Buckling of columns with top and distributed loading using Rayleigh's method: assessment of two cases of boundary conditions

Ana P. F. Borges[1], Alexandre de M. Wahrhaftig[1,2], M. Eisenberger[3]

[1]*Graduate Program in Civil and Environmental Engineering, State University of Feira de Santana, Avenida Transnordestina, s/n, Novo Horizonte, Feira de Santana, CEP: 44036-900, Bahia, Brazil*
a_paulaborges@hotmail.com
[2]*Department of Construction and Structures, Polytechnic School, Federal University of Bahia, Rua Aristides Novis, nº 02, 5º andar, Federação, Salvador, CEP: 40210-630, Bahia, Brazil*
alixa@ufba.br
[3]*Department of Civil Engineering, Technion-lsrael Institute of Technology, Technion City 32000, Haifa, Israel.*
cvrmosh@technion.ac.il

Abstract.
The problem of buckling in compressed parts has been the subject of study by many researchers due to the importance that these structural elements have for different areas of engineering. The compressive capacity of columns can be quantified using the so-called critical buckling load. The first studies date back to the work of Euler and Greenhill that began in the 18th century. Other contributions have been made since then. One of the most relevant for this field of study is the Rayleigh solution formulated to understand the vibration of elastic systems, whose equations can be directly used in determining the critical buckling load. The central feature of this method is the possibility of associating continuous systems with an equivalent system with a single degree of freedom. The Rayleigh method is used to define the critical buckling load of two columns under the simultaneous action of concentrated and distributed load along their length. Prismatic section parts are evaluated considering two boundary conditions. The values suggested in the literature are compared to the current results.

Keywords: critical buckling load, distributed load, columns, boundary conditions, Rayleigh method.

1 Introduction

When designed, structural elements must satisfy both strength and stability requirements. In the case of slender columns, the last aspect governs. The first studies to understand buckling are attributed to Euler [1], who found it difficult to consider the effect of self-weight in determining the critical buckling load. This problem was later solved by Greenhill [2] with the use of elliptic integrals. Using Rayleigh's method [3] it is possible to calculate the critical buckling load with the simultaneous action of an end concentrated force and distributed loading without the need to use elliptical integrals. A central feature of Rayleigh's method is the possibility of transforming structural elements with infinite number of degrees of freedom into an equivalent one with only one degree of freedom. The scientific basis behind the Rayleigh method is the principle of energy conservation, in which the maximum deformation energy equals the work done by external forces. The solutions are obtained directly in the continuum and without the need for discretizations or interactive processes. However, the results from Rayleigh's method are approximate compared to other methods, such as those proposed by Timoshenko and Gere [4]. In most cases the approximate results obtained by Rayleigh's method remain valid given the precision required by the problems of engineering practice (see Temple [5]).

Based on the properties of Rayleigh's method, this work aims to analyze the critical buckling loads of columns under the simultaneous action of a concentrated load at the upper end and distributed along its height. In the present approach, the columns have a prismatic cross section and two sets of boundary conditions. The results from Rayleigh's method are compared with the results found using Timoshenko and Gere [4].

2 Rayleigh method for calculating the critical buckling load

Rayleigh initially employed his technique to solve a boundary value problem, using variational calculus and linear approximation for the basic functions he chose for the form of vibration of elastic systems. The objective was to minimize a special class of analytical functions that satisfied the essential, or geometric, boundary conditions (displacements and rotations) of the problem (see Ilanko et al. [6]).

The application of the Rayleigh method to determine the critical buckling load necessarily involves defining the two stiffness contributions, namely, the flexural stiffness term and the geometric stiffness term. The first depends on the bending stiffness, which involves the properties of the material and the cross-section geometry. The second depends on the acting normal force, whether concentrated or distributed, in addition to the length of the column. An important aspect to be observed in Rayleigh's formulation is that it is possible to consider, directly in it, a force distributed along the length, which may be associated with the self-weight of the structural element, or not. In this approach, the lateral buckled configuration does not alter the direction of the axial force and the displacement assumed for the movement due to the buckling occurs nearby the original, unloaded form of the column.

Consider the case of two columns with a prismatic cross-section, as indicated in Fig. 1. In these, the area A and I inertia of the cross-section are constant along the height of the column, with length l and modulus of elasticity E. The gravity acceleration is g. Two members are analyzed using Rayleigh's method. The first is a column fixed at the base and free at the upper end, (*CF*), shown in Fig 1(a), and the second is a column hinged at both ends (*HH*) shown in Fig. 1(b).

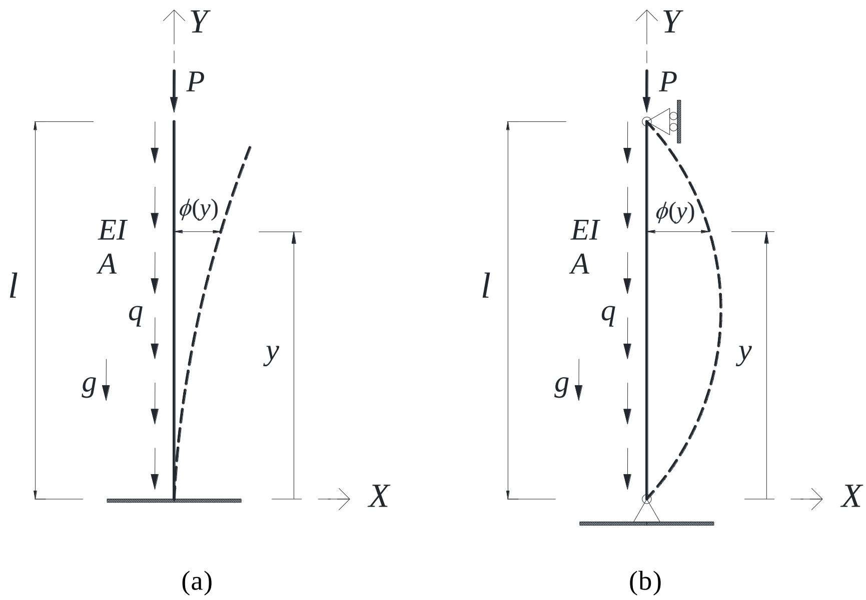

Figure 1. (a) clamped-free case (*CF*), (b) hinged-hinged case (*HH*)

For the *CF* column the assumed shape function $\phi(y) = 1 - \cos(\pi y/(2l))$ is used, and for the *HH* column the function $\phi(y) = \sin(\pi y/l)$ is used. The axial force function along the height of the column, in both cases, is given by:

$$N(y) = P + \int_{y}^{l} q dy \,, \tag{1}$$

with q representing the distributed loading and P is the concentrated force at the upper end of the column, i.e., P

at the top to be represented as P_{top}. If q is the result of considering the self-weight, this loading is $q = A\gamma$, with γ being the specific weight of the material, that is, the product of its density, in terms of specific mass, by the acceleration of gravity. Given that the distributed load q is constant along the height, the solution of eq. (1) leads to the variation of the normal force in the form of eq. (2):

$$N(y) = P + q(l - y)\,. \tag{2}$$

According to Wahrhaftig [7] ,the geometric part of the stiffness is obtained by:

$$K_g = \int_0^l N(y)(\phi'(y))^2 dy\,, \tag{3}$$

and the bending stiffness part by:

$$K_0 = \int_0^l EI(\phi''(y))^2 dy\,, \tag{4}$$

in which EI is the bending stiffness product, where I is the moment of inertia of the cross-section in relation to the buckling mode considered, and E is the modulus of elasticity of the material. In the previous equations $\phi(y)$ is the function established to describe the first buckling mode, with $\phi(y)'$ and $\phi(y)''$ characterizing its first and second derivatives.

The critical buckling load condition is established for the zero stiffness state of the structural system, that is, $K_0 - K_g = 0$, with the axial force considered to be positive. With this equality established and the previous conditions assumed for the axial force and the buckling shapes, it is possible to write expressions for the critical buckling load, which takes into account the variation of q or P.

2.1 Clamped-free column

For the clamped-free column (*CF*) that expression is given by

$$P_{top,cr}(q) = \frac{\pi^4 EI - ql^3(2\pi^2 - 8)}{4\pi^2 l^2}, \tag{5}$$

for $q = 0$

$$P_{cr} = \frac{\pi^2 EI}{4l^2} \rightarrow P_{cr} = 2,467\frac{EI}{l^2}\,. \tag{6}$$

for $P = 0$

$$q_{cr} = \frac{\pi^4 EI}{2(\pi^2 - 4)l^3} \rightarrow q_{cr} = 8,298\frac{EI}{l^3}\,. \tag{7}$$

2.2 Hinged-hinged column

For the hinged-hinged column (*HH*) by

$$P_{top,cr}(q) = \frac{2\pi^2 EI - ql^3}{2l^2}\,. \tag{8}$$

for $q = 0$

$$P_{cr} = \frac{\pi^2 EI}{l^2} \rightarrow P_{cr} = 9,870\frac{EI}{l^2}\,. \tag{9}$$

for $P = 0$

$$q_{cr} = \frac{2\pi^2 EI}{l^3} \rightarrow q_{cr} = 19,739\frac{EI}{l^3}. \tag{10}$$

3 Results and discussion

The variation of the expressions described in the previous item, considering their parameters unitary (in the international system of units - S.I.), can be seen in the following graphs, specifically designed for the cases studied.

It is worth noting that if q is associated with self-weight, the product of q by the length of column l is the total weight of the structural element. Using the Rayleigh method, the critical buckling load must be reduced by $0.297ql$ if it is clamped-free and of $0.500ql$ if it is hinged-hinged. According to Timoshenko and Gere [4], the first quantity should be $0.3ql$ ($0.315ql$ if it is observed by the relationship of P_{cr} by q_{cr}). Therefore, the difference in the first case is only 0.88% if the value recommended by Timoshenko and Gere [4] is considered, and 5.4% if the tabled value is taken as a reference. In the second condition, there is no difference between both mathematical procedures.

The relationships between q_{cr} and P_{cr} given by eq. (11), with results from Rayleigh method compared to that recommended by Timoshenko and Gere [4] can be seen in Tabs. 1 and 2. For a better understanding, in the current work (Rayleigh method), the results are obtained by introducing $ql = nP_{cr}$ in eqs. (5) and (8) according to the case.

$$n = \frac{q_{cr} l}{P_{cr}}, \tag{11}$$

Table 1. Results by Rayleigh method, and Timoshenko and Gere [4] (Table 2-7, pg. 104) – Column *CF*

n	Rayleigh	Timoshenko	Diff. Absolute	Diff. Relative (%)
0.00	2.467	2.467	0.000	0.00%
0.25	2.284	2.280	-0.004	-0.17%
0.50	2.101	2.080	-0.021	-0.99%
0.75	1.917	1.910	-0.007	-0.37%
1.00	1.734	1.720	-0.014	-0.80%
2.00	1.000	0.960	-0.040	-4.17%
3.00	0.266	0.150	-0.116	-77.53%
3.18	0.134	0.000	-0.134	-
4.00	-0.467	-0.690	-0.223	32.26%
5.00	-1.201	-1.560	-0.359	23.01%
10.00	-4.870	-6.950	-2.080	29.93%

Diff. = Difference, Timoshenko and Gere [4] as the reference

Table 2. . Results by Rayleigh method, and Timoshenko and Gere [4] (Table 2-8, pg. 107) – Column *HH*

n	Rayleigh	Timoshenko	Diff. Absolute	Diff. Relative (%)
0.00	9.870	9.870	0.000	0.00%
0.25	8.636	8.630	-0.006	-0.07%
0.50	7.402	7.360	-0.042	-0.57%
0.75	6.169	6.080	-0.089	-1.46%
1.00	4.935	4.770	-0.165	-3.46%
2.00	0.000	-0.657	-0.657	100.00%
3.00	-4.935	-4.940	-0.005	0.10%

Diff. = Difference, Timoshenko and Gere [4] as the reference

4 Conclusions

In this work, an analysis based on the Rayleigh method was conducted to obtain the critical buckling load of two prismatic columns with different boundary conditions. The columns were subjected to a top force concentrated at the upper end and uniformly distributed along their length. The two boundary conditions used were a cantilevered column, clamped to the base, and a column simply supported at the ends. For comparative purposes, results presented by Timoshenko and Gere [4] was used as reference.

It could be verified that in the absence of distributed force, the solution obtained by the Rayleigh method closely approximates the value provided in Timoshenko and Gere [4]. However, with the presence of an axially distributed force along the column, the results found by Rayleigh's method differ slightly from those suggested in Timoshenko and Gere [4] for the cantilever column. For the simply supported column, the values obtained by both methods coincide.

Acknowledgements. The authors would like to thank CNPq, the Brazilian National Council for Scientific and Technological Development; and CAPES, the Coordination of Superior Level Staff Improvement, for the research scholarship and support provided.

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