Isogeometric Analysis of Functionally Graded Plates using High-Order Theories

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Abstract. Isogeometric Analysis is a numerical method that integrates the concepts of geometric modeling and structural analysis. It approximates the displacement field using the same basis functions employed by Computer-Aided Design (CAD) systems to describe the structure's geometry. This paper presents an isogeometric formulation for the analysis of functionally graded plates based on Higher-Order Shear Deformation Theories (HSDTs), that require a C^1 continuity displacement field, and Non-Uniform Rational B-Splines (NURBS), that allow high continuity elements. A series of tests were conducted to assess the accuracy of this formulation considering examples available in the literature. The obtained results present excellent agreement with the reference solutions for thin and thick plates.

Keywords: Isogeometric Analysis, Functionally Graded Plates, High-Order Shear Deformation Theories.

1 Introduction

Functionally Graded Materials (FGM) are advanced composites, typically composed of ceramic and metal, in which the volume fraction of their constituents varies smoothly along an interest direction. These materials were initially proposed as a solution for thermal barriers development, but your applications are now diverse and numerous [1]. The composition variation results in a gradual change in their properties, enhancing the performance of structures subjected to thermal and mechanical loads, but making structural analysis more complex.

Plates are three-dimensional flat structures in which the thickness is much smaller than their other two dimensions. Due to their wide application, different theories have been proposed for the structural analysis of plates. These theories can be categorized based on their treatment of transverse shear strains. The Kirchoff-Love Theory [2] (also known as the Classical Plate Theory - CPT) disregards these strains, the Reissner-Mindlin Theory [3] (also known as the First-Order Shear Deformation Theory – FSDT) assumes shear strains are constant through the plate thickness, and Higher-Order Shear Deformation Theories (HSDTs) [4] account for nonlinear variations of shear strain through different approaches.

Among these theories, the HSDTs are the most robust and accurate ones. However, they require the displacement field to have a C^1 continuity, which is complex for isoparametric finite elements. Thus, Isogeometric Analysis emerges as a more feasible alternative by employing high continuity elements based on Non-Uniform Rational B-Splines (NURBS). Therefore, this paper presents and assesses a NURBS-based isogeometric formulation for thermal buckling and free vibration analyses of functionally graded plates using HSDTs.

2 Analysis of functionally graded plates

In functionally graded plates, the volume fraction variation of their constituents generally occurs in the thickness direction. This work considers a power-law variation:

$$V_c = \left(\frac{1}{2} + \frac{z}{h}\right)^N \quad , \quad V_m = 1 - V_c \tag{1}$$

where V_c is the volume fraction of the ceramic constituent, V_m is the volume fraction of the metallic constituent, z is the spatial coordinate starting from the plate mid-plane in the thickness direction, h is the plate thickness and N is an arbitrary rational exponent. The calculation of the effective properties, in turn, was carried out based on Voigt's micromechanical model, also known as the Rule of Mixtures:

$$P_{ef} = V_c \cdot P_c + V_m \cdot P_m \tag{2}$$

where P_{ef} is the effective value of a given property at a point along the thickness, P_c is the property value corresponding to the ceramic constituent and P_m is the property value corresponding to the metallic constituent.

2.1 High-order shear deformation theories

In general, the displacement field described by higher-order theories is given by [4]:

$$\begin{aligned} \bar{u}(x,y,z) &= u(x,y) - z \cdot w_x + f(z) \cdot (w_x - \beta_x(x,y)) \\ \bar{v}(x,y,z) &= v(x,y) - z \cdot w_y + f(z) \cdot (w_y - \beta_y(x,y)) \\ \bar{w}(x,y,z) &= w(x,y) \end{aligned} \tag{3}$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the global displacements, u, v and w are the displacements in the plate mid-plane, β_x and β_y are the rotations in the undeformed mid-plane and normal to the xz and yz planes, respectively, and $f(z)$ is a characteristic function that defines the distribution of the shear strains along the thickness. Three shear deformation theories are considered in this work: the first-order theory (FSDT), which can be represented by $f(z) = z$, the Third-Order Shear Deformation Theory (TSDT), in which $f(z) = z - 4z^3/3h^2$, proposed by Reddy [5], and the Exponential Shear Deformation Theory (ESDT), in which $f(z) = ze^{-2(z/h)^2}$, proposed by Mantari et al. [6].

2.2 Isogeometric Analysis

Isogeometric analysis is a concept introduced by Hughes et al. [7] and consists of a numerical method similar to the Finite Element Method (FEM). However, the shape functions used in this approach are not polynomials, such as Lagrange and Hermite polynomials. Instead, the same functions used in Computer-Aided Design systems for geometry description, such as Non-Uniform Rational B-Splines (NURBS), are employed. Thus, the isogeometric formulation presented in this work starts from the geometric description of plates using a NURBS surface, defined by:

$$S(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,j,p}(\xi,\eta) \cdot \boldsymbol{p}_{i,j} \quad , \quad \boldsymbol{p}_{i,j} = [x_{i,j} \quad y_{i,j} \quad z_{i,j}]^T \tag{4}$$

where $S(\xi,\eta)$ is the function that maps the NURBS surface from a parametric space $\xi\eta$ to the Cartesian space, p is the surface's degree, $\mathbf{p}_{i,j}$ is a control point in the Cartesian space of a grid $n \times m$ and $R_{i,j,p}(\xi,\eta)$ is the basis function of degree p associated with that point.

Once the geometry is defined, similarly to the isoparametric formulation of finite elements, the central idea of isogeometric analysis is that the same basis functions used for the exact geometric description of the structure are also used to approximate the displacement field. Therefore, the displacement field is approximated by:

$$u(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,j,p}(\xi,\eta) \cdot \boldsymbol{u}_{i,j} \quad , \quad \boldsymbol{u}_{i,j} = [u_{i,j} \quad v_{i,j} \quad w_{i,j} \quad \beta_{x,i,j} \quad \beta_{y,i,j}]^T \tag{5}$$

where $u(\xi,\eta)$ is the displacement field in the plate mid-plane, and $\mathbf{u}_{i,j}$ is a vector of displacements and rotations associated with one of the mesh control points. With this concept understood, to calculate the displacement vectors $\mathbf{u}_{i,j}$, the system of equations is assembled in a manner analogous to FEM as demonstrated by Praciano et al. [3].

3 Numerical Examples

Two well-known examples of functionally graded plate are considered to assess the accuracy of the presented formulation. In both cases, the plate geometry is square, as shown in Fig. 1, and 7 different discretization schemes were adopted (2, 3, 4, 6, 8, 12, and 16 cubic NURBS elements per side of the plate). The materials used were SUS304 and Si_3N_4, whose properties vary depending on the example, as shown in Tab. 1.

Table 1. Material properties

Example	Material	E (GPa)	ν	ρ (kg/m³)	α (°C^{-1})
1	SUS304	201.04	0.30	8166	-
	Si_3N_4	348.43	0.30	2370	-
2	SUS304	207.79	0.28	-	1.5321e-05
	Si_3N_4	322.27	0.28	-	7.4746e-06

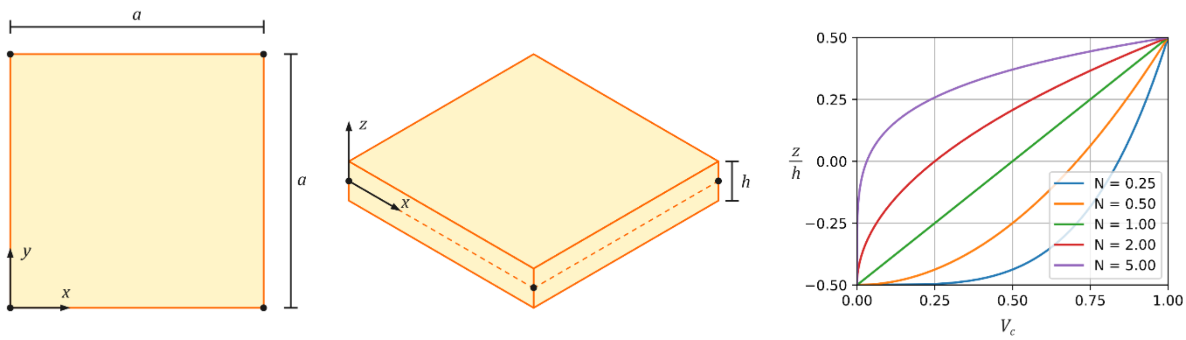


Figure 1. Functionally Graded Square Plate Geometry

3.1 Example 1 – Free vibration analysis

This example concerns a free vibration analysis of a plate with $a/h = 10$ ratio. Its boundary conditions are defined by:

$$\begin{aligned} \forall(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid y = 0 \vee y = a\} \quad &: \quad u = w = \beta_x = 0 \\ \forall(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid x = 0 \vee x = a\} \quad &: \quad v = w = \beta_y = 0 \end{aligned} \tag{6}$$

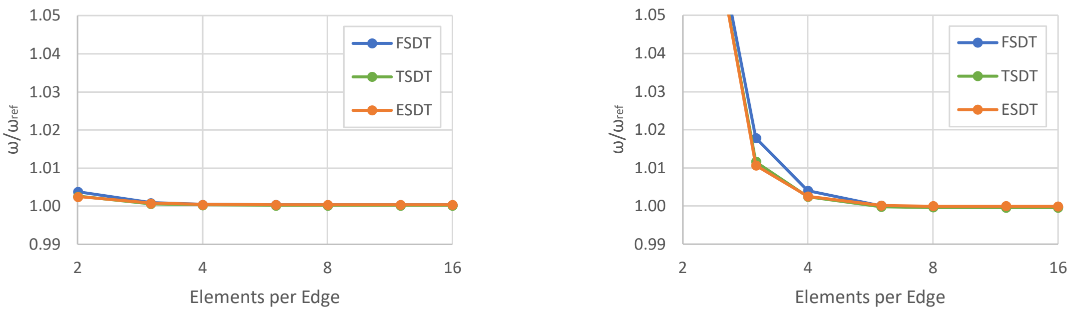
The normalized frequency is computed as:

$$\bar{\omega}_i = \omega_i h/a\sqrt{\rho_c/G_c} \tag{7}$$

where ω_i is the natural vibration frequency of order i in rad/s, ρ_c and G_c are the density and shear modulus of the ceramic constituent (Si_3N_4), respectively. The results obtained are compared with those achieved by Nguyen et al. [8] from a NURBS-based isogeometric formulation but using a novel quasi-3D theory. In this sense, a convergence study of the first two vibration modes, as shown in the graphs in Fig. 2, and a direct comparison using the most discretized mesh (16x16 elements), available in Tab. 2, were performed. Regardless of the theory and the value of N, the results converge to given by the reference. This convergence is evidently monotonic, a behavior explained by the use of full integration scheme. Furthermore, all theories presented practically the same frequency values in both vibration modes.

Table 2. Normalized natural frequencies of FGM square plate

Mode	1				2			
N	1.0	2.0	5.0	10.0	1.0	2.0	5.0	10.0
FSDT	0.0542	0.0485	0.0438	0.0416	0.1293	0.1155	0.1044	0.0991
TSDT	0.0542	0.0485	0.0438	0.0416	0.1292	0.1154	0.1042	0.0990
ESDT	0.0542	0.0485	0.0438	0.0416	0.1293	0.1154	0.1042	0.0990
[8]	0.0542	0.0485	0.0438	0.0416	0.1293	0.1154	0.1042	0.0990



(a) Mode 1 (b) Mode 2

Figure 2. Convergence of normalized natural frequencies for $N = 1.0$

3.2 Example 2 – Thermal buckling analysis

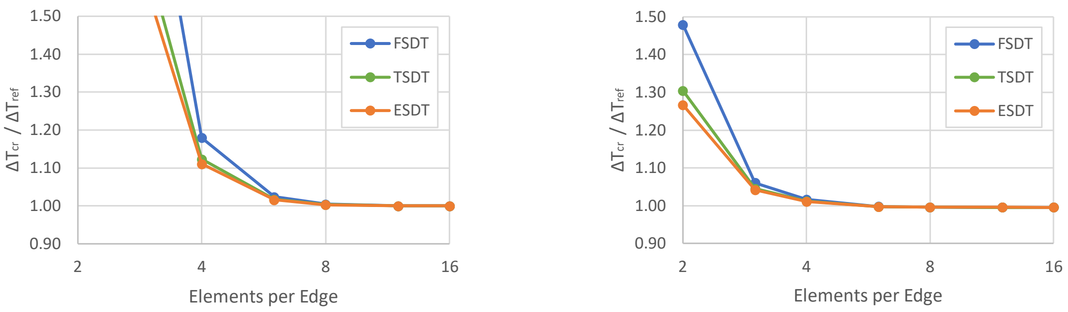
This example concerns a thermal buckling analysis under a uniform temperature rise. The boundary conditions are defined by:

$$\forall(x, y) \in \{(x, y) \in \mathbb{R}^2 \mid x \in \{0, a\} \vee y \in \{0, a\}\} \quad : \quad u = v = w = \beta_x = \beta_y = 0 \tag{8}$$

The result consists of the critical temperature increment ΔT_{cr} (K), considering that the material properties are temperature independent. The obtained results are compared with those obtained by Bateni et al. [9] using the multi-term Galerkin method in conjunction with a refined plate theory with a parabolic distribution of shear strains. In this context, a convergence study, as shown in Fig. 3, and a direct comparison of the critical temperature increments of the most discretized mesh (16x16 elements), available in Tab. 3, were performed. The critical temperatures are in very good agreement with the reference results, with an average difference of 0.2% for $a/h = 100$ and 0.8% for $a/h = 20$. Considering the use of the full integration scheme and the resulting monotonic convergence, these differences are probably due to the more refined mesh used in this work. Another explanation is the difference between the analysis methods and plate theories employed in both studies. Finally, as in the previous example, the difference between the values of the considered theories is slight.

Table 3. Critical buckling temperatures of FGM square plate

a/h	100					20				
N	0.0	0.5	1.0	2.0	5.0	0.0	0.5	1.0	2.0	5.0
FSDT	45.520	33.496	30.135	27.838	25.936	1094.67	806.235	725.117	669.113	622.574
TSDT	45.519	33.495	30.134	27.836	25.934	1094.70	806.444	725.133	668.707	621.813
ESDT	45.519	33.496	30.135	27.836	25.934	1094.97	806.643	725.310	668.840	621.922
[9]	45.528	33.501	30.140	27.842	25.940	1099.90	788.574	728.532	671.962	624.970



(a) Convergence for $a/h = 100$

(b) Convergence for $a/h = 20$

Figure 3. Convergence of critical buckling temperature for $N = 1.0$

4 Conclusions

This paper presented and assessed a NURBS-based isogeometric formulation for thermal buckling and free vibration analyses of functionally graded plates using HSDTs. In this regard, this formulation has proven effective in performing the targeted analyses. The results obtained exhibit excellent agreement with those available in the literature and converge regardless of the scenarios tested. Furthermore, the convergence of the results does not appear to be significantly affected by the shear deformation theory employed. However, this behavior may not hold for more complex geometries, different FGM constituents and lower a/h ratios, which can be investigated in future studies.

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