Nonlinear normal modes to analyze the nonlinear forced vibrations of cylindrical shells with internal resonances

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Abstract. In this work, the nonlinear normal modes are applied to analyze the nonlinear forced vibrations of a simply supported cylindrical shell with internal resonances. The nonlinear equilibrium equations are obtained considering the Donnell's nonlinear shallow shell theory. The modal solution to the transversal displacement field, used to discretize the equilibrium equations, is obtained by perturbation techniques that consider an internal resonance between the linear vibration modes. The discretized equations of the reduced order model are obtained using an invariant manifold approach. The nonlinear dynamic behavior is analyzed from the resonance curves that are obtained by the continuation method. The resonance curves are obtained for both full and reduced order models, and these results are compared to determine the level of a harmonic load that can be reliably represented by a reduced order model. Several multi-modes are considered to assemble the best nonlinear normal modes basis that contains the most important information of the interactions that occur between the modes of the transversal displacement field. Time responses and phase portraits, with mapping Poincaré sections, are also used to analyze the nonlinear dynamic behavior of the cylindrical shell and to check the accuracy of the reduced order models.

Keywords: Nonlinear normal modes, Nonlinear forced vibrations, Cylindrical shells, Internal resonances

1 Introduction

Nonlinear normal modes (NNMs) extend the concept of linear normal modes to systems where nonlinearities can significantly influence dynamics where NNMs take into account for nonlinear interactions between the vibration modes which, in their turn, can lead to complex behaviors such as mode coupling, energy transfer between modes, and changes in mode shapes with amplitude variations. On the other hand, Linear normal modes (LNMs) describe the behavior of linear systems where all modes are independent, oscillating at their natural frequencies without interaction between them.

Many works have been developed in this research field, with some interesting findings as the work Nayfeh et al. [1] that explores weakly nonlinear discrete systems with cubic geometric nonlinearities, revealing that NNMs can out perform linear modes in the presence of internal resonances. Renson and Kerschen [2] introduce a computational method to compute the NNMs in non-conservative systems, solving partial differential equations to capture the modes' geometry and the frequency-energy relationships without using time integration. Hill et al. [3] focus on the forced responses of nonlinear systems and analysis of significance of the NNMs, identifying which NNMs are pertinent to understanding the forced response through analytical investigations and energy arguments to relate NNMs to the forced responses.

In general, NNMs are characterized by their ability to describe the motion of nonlinear systems as invariant manifolds in phase-space, where the system's behavior can be reduced to a lower-dimensional representation. This approach captures the essence of the system's dynamics, including phenomena such as internal resonances which occur when the natural frequencies of the system satisfy certain commensurate relations, leading to strong modal interactions.

So, this work deduces a reduced order models (ROMs) through a invariant manifold method to evaluate the dynamics of a simply-supported cylindrical shell with internal resonance harmonically excited. Several multi-modes ROMs are considered to assemble the best nonlinear normal modes basis that contains the most important information of the interactions that occur between the modes of the transversal displacement field. Time responses and phase portraits, with mapping Poincaré's sections, are also used to analyze the nonlinear dynamic behavior of the cylindrical shell and to check the accuracy of the reduced order models. The obtained results show a complex behavior of forced response of shell with strong influence of the master variables used to derive the ROMs and the magnitude of the transversal time-dependent load.

2 Problem formulation

A perfect cylindrical shell with radius R, length L, and thickness h is illustrated in Figure 1, where $h << R$. In this figure, the displacement fields in the directions axial u, circumferential v, and transversal w are also presented, which correspond to the cylindrical coordinate system x, θ, and z, respectively. The cylindrical is made by an isotropic, homogeneous and linear material with Young's modulus E, Poisson's ration ν and density ρ. The nonlinear equilibrium equations of the cylindrical shell are derived using the well-known Donnell's nonlinear shallow shell theory. These equations can be written in terms of the transversal displacement field w and the Airy's stress function Φ [4]:

$$\rho h\ddot{w} + 2\eta_1 \rho h \omega_0 \dot{w} + D\nabla^4 w = \frac{1}{R}\Phi_{,xx} + \frac{1}{R^2}(\Phi_{,\theta\theta}w_{,xx} - 2\Phi_{,x\theta}w_{,x\theta} + \Phi_{,xx}w_{,\theta\theta}) - P(t) \tag{1}$$

$$\frac{1}{Eh}\nabla^4\Phi = -\frac{1}{R}w_{,xx} + \frac{1}{R^2}\left(w_{,x\theta}^2 - w_{,xx}w_{,\theta\theta}\right) \tag{2}$$

where $D(= Eh^3/12(1-\nu^2))$, ω_0, η_1 and $P(t)$ are the flexural stiffness, the lowest natural frequency, the viscous damping factor and the transversal time-dependent load, respectively.

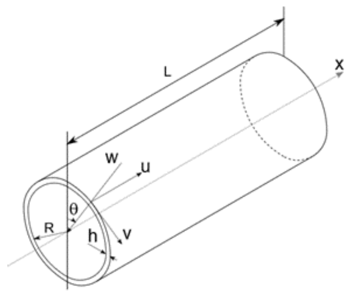

Figure 1. Geometry of cylindrical shell and coordinates systems

To discretize the eq. (1) by a Galerkin method, the Airy's stress function Φ is analytically solved through the compatibility and continuity geometric equation, eq. (2). To solve this indeterminate stress function, many techniques can be applied. As identified in eq. (2), this equation is a biharmonic equation which any harmonic function is also a biharmonic solution. In this work, a particular transversal displacement field, that is composed by harmonic functions, is chosen a priori. To obtain the solution of the stress function Φ, it is required to substitute this displacement field into eq. (2), and the stress function is determined by solving Φ through the indeterminate coefficients method.

3 ROM through invariant manifold techniques

Deriving a consistent reduced-order model for a cylindrical shell using invariant manifolds involves several steps. Firstly, using the following general modal solution for the transversal displacement field, that was obtained by perturbation method as presented in [5, 6],

$$\begin{aligned} w = &\sum_{\beta=1,3}\sum_{i=1,3}\left\{\left[W_{i,\beta}(\tau)\cos(in\theta) + W_{i,\beta}^C(\tau)\sin(in\theta)\right]\sin(\beta q\xi)\right\}+ \\ &\sum_{i=0,2}\left\{\left[W_{i,2}(\tau)\cos(in\theta) + W_{i,0}^C(\tau)\sin(in\theta)\right]\left[\frac{3}{4} - \cos(2q\xi) + \frac{1}{4}\cos(4q\xi)\right]\right\}, \end{aligned} \tag{3}$$

the nonlinear equilibrium equation, eq. (1) is discretized by a Galerkin method, and a full set of second-order differential equations for the dynamics of the cylindrical shell is derived in terms of the the modal amplitudes W, W^C that are related with driven and companion modes, respectively. These modal amplitudes provoke an internal resonance of type 1:1 and from a vast literature on this topic, these modes need to be considered to describe the correct nonlinear dynamic behavior of the shell [7, 8]. In eq. (3) the following parameters are used: $q = m\pi$, $\eta = x/L$, with $0 \leq \eta \leq 1$, $\tau = t\omega_0$, where m and n are, respectively, the number of axial half-waves and circumferential waves associated with the vibration modes of the lowest natural frequency.

The invariant manifolds are lower-dimensional surfaces in the phase space of the dynamical system where the system's trajectories evolve. To identify these manifolds, one typically looks for slow and fast dynamics separation. The idea is that the system's behavior can be described by a few dominant modes (slow dynamics), while the rest of the modes (fast dynamics) quickly decay and can be approximated as slaved to the dominant modes. This is achieved using Center Manifold Theory which can be used to rigorously justify the existence of an invariant manifold where the center manifold is a low-dimensional manifold that attracts trajectories starting near it. Thus, to project the dynamics onto the invariant manifold, it is need to express the high-dimensional state vector in terms of a few coordinates that describe the motion on the manifold. This step often uses a parameterization method in which this work uses a function of a nonlinear invariant manifold, derived using methodology as shown in [9].

Assuming a discretized system of first order differential equilibrium equations in the following manner:

$$F_i\left(t, y_i, \dot{y}_i\right) = 0 \tag{4}$$

where y_k and $\dot{y}_k$ corresponds to k^{th} displacements and velocities fields of the set of differential equations, respectively, and i is i^{th} first order equilibrium equation. An isomorphism is also assumed to exist between the motion of one or multiple master pairs of displacement and velocity which controls the dynamics of the set of equilibrium equations. Then, this nonlinear transformation defines a nonlinear normal mode in space configuration and can be expressed as:

$$\begin{aligned}
&y_k = X_k(u_j, v_j) = a^i_{1k}u_j + a^i_{2k}v_j + a^i_{3k}u_ju_j + a^i_{4k}u_jv_j + a^i_{5k}v_jv_j+\\
&a^i_{6k}u_ju_ju_j + a^i_{7k}u_ju_jv_j + a^i_{8k}u_jv_jv_j + a^i_{9k}v_jv_jv_j\\
&\dot{y}_k = Y_k(u_j, v_j) = b^i_{1k}u_j + b^i_{2k}v_j + b^i_{3k}u_ju_j + b^i_{4k}u_jv_j + b^i_{5k}v_jv_j+\\
&b^i_{6k}u_ju_ju_j + b^i_{7k}u_ju_jv_j + b^i_{8k}u_jv_jv_j + b^i_{9k}v_jv_jv_j
\end{aligned} \tag{5}$$

where X_k and Y_k are surfaces that describe a motion near an equilibrium point for the displacement and velocity fields, respectively. u_j and v_j are j^{th} chosen pair of displacement and velocity called in the technical literature as master variables, also a^i_k and b^i_k are coefficients to be calculated using a invariant manifold equation (6).

Taking the expansions of eq. (5) and using the differentiation chain rule, the explicit time dependence is eliminated [10]:

$$\begin{aligned}
\dot{y}_k &= \frac{\partial X_k(u_j, v_j)}{\partial u_j}u_j + \frac{\partial X_k(u_j, v_j)}{\partial v_j}v_j\\
\ddot{y}_k &= \frac{\partial Y_k(u_j, v_j)}{\partial u_j}u_j + \frac{\partial Y_k(u_j, v_j)}{\partial v_j}v_j
\end{aligned} \tag{6}$$

To obtain the ROMs, applying the nonlinear normal modes, this work considers as the full model (Model A) the set of discrete equilibrium equations obtained using the transversal displacement field using the following modal amplitudes: $W_{1,1}, W^C_{1,1}, W_{0,2}$, $W_{2,2}, W^C_{2,2}, W_{1,3}, W^C_{1,3}, W_{3,1}, W^C_{3,1}, W_{3,3}, W^C_{3,3}$ in eq. (3). The ROMs are obtained considering different pairs of master variables, and these pairs are selected from the modes of the full model. Two different ROMs, named Model B and C, are derived, considering the presence of the driven and companion modes of the transversal displacement field, eq. (3). Model B is assembled with solely two pairs of master variables, namely (u_1, v_1) and (u_2, v_2) that refers to $W_{1,1}, W^C_{1,1}$. This means that the modal amplitudes used in this model take into account the internal resonance 1:1, emerging from the consideration of both driven and companion modes of eq. (3). Model C is built by choosing the pairs (u_1, v_1), (u_2, v_2) and (u_3, v_3) as the master coordinates that are related to modal amplitudes $W_{1,1}, W^C_{1,1}$, and $W_{0,2}$, respectively.

4 Numerical results

To obtain the following numerical results, the physical and geometric parameters are a cylindrical shell of radius $R = 0.2m$, length $L = 0.4m$ and thickness $h = 0.002m$, with an elastic isotropic material that follows the properties $E = 2.1 \times 10^{11} N/m^2$, $\nu = 0.3$ and $\rho = 7850 kg/m^3$. Also, the viscous damping is set to η_1 = 0.001. To describe the time-dependent lateral harmonic pressure, it is assumed that the pressure has the same shape of the driven mode, i.e.

$$P\left(t\right) = P_L \cos\left(n\theta\right) \sin\left(q\xi\right) \cos\left(\frac{\omega_1}{\omega_0}\tau\right) , \quad (7)$$

where ω_1 is the excitation frequency of the lateral pressure, and P_L is the amplitude of the lateral pressure, that directly excites the driven vibration mode. For this geometry and boundary conditions, the lowest natural frequency of the cylindrical shell occurs for the vibration mode (m, n) = (1, 5).

A parametric analysis of the resonance curves is conducted. For that, the maximum amplitude of the driven mode is computed by varying the external excitation frequency and for different amplitude of the lateral pressure. These numerical results (resonance curves) presented in this section are obtained by AUTO that is a continuation and bifurcation identification software for dynamical systems [11].

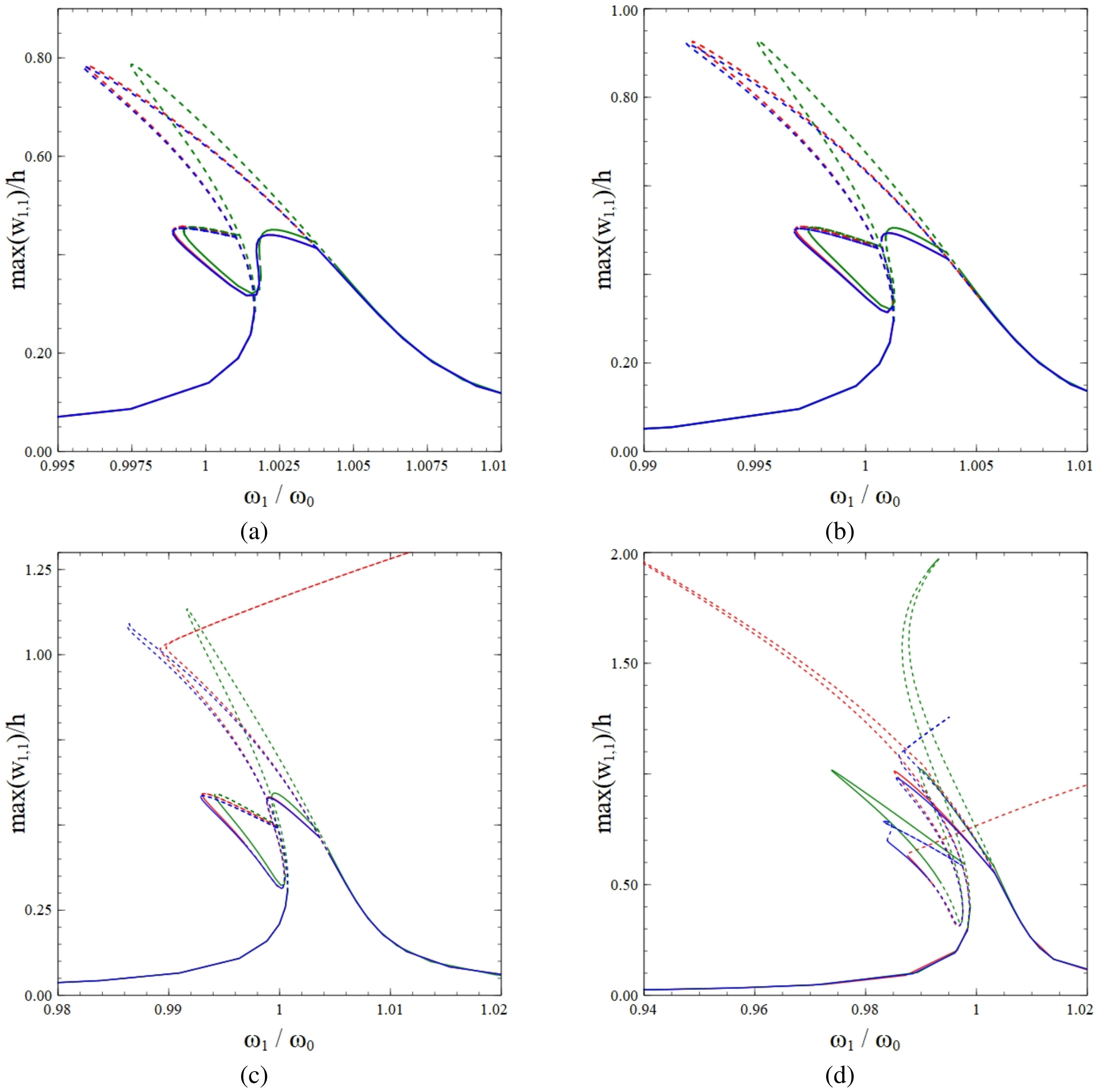

Figure 2. Comparison of the resonance curves of drive mode ($W_{1,1}$) for (a) $P_L = 0.0012566$, (b) $P_L = 0.001478$, (c) $P_L = 0.001795$ and (d) $P_L = 0.003141$. Blue line - Model A (Full model), Green line - Model B ($W_{1,1}, W^C_{1,1}$), Red line - Model C ($W_{1,1}, W^C_{1,1}$, and $W_{0,2}$). Continuous line (stable path), Dashed line (unstable path).

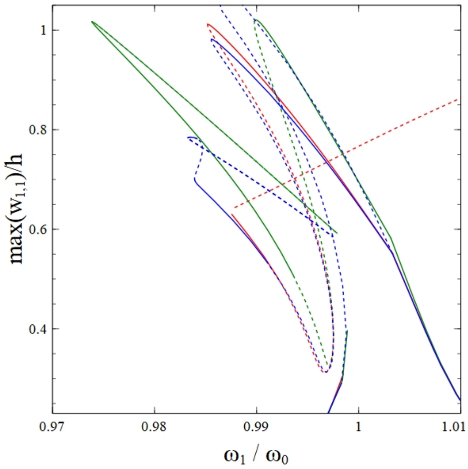

Figure 3. Zoomed view of Fig 2 (d) in the resonance peak for $P_L = 0.001478$. Blue line - Model A (Full model), Green line - Model B ($W_{1,1}, W^C_{1,1}$), Red line - Model C ($W_{1,1}, W^C_{1,1}$, and $W_{0,2}$). Continuous line (stable path), Dashed line (unstable path).

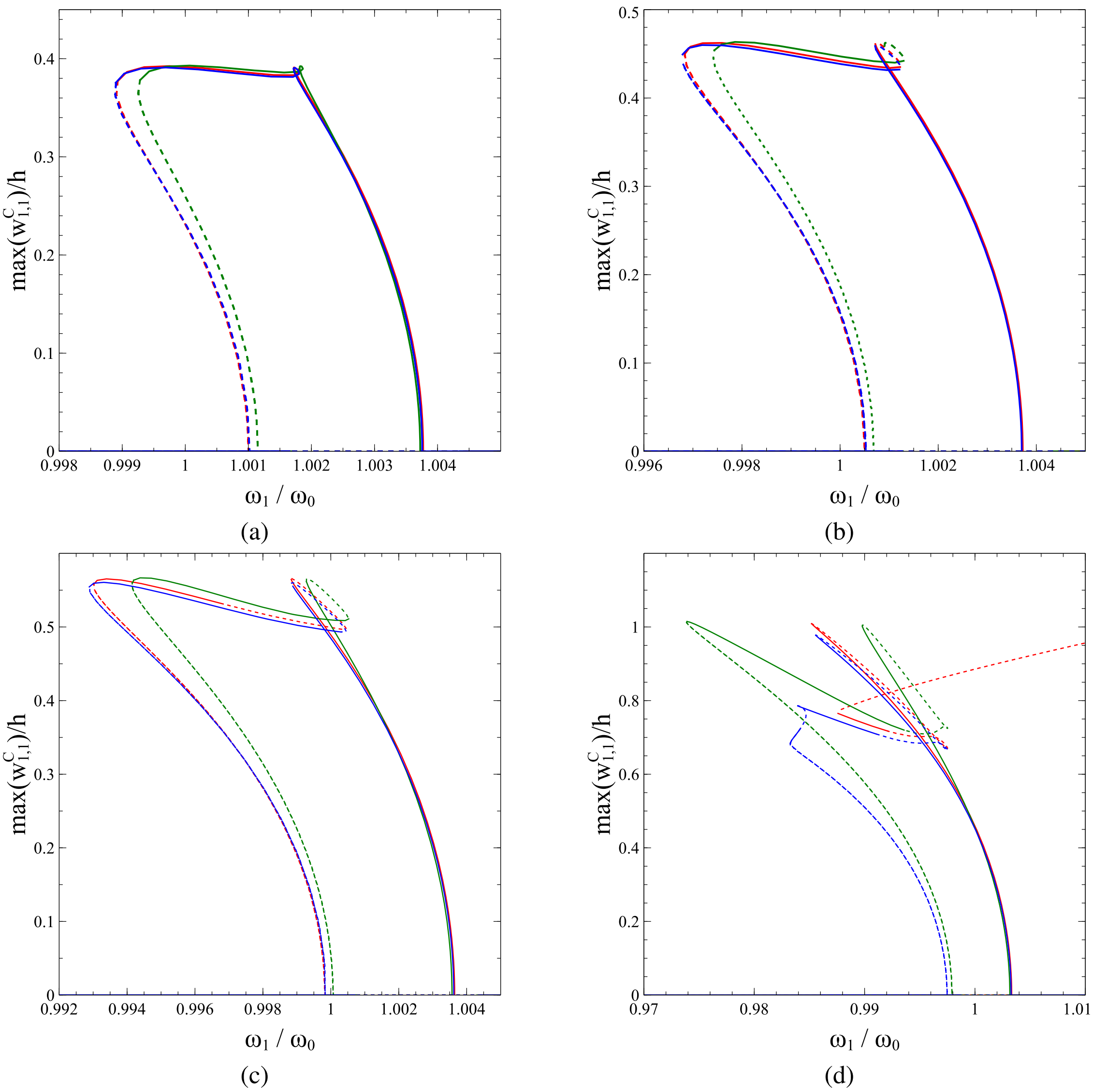

Figure 4. Comparison of the resonance curves of companion mode ($W^C_{1,1}$) for (a) $P_L = 0.0012566$, (b) $P_L = 0.001478$, (c) $P_L = 0.001795$ and (d) $P_L = 0.003141$. Blue line - Model A (Full model), Green line - Model B ($W_{1,1}, W^C_{1,1}$), Red line - Model C ($W_{1,1}, W^C_{1,1}$, and $W_{0,2}$). Continuous line (stable path), Dashed line (unstable path).

Figure 2 shows a comparison of the resonance curves obtained by all three models for successively increments of the amplitude of the lateral pressure. As show in Figure 2 (a) and (b), Model B can accurately match the full model only for small loads that generates vibrations amplitude up to order of thickness of the shell. However, it can accurately capture the general dynamic behavior of shell such as the limits points, changes of stability, bifurcation points and the secondary nonlinear equilibrium path. This result was expected since it is a fundamental property of the time-invariant manifold to recover the essential dynamical properties and the qualitative behaviour (number and nature of bifurcations) will always be predicted by the ROM [12].

Model B, which are composed only of fundamental modes ($W_{1,1}, W_{1,1}^C$), shows your limitations for very large amplitudes displacements where vibrations amplitudes above thickness of the shell leads to lost of accuracy, but maintaining the correct type of the nonlinearity as well as the bifurcations points. On the other hand, Model C also lost accuracy but with a better match with the full model (Model A). In regards of losing accuracy on the secondary nonlinear equilbrium branch, Figure 3 shows that Model C was able to describe stable paths accurately in some extent, which are most important to analyse global stability.

Observing the accuracy of the ROMs in the resonace curves of the companion mode, Figure 4 shows that the accuracy of ROM is improved when the ROM includes the modal amplitude $W_{0,2}$ as master variable in conjunction with ($W_{1,1}, W_{1,1}^C$), allowing to recover the vibration amplitudes with high accuracy. As observed in this figure, there is a lost of accuracy with the increasing of the amplitude of the lateral load.

5 Conclusions

In this work, the time-independent invariant manifold was assembled to determine a reduced-order model (ROM) of a cylindrical shell with 1:1 internal resonance. The nonlinear equilibrium equation displays quadratic and cubic nonlinearities due to Donnell's nonlinear shallow shell theory. This computation allows to express of the dynamics in an invariant-based span of the phase space used as master variables for the derivation of ROMs of continuous structures with geometrical nonlinearities. The main focus of this work relied on the influence of energy on the system and which master variables are better suited to the assembly of the representative ROM for a cylindrical shell. For the derivation of the ROMs of externally forced structure, the approximation, consisting of using time-independent manifolds for reducing the dynamical systems, has been discussed, with the inclusion of an axisymmetrical mode of the transversal modal solution of the cylindrical shell. The drawback of losing accuracy for high values of the forced vibration amplitude was solved in this ordinary differential equation system, obtaining numerical results that are slightly different from the full dynamical system model depending on the inclusion of the master variables that span a subspace of invariant manifold. The numerical results of the cylindrical shell show that the ROMs predicted the correct nonlinear behavior. On the other hand, there were discrepancies between the ROMs and the full dynamical system with the increasing of the forcing amplitude, leading to a loss of accuracy when the vibration amplitude was up to the order of two times the thickness of the cylindrical shell. It's worth noting that these results were obtained by reducing a discrete model with eleven degrees of freedom (DOF) to a ROM with only three DOF, which underlines the main advantage that a correct multi-mode ROM can bring a very close agreement between full and reduced models with high forced vibration amplitudes.

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