Development of an academic finite element program for acoustics

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Abstract. Acoustic is the study of mechanical waves in solids, liquid and gases. Altougth the differential equations describing these problems are well known, with analytical solutions for simple geometries and/or loads, there are no closed solution for general problems. Thus, it is common to sougth on numerical procedures to address such problems. This work addresses the simulation of mechanical waves in loseless air using the linear finite element method. Solution in both time and frequency domains are discussed and compared to theoretical results.

Keywords: Finite element, acoustic, transient, modal.

1 Introduction

Acoustics is the science of sound and its behavior in various environments. The goal is to study how sound is produced, transmitted, and perceived by humans and other organisms, by understanding the principles governing sound waves, their propagation through different mediums, and their interaction with surfaces and structures. By understanding acoustics, engineers can design spaces that optimize sound quality, musicians can create compositions that resonate deeply, and researchers can unravel the mysteries of how sound impacts our lives and surroundings.

The linear partial differential equation for acoustics is [1]

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} p(\mathbf{x}, t)) - \frac{1}{c^2}\left(\frac{\partial^2 p(\mathbf{x}, t)}{\partial t^2}\right) = 0, \quad \mathbf{x} \in \Omega, t \in [t_0, t_f] \tag{1}$$

where $p(\mathbf{x}, t)$ is pressure fluctuation around the mean value p_0, $\mathbf{x}$ are the coordinates of a point in the domain Ω, t is the time and c is the propagation velocity of the sound in the medium. This hyperbolic partial differential equation is known as the wave equation [2]. The analytical solution for this differential equation is known for some specific geometries, loads and boundary conditions, but complete solutions for general cases are hard to obtain without resorting to some numerical method [3].

Thus, this research aims to study the differential equation 1 and its solution by means of the Finite Element Method [4].

2 Finite Element Model

The complete solution for eq.(1) is the pressure $p(\mathbf{x}, t)$ that satisfies the differential equation for all $\mathbf{x} \in \Omega$ and for all t, along with the satisfaction of all boundary and initial conditions. Finding this function can be very hard for general Ω and boundary conditions [2]. Thus, one may aim for some form of approximation $\tilde{p}(\mathbf{x}, t)$. The weighted residual method [4] states that

$$\int_{\Omega} w(\mathbf{x}, t) \cdot r(\mathbf{x}, t) \, d\,\Omega = 0 \tag{2}$$

where

$$r(\mathbf{x}, t) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \tilde{p}(\mathbf{x}, t)) - \frac{1}{c^2}\left(\frac{\partial^2 \tilde{p}(\mathbf{x}, t)}{\partial t^2}\right) \tag{3}$$

is the residual due to the approximate solution $\tilde{p}$ and w is a weight function. In this work, we assume that both $\tilde{p}$ and w respect the Dirichlet boundary conditions for all t. We also assume that both functions are written using the same set of base functions. In the Finite Element Method we start by defining a partition $\Omega_e \subseteq \Omega$ and a set of base functions $N(\mathbf{r})$ with compact support, such that

$$\tilde{p}(\mathbf{r},t) = \sum_{i=1}^{n} N_i(\mathbf{r}) P_i(t) = \mathbf{N}(\mathbf{r})\mathbf{P}_e(t) \tag{4}$$

and

$$w(\mathbf{r},t) = \sum_{i=1}^{n} N_i(\mathbf{r}) W_i(t) = \mathbf{N}(\mathbf{r})\mathbf{W}_e(t) \tag{5}$$

where $\tilde{p}(\mathbf{r})$ and $w(\mathbf{r})$ are, respectively, the approximated pressure and weight inside ω_e, as a function of the base functions N_i and discrete (nodal) values P_i and W_i at the n nodes of the element. It is assumed that the coordinates $\mathbf{r}$ inside Ω_e are defined in the range $[-1, 1]$. Matrix $\mathbf{N}$ is $1 \times n$ and vectors $\mathbf{P}_e$ and $\mathbf{W}_e$ are $n \times 1$. It is assumed that the base functions N_i are not function of time, such that

$$\frac{\partial^2 \tilde{p}(\mathbf{x},t)}{\partial t^2} = \mathbf{N}(\mathbf{r}) \frac{\partial^2 \mathbf{P}(t)_e}{\partial t^2} = \mathbf{N}(\mathbf{r}) \ddot{\mathbf{P}}_e(t). \tag{6}$$

Substituting eq.(3) into eq.(2) and integrating by parts with respect to the spatial coordinates results in

$$\int_\Gamma w \underbrace{(\boldsymbol{\nabla}\tilde{p} \cdot \boldsymbol{n})}_{p_n(t)} d\Gamma - \int_\Omega (\boldsymbol{\nabla} w) \cdot (\boldsymbol{\nabla}\tilde{p}) d\Omega - \frac{1}{c^2} \int_\Omega w \ddot{\tilde{p}} d\Omega = 0 \tag{7}$$

where $p_n(t)$ is the normal pressure on boundary Γ_e. Using the local interpolations and the fact that $\mathbf{W}_e(t)$, $\mathbf{P}_e(t)$ and $\ddot{\mathbf{P}}_e(t)$ do not depend on $\mathbf{r}$

$$\mathbf{W}_e^T \int_{\Gamma_e} \mathbf{N}^T p_n(t)\, d\,\Gamma_e - \mathbf{W}_e^T \int_{\Omega_e} \mathbf{B}^T \mathbf{B}\, d\,\Omega_e \mathbf{P}_e(t) - \mathbf{W}_e^T \frac{1}{c^2} \int_{\Omega_e} \mathbf{N}^T \mathbf{N}\, d\,\Omega_e \ddot{\mathbf{P}}_e(t) = 0 \tag{8}$$

where $\mathbf{B} = \boldsymbol{\nabla}\mathbf{N}$, such that

$$\underbrace{\int_{\Omega_e} \mathbf{B}^T \mathbf{B}\, d\,\Omega_e}_{\mathbf{K}_e} \mathbf{P}_e(t) + \underbrace{\frac{1}{c^2} \int_{\Omega_e} \mathbf{N}^T \mathbf{N}\, d\,\Omega_e}_{\mathbf{M}_e} \ddot{\mathbf{P}}_e(t) = \underbrace{\int_{\Gamma_e} \mathbf{N}^T p_n(t)\, d\,\Gamma_e}_{\mathbf{F}_e(t)} . \tag{9}$$

As the base functions have compact support and assuming non overlapping elements, it is possible to represent the original domain Ω as a superposition of the individual elements Ω_e as

$$\mathbf{K}\mathbf{P}(t) + \mathbf{M}\ddot{\mathbf{P}}(t) = \mathbf{F}(t) \tag{10}$$

where

$$\mathbf{K} = \sum_{e=1}^{n_e} \mathbf{H}_e^T \mathbf{K}_e \mathbf{H}_e, \quad \mathbf{M} = \sum_{e=1}^{n_e} \mathbf{H}_e^T \mathbf{M}_e \mathbf{H}_e, \quad \mathbf{F}(t) = \sum_{e=1}^{n_e} \mathbf{H}_e^T \mathbf{F}_e(t) \tag{11}$$

are, respectively, the global stiffness matrix, the global inertia matrix and the global source vector. Matrices $\mathbf{K}$ and $\mathbf{M}$ are $n_{tot} \times n_{tot}$, where n_{tot} is the number of nodes in Ω. Vectors $\mathbf{P}$ and $\mathbf{F}$ are the global vector of nodal pressures and the global loading vector, both with dimension $n_{tot} \times 1$. $\mathbf{H}_e$ is an $n \times n_{tot}$ localization matrix. The current computer code uses four node bilinear isoaparametric elements (it is possible to fuse two nodes to obtain a triangular element). Equation 10 is a linear system of Ordinary Differential Equations, where the time t is the independent variable.

2.1 Modal Problem

Modal analysis can be used to determine the resonance frequencies, damping coefficients, and modes of a system. By predicting and validating behaviors, it can also be used to reduce the computational effort of modeling the acoustic field [3].

Assuming a domain Ω with given Dirichlet boundary conditions, represented by the discrete eq.(10) and without source terms.

If the domain is subjected to an initial condition $\mathbf{P}(0) = \mathbf{P}_0$, the response is $\mathbf{P}(t) = \mathbf{P}_0 \cos(i\omega t + \phi)$, where ϕ is the phase. Equation 10 can then be written as

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{P}_0 = \mathbf{0}. \tag{12}$$

For physically consistent problems, eq.(12) has n_{tot} solution pairs ω and $\mathbf{P}_0$, which identify a specific vibration mode of the acoustic structure, where ω is a fundamental or natural frequency and $\mathbf{P}_0$ is the associated eigenvector or mode vector, describing the spatial distribution of (pressure) amplitudes during vibration at this frequency ω.

2.2 Transient Solution

When loading $\mathbf{F}(t)$ is known and not null, one must solve eq.(10) for each time $t \in [t_0, t_f]$. The most common approach is to use some sort of discretization procedure, like the Newmark-β method or some analytical procedure, like [5, 6]. The Newmark-β method [4, 7] assumes that both $\mathbf{P}(t)$ and $\dot{\mathbf{P}}(t)$ varies as

$$\mathbf{P}_{t+\Delta t} = \mathbf{P}_t + \dot{\mathbf{P}}_t(\Delta t) + \frac{(\Delta t)^2}{2}\left[(1-2\beta)\ddot{\mathbf{P}}_t + 2\beta\ddot{\mathbf{P}}_{t+\Delta t}\right], \tag{13}$$

and

$$\dot{\mathbf{P}}_{t+\Delta t} = \dot{\mathbf{P}}_t + (\Delta t)\left[(1-\gamma)\ddot{\mathbf{P}}_t + \gamma\ddot{\mathbf{P}}_{t+\Delta t}\right] \tag{14}$$

where Δt is a discrete time step and sub-indexes t and $t+\Delta t$ denote the values at the discrete times. Constants γ and β are used to adjust the characteristics of the approximation and the solution, like accuracy, numerical dissipation, period elongation and stability. Substituting both approximations in the discrete ODE of eq.(10) results in

$$\left[\mathbf{M} + \mathbf{K}\beta(\Delta t)^2\right]\ddot{\mathbf{P}}_{t+\Delta t} = \mathbf{F}_{t+\Delta t} - \mathbf{K}\mathbf{P}_{t+\Delta t}, \tag{15}$$

where in $t = 0$,

$$\mathbf{M}\ddot{\mathbf{P}}_0 = [\mathbf{F}_0 - \mathbf{K}\mathbf{P}_0]. \tag{16}$$

The Newmark-β method is known for its flexibility and it is very easy to implement. For $\gamma = 1/2$ and $\beta = 1/4$ the method is unconditionally stable and does not induce numerical damping [4]. Nonetheless, it is known that for some problems where high frequencies are present, there are strong oscillation in the numerical response. To this end, various authors proposed modifications to the baseline method to control large frequencies in the response. One of such methods is the β_1/β_2 method proposed by [7]. This method uses an additional sub-step and introduced new algorithm parameters β_1 and β_2. Both methods were implemented in the Julia language [8].

GMSH [9] was used as a post-processing environment through Lgmsh (https://github.com/CodeLenz/Lgmsh), a package designed to provide several subroutines that facilitate the export of data to the GMSH post-processing software.

The current implementation accounts for two different "source" terms (r.h.s of equation 9) [3]: an applied normal velocity $v_n(t)$ at the boundary

$$p_n(t) = -\rho\frac{\partial v_n(t)}{\partial t} \tag{17}$$

where ρ is the density of the acoustic propagation medium or an impedance condition

$$p_n(t) = -\frac{1}{c\zeta}\frac{\partial p(t)}{\partial t} \tag{18}$$

in which $\zeta = \frac{Z}{\rho c}$ is the normalized impedance and Z is the specific acoustic impedance [3].

3 Results

The first test case is the evaluation of the fundamental frequencies of a "tube" with open-open boundary conditions. The tube is 1m long, with 0.1m of height, and is modelled as a 2D mesh of 100×10 bilinear isoparametric elements. The mesh size was found after the convergence of the first four fundamental frequencies. For $c = 340$m/s, the theoretical fundamental frequencies are 170, 340, 680 and 1020Hz. The frequencies obtained with this mesh were 170.007, 340.05, 680.04 and 1021.51Hz. Figure 1 shows the pressure distribution (eigenvector) associated to these fundamental frequencies, from top to bottom. These pressure distributions are in accordance with the theoretical results [1].

The second test case is the application of an initial condition (unitary pressure) in the middle portion of the tube as shown in the first graph of Fig. 2. This problem has 1D analytical solution given by [2]. A mesh comprised of 1000 four node bilinear isoparametric elements in the longitudinal direction was used, with $c = 340$m/s. The CFL number is set to 1.0.

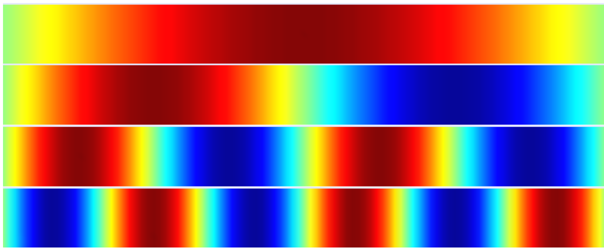

Figure 1. First four modes of an open-open tube. Each row corresponds to the pressure distribution of the mode, arranged from top to bottom (170, 340, 680 and 1020Hz). Pressure intensity scale varies linearly from -1 (blue) to 1 (red).

Figure 2 shows the pressure pattern along the length of the tube for different time steps and for the two solution methods implemented in this work. The results are in accordance with the analytical solution, but the well known artifacts associated to the discretization are present in the Newmark method (left column). Parameters used for the β_1/β_2 were $\gamma = 1/2$, $\delta = 1/4$, $\beta_1 = 0.39$ and $\beta_2 = 0.78$ [7]. From the results, one can observe that the β_1/β_2 method is effective in suppressing the high artificial frequency artifacts in the response.

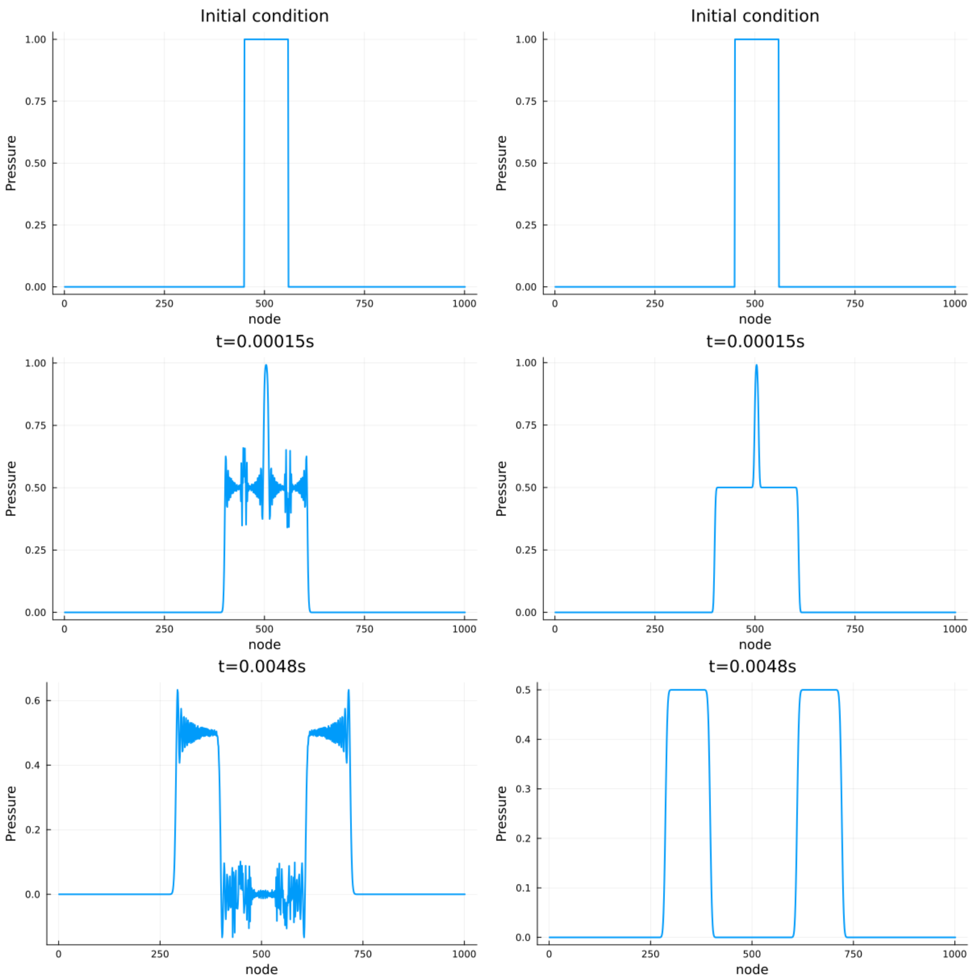

Figure 2. Initial condition and two pressure patterns at different times, obtained using the traditional Newmark-β method (left column) and the β_1/β_2 method (right column).

4 Conclusions

This work addressed the solution of the linear acoustic equation using the Finite Element Method. Both modal and transient solutions were discussed and two examples with analytical solutions were studied. The obtained results validate the formulation and the numerical implementation. Two different transient solvers were implemented and tested.

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