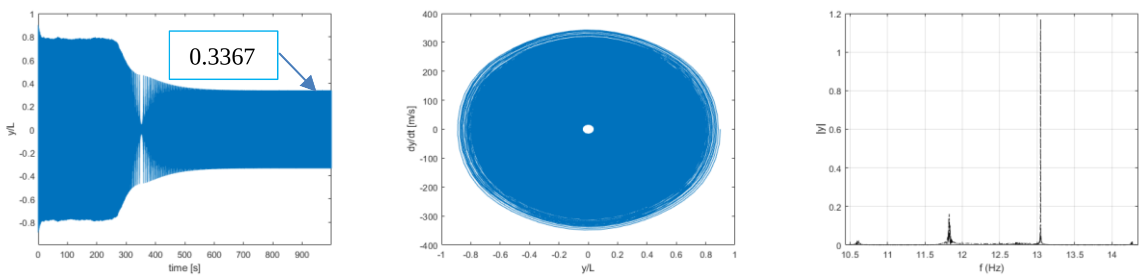
(a) (b)

Figure 5. Transversal displacement time history (a), phase space (b). $d_0 / L = 0.1$, $P_0L^2 / EI = 22.3$



(a) (b)

Figure 6. Transversal displacement time history (a), phase space (b). $d_0 / L = 0.2$, $P_0L^2 / EI = 22.3$



(a) (b) (c)

Figure 7. Transversal displacement time history (a), phase space (b) and FFT (c). $d_0 / L = 0.9$, $P_0L^2 / EI = 20$

4 Conclusions

This work investigated the nonlinear dynamic behavior of a clamped-free beam under a transverse follower force. This generates a partial differential equation with time-varying parameters, leading to a type of parametric excitation. This type of oscillation is very sensitive to the load parameters and characteristics of the structure, as well as the initial boundary condition. Because the external load is linked to the beam slope at the application point, the stability of the solution depends on the load parameters and initial conditions. The solution obtained by the Galerkin-Urabe method depicts a saddle-node bifurcation separating the stable and unstable branches.

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