Numerical approximation of the Womersley problem in OpenFOAM

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Abstract. Replicating the pulsatile blood flow in simulations under various physiological conditions is essential for understanding hemodynamic behavior and its relationship with cardiovascular diseases, as well as for developing innovative clinical approaches. The Womersley problem, which describes the relationship between pulsatile flow and mean flow in a blood vessel, is at the core of every hemodynamic modeling problem. Computational Fluid Dynamics (CFD) simulation provides a detailed analysis of blood flow, allowing for precise representation of vascular geometry and the inclusion of pathological conditions such as stenoses and aneurysms. This study aims to find approximate solution to the Womersley problem through CFD simulations using the OpenFOAM software. Cases will be developed to obtain Womersley velocity profiles in vascular geometries using a periodic pressure difference. Results will be compared with the analytical solution to ensure the accuracy and reliability of the simulations.

Keywords: Hemodynamic, Womersley problem, Computational Fluid Dynamics (CFD).

1 Introduction

Around 600-700 mL of blood flows through the carotid arteries every minute to sustain a vital organ. Therefore, hemodynamics, that is, blood flow and its characteristics, is of fundamental importance for maintaining human life. However, this transport mechanism in the carotid arteries is sometimes altered by a medical condition characterized by the formation of plaques that hinder blood passage. At this stage, the patient may develop symptoms of a stroke, which is the third leading cause of death worldwide, according to Oyejide et al. [1]. As a result, understanding the functioning of the circulatory system and its relationship with various aspects of human physiology and pathology directly influences the effectiveness of medical treatments.

Computational Fluid Dynamics (CFD) has been widely used to simulate and analyze fluid behavior in various scientific and engineering applications. In hemodynamics, the combination of CFD with imaging techniques allows for the reconstruction of blood flow patterns in realistic and specific simulations as studied by Gallo et al. [2]. In this way, it is possible to assess a patient's hemodynamics non-invasively, determine the most effective treatment, and, for example, evaluate the risk of rupture and pressure in the intracranial aneurysm channel based on individual physiological models. This was conducted by Xie et al. [3] in a recent study. Existing studies use the

Womersley number as to characterized the inlet velocity boundary condition or aim to find an approximation or method that replicates the oscillation frequency, such as the works developed by Impiombato et al. [4] and Daidzic [5].

The present study uses a periodically varying pressure difference to induce pulsatile flow in the vascular domain, providing a more accurate representation of physiological conditions where arterial pressure is the main driving force of blood flow. The periodicity of the flow significantly impacts the results for velocity values, pressures, viscosities, and wall shear stress. This periodic behavior can lead to notable differences in simulation outcomes, emphasizing the importance of accounting for flow oscillations when modeling blood flow. This research aimed at finding approximate solutions for the Womersley problem through two-dimensional CFD simulations using OpenFOAM, developing specific cases to obtain and validate Womersley velocity profiles. The aim is to advance scientific knowledge in medicine, life engineering, and computational modeling.

2 Numerical simulations

2.1 Governing equations

The main equations that model fluid flows, representing the conservation laws of physics mathematically, are:

Conservation of Mass.

The mass conservation equation, or continuity equation, is expressed as eq. (1):

$$\nabla \cdot u = 0, \tag{1}$$

Conservation of Momentum.

The momentum conservation equation, known as the Navier-Stokes equation, is given by eq. (2):

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{1}{\rho} \; - \nabla P + \frac{\mu}{\rho} \nabla^2 u + F, \tag{2}$$

where u is the velocity vector, t is time, ρ is the fluid density, P is pressure, μ is the dynamic viscosity and F represents external forces.

In the performed simulations, blood is approximated as a Newtonian fluid for simplicity. Modeling blood as a non-Newtonian fluid would require implementing rheological models to accurately capture its shear-thinning behavior, which significantly increases computational demands and complexity. Such detailed modeling of blood's non-Newtonian properties, including various rheological parameters, is beyond the scope of this study.

2.2 Reynolds number

The Reynolds number, denoted by Re, is a dimensionless parameter that quantifies the ratio of inertial forces to viscous forces in fluid flow. It is calculated using the eq. (3):

$$Re = \frac{\rho \cdot U \cdot D}{\mu}, \tag{3}$$

where U is the characteristic velocity of the flow and D is the diameter of the tube.

2.3 Womersley number

The Womersley number, denoted by Wo, is a dimensionless parameter that quantifies the relationship between inertial and viscous forces in pulsatile fluid flow within a vessel, as shown in the work conducted by Womersley [6]. It is calculated using eq. (4):

$$Wo = r \cdot \sqrt{\frac{\omega \cdot \rho}{\mu}}, \tag{4}$$

where r is the radius of the vessel and ω is the angular frequency of the pulsatile flow.

2.4 Modeling

The geometry of the two-dimensional simulation was based on a blood vessel in order to represent a longitudinal section plane of a cylindrical tube, where the length (L) is 10 times the diameter (D). A uniform mesh was used in order to ensure a consistent and simplified representation of the domain. A total of 36,000 square cells was deemed suitable through a mesh convergence analysis, providing an adequate balance between result accuracy and computational cost. The total simulated time was 50 seconds, and the computational time required for the simulation was approximately 18 hours.

After establishing the geometry, the data used for all simulations were fixed, as shown in Tab. 1, where only Wo was changed. A time step of 10^{-4} s was used to keep the Courant number below 1, thus ensuring numerical stability and capturing the necessary phenomena.

Table 1. Properties and parameters adopted in the simulations

Property	Nomenclature	Value
Density (kg/m³)	ρ	1060
Dynamic viscosity (Pa·s)	μ	$3.45 \cdot 10^{-3}$
Diameter (m)	D	0.01
Amplitude (Pa)	A	31.8

Moreover, the simulations were conducted using the icoFoam solver, which is designed for laminar and in-compressible flow of Newtonian fluids. A Linear Gaussian scheme was used for variable discretization. For solving linear systems, the Preconditioned Conjugate Gradients method was applied for pressure, while the smoothSolver with the Gauss-Seidel method was used for velocity, both with relative tolerance. Temporal integration was carried out using the default Euler scheme provided by the solver.

The inlet boundary condition for all simulations was set as a periodic pressure difference, as described by eq. (5).

$$P_{in} - P_{out} = A \cdot sin(\omega \cdot t), \tag{5}$$

here, P_{in} is the inlet pressure, P_{out} is the outlet pressure (fixed at 0), A is the amplitude, ω is the angular frequency and t is the time instant. This setup ensures that the pulsatile velocity profile results from the inlet condition.

3 Results and Discussion

Figure 1 presents the blood flow velocity field in 3 time instants within a simulation period for Wo equal to 5.

To validate the results, the computationally obtained velocity profile was compared with the analytical solution from the work of Loudon and Tordesillas [7], where the Womersley velocity profile is obtained through eq. (6):

$$\begin{aligned} u\,(y,t) = \frac{A}{\omega\rho\gamma} \cdot \{&[sinh\,\Phi_1(y) \cdot sin\,\Phi_2(y) + sinh\,\Phi_2(y) \cdot sin\,\Phi_1(y)] \cdot cos\,(\omega \cdot t) + \\ &[\gamma - cosh\,\Phi_1(y) \cdot cos\,\Phi_2(y) - cosh\,\Phi_2(y) \cdot cos\,\Phi_1(y)] \cdot sin\,(\omega \cdot t)\}. \end{aligned} \tag{6}$$

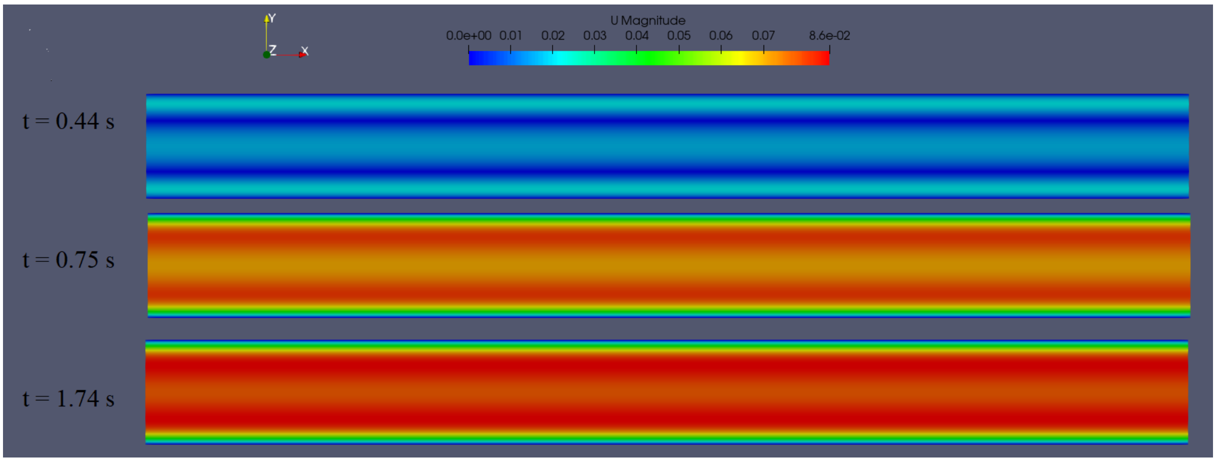

Figure 1. Velocity field (m/s) for 3 instants of a period (1.93 s), for Wo equal to 5

Function Φ_1 is given by eq. (7):

$$\Phi_1 = \frac{Wo}{\sqrt{2}} \cdot (1 + \frac{y}{D}), \tag{7}$$

where y is the coordinate relative to the center of the tube.

Function Φ_2 is given by eq. (8):

$$\Phi_2 = \frac{Wo}{\sqrt{2}} \cdot (1 - \frac{y}{D}). \tag{8}$$

Function γ is derived from eq. (9):

$$\gamma = \cosh(\sqrt{2}\, Wo) \cdot \cos(\sqrt{2}\, Wo). \tag{9}$$

That said, Womersley numbers 0.5 and 10 were also simulated, which can be seen in Fig. 2. This figure presents 3 time instants of the numerical solution obtained in comparison with the analytical solution derived from eqs. (6-9). At the 49-second mark of each simulation in Fig. 2, the maximum Reynolds numbers achieved were 2350, 247, and 74, respectively.

The results in Fig. 2 demonstrate a strong agreement between the numerical and analytical solutions. At different instants, the velocity profiles derived from the numerical simulations closely match those given by the analytical solution. This alignment validates the accuracy of the numerical model for different values of the Womersley number.

4 Conclusions

The performed simulations effectively demonstrated the capability to model pulsatile velocity profiles through a constant pressure difference, as shown by the strong agreement between the numerical and analytical solutions of the Womersley problem. This highlights the implementation's ability to capture the dynamics of blood flow across a range of Womersley numbers, reflecting both low and high pulsatility scenarios, thereby reinforcing its applicability to various physiological conditions within the human vascular system. Furthermore, the use of OpenFOAM to adjust the Womersley number for each scenario makes the simulation more personalized, potentially leading to increasingly accurate results. Since the dominant Womersley numbers can be experimentally obtained for individual patients, this approach allows for tailored simulations that enhance the precision of hemodynamic assessments and interventions.

Future work will involve simulating blood flow in different geometries under various physiological conditions and according to complex rheological models to expand the applicability of the current model. Additionally, the development and validation of three-dimensional simulations will be carried out to provide a more comprehensive and detailed understanding of hemodynamics in complex vascular structures.

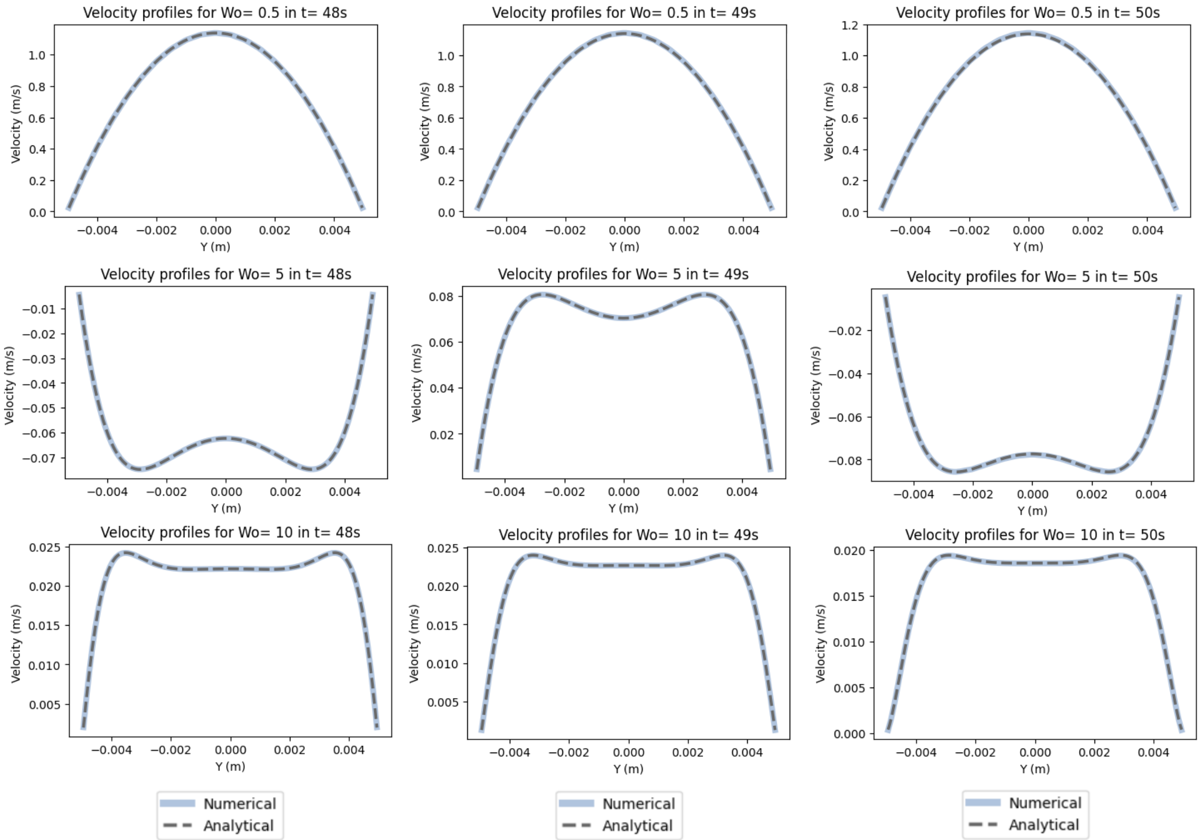

Figure 2. Comparison of the numerical solution with the analytical solution for different values of Wo

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