Reinterpretation of the Cross Method: Gauss-Seidel approximation for the solution of transverse displacement

Daniel G. Oliveira[1], Rafael A. de Souza[1]

[1]*Faculdade de Engenharias, Arquitetura e Urbanismo e Geografia, Universidade Federal de Mato Grosso do Sul*
Av. Costa e Silva, s/nº | Bairro Universitário | 79070-900 | Campo Grande, MS, Brazil
daniel.gomes@ufms.br, *rafael.a.sousa@ufms.br*

Abstract. The Cross Method was very used for analysis of structures, this was to the agility of the method. But he lose his use by the limitations of metho, such as transversal displacements. So, a general method was proposed, able of analyzing structures with transversal displacements and have the classical method as a particular case. The methods found to get these objective was to use displacement method and Gauss-Seidel approximation together and compare with the classical method. The results showed the Gauss-Seidel approximation was able to obtain an approximate solution to for structures with transverse displacements maintaining part of the agility of the traditional method. With these results, can be concluded that Cross's reinterpretation is able to analysis of structures with transverse displacement more easily than the classical method.

Keywords: Cross Method, Gauss-Seidel, transversal displacements

1 Introduction

Cross method was presented by Professor Hardy Cross in 1930 in the article “Analysis of Continuous Frames by Distribution Fixed-End Moments”, published in the American Society of Civil Engineers Transactions, to analyze hyperstatic structures with inextensible bars and without transverse displacements. your advantage is it doesn't need a directly solve a system of equations and find the bending moment diagram of the structure. (SORIANO [1])

But the Cross Method lose ground with the evolution and popularization of structural analysis software, its use is reserved only for undergraduate students, because of its historical retrospective and aid in understanding structural analysis (SOUZA [2]).

2 Cross Method

The original Cross Method, referred to in this work as the Classic Cross Method, determines the bending moment diagram and the support reactions of hyperstatic structures through the distribution of bending moments at the nodes, according to the stiffness of each bar. All of this is done using two tables. Stiffness Coefficient Table and Perfect Embedment Table (SORIANO [1]).

However, it has limitations, “the direct application of the method is restricted to cases in which the joints do not move during the moment distribution process.But the method can be applied indirectly in cases where joints are displaced during moment distribution” (CROSS [3]).

A matrix approach is presented by Moureira [4] with the aim of automating the iterative process of the classical method. All processes are automated through matrix operations, using the Distribution and Transmission Operator to perform the functions of the Distribution and Transmission Coefficients. For more details, see Moureira [4].

From a numerical point of view, the Cross Method is an iterative solution process for the equilibrium equation systems of the Displacement Method, according to White, Gergely, and Sexsmith [5]. This occurs regardless of the approach used (Classical, Gauss-Seidel, or Matrix).

3 Objectives

Propose a general method that includes the classical method as a particular case and without transverse displacements limitations.

4 Methodology

Use the Displacement Method in conjunction with the Gauss-Seidel approximation and physically interpret the iterative process.

5 The Cross Method by Gauss-Seidel

The method begins with the assembly of the equilibrium equation system of the displacement method. The state E0 is created, where all displacements of the structure are restrained, and the support reactions at each node are calculated using the Fixed-End Moments Table. In E0, states are created by applying a unit prescribed displacement at a support. The reactions caused by this displacement are found in the Stiffness Coefficients Table.

Using the superposition of effects, the equilibrium equations and the stiffness matrix are assembled, including all axial, transverse, and rotational displacements. The equilibrium equation is compactly expressed as:

$$K.d = E_0 \tag{1}$$

With:
K = Stiffness Matrix
d = Displacement Vector
E_0 = Force Vector in state E0

This system can be solved either directly or indirectly. In this work, the indirect approach of the Gauss-Seidel approximation will be used. The equation can be solved directly by inserting the nodal forces into the equilibrium equations, or indirectly by adding the reactions from state E0 to the nodal forces, with no significant difference between the methods.

5.1 Gauss-Seidel Method

Gauss-Seidel is a systematic approximation method. Given a generic system:

$$[A].[X] = [B] \tag{2}$$

If the diagonal elements of matrix $[A]$ are non-zero, it is possible to isolate x_i in row "i", for example x_1 in equation (2):

$$x_1 = \frac{b_1 - a_{12}.x_2 - a_{13}.x_3 - \cdots a_{1n}.x_n}{a_{11}} \tag{3}$$

The iterative process begins by choosing an initial approximation for the vector $[X]$. A simple case is to assume $[X] = [0]$.The initial approximation is substituted into equation (3) to find a new value for x_1, which will be used in the approximation of x_2. The process continues using the values of x_{i} to calculate the approximation of $x_{\mathrm{i+1}}$. After the first iteration, the process is repeated until the approximation converges to values sufficiently close to the exact solution. Assuming iteration 1, $[X] = [0]$, the i-th first variable is being calculated.

$$x_{i+1} = \frac{b_{i+1} - a_{i+1,1}.x_1 \ldots - a_{i+1,i}.x_i - a_{i+1,i+2}.\underbrace{0}_{x_i+2} \ldots - a_{i+1,n}.\underbrace{0}_{x_n}}{a_{i+1,i+1}} \tag{4}$$

5.2 Application of Gauss-Seidel in equilibrium equations

The Cross Method by Gauss-Seidel aims to find a solution to the equilibrium equations using a Gauss-Seidel approximation. The condition for applying Gauss-Seidel is naturally met by the Stiffness Matrix. The iterative process of equation (1) begins by assuming the initial approximation $[d] = [0]$, calculating d_1 according to equation (5):

$$d_1 = \frac{E_{0_1} - k_{12}.d_2 - k_{13}.d_3 - \cdots k_{1n}.d_n}{k_{11}} \quad (5)$$

With $[d] = [0]$.

$$d_1 = \frac{E_{0_1}}{k_{11}} \quad (6)$$

Using $[d] = [0]$ corresponds to restraining all displacements, while equation (5) is equivalent to releasing displacement 1, allowing its movement due to the loads of E0. The remaining nodes are kept restrained, similar to the Classical Cross Method, but transverse displacements can also be released.

At this stage, a modification is made to the Gauss-Seidel iterative process by adding the Residual matrix, equation (7):

$$Res_{1.it1} = K.\begin{bmatrix} d_{21it1} \\ \vdots \\ 0 \end{bmatrix} + E0 \quad (7)$$

This matrix stores the support reactions in the displacements. The first term represents the effect of the new value of d_1 on the remaining displacements, similar to the moment transmitted in the Cross Method. The second term is the existing support reaction in state E0, analogous to the sum of moments acting on each node in the Classical Cross Method.

The iteration continues from this stage, using the Residual to find new values of $[d]$. Since the effect of d_1 is already included in the Residual, it should not be directly accounted for in the approximation of other displacements. Thus, $d_1 = 0$ in the vector $[d]$, meaning displacement 1 is restrained again to release another displacement, similar to the classical method. This simplifies the calculations for other approximations, as there will be only one module, equation (8):

$$d_{2.it1} = -\frac{Res_{1.it1_{2,1}}}{K_{2,2}} \quad (8)$$

Using the Residual in equation (8) will find only successive increments for$[d]$, instead of calculating the total value of the current approximation. It is necessary to sum all iterations at the end of the process, just like in the Cross Method. The process continues for other displacements until the iteration is complete:

$$Res_{2.it1} = K.\begin{bmatrix} 0 \\ d_{2.it1} \\ \vdots \\ 0 \end{bmatrix} + Res_{1.it1} \quad (9)$$

From this stage, the residual from the previous approximation is used instead of E0. This is done to account for the effects of all increments calculated so far. A new iteration is performed until the increments of $[d]$ are sufficiently close to zero.

6 Example

The beam in Figure 1, whose bars have an elasticity modulus E = 210 GPa, a width of 0.20m, and a height of 0.80m, was analyzed using the Cross Method by Gauss-Seidel.

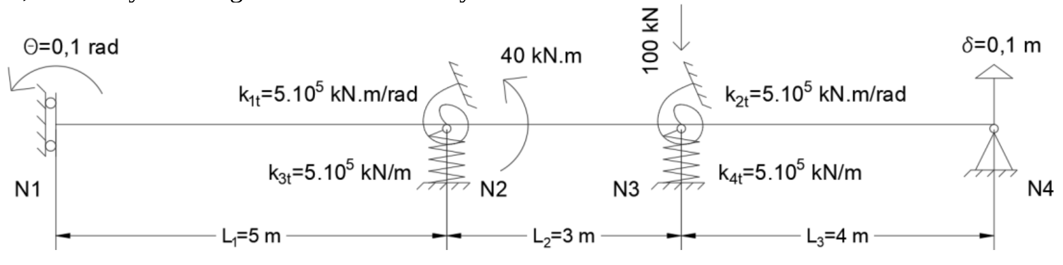

Figure 1. Example beam.

6.1 The Cross Method by Gauss-Seidel

The method starts with state E0 from Figure 1. From there, a state is created for each displaceability, and the equilibrium equations, equation (1), and the stiffness matrix are assembled using the Stiffness Coefficients Table

$$E_0 + K.d = 0 \longrightarrow d = -K^{-1}.E_0$$

The stiffness matrix, *K*.

$$K = \begin{bmatrix} k_{11} + k_{12} + k_{2t} & \frac{k_{12}}{2} & k_{13} & k_{14} \\ \frac{k_{12}}{2} & k_{22} + k_{23} + k_{2t} & k_{32} & k_{14} + k_{42} \\ k_{31} & k_{23} & k_{33} + k_{3t} & k_{34} \\ k_{41} & k_{41} + k_{42} & k_{43} & k_{33} + k_{44} + k_{4t} \end{bmatrix}$$

Stiffness coefficients, k_{ij}:

$$k_{13} = k_{23} = k_{31} = k_{32} = \frac{6.E.I}{L_2^2}, k_{33} = \frac{12.E.I}{L_3^2},$$

$$k_{14} = k_{41} = \frac{-6.E.I}{L_2^2}, k_{34} = k_{43} = \frac{-12.E.I}{L_3^2}, k_{44} = k_{42} = \frac{3.E.I}{L_2^3}$$

Stiffness coefficients of transverse and rotational springs:

$$k_{1t} = k_{2t} = 5\,.\,10^5\ kN.m/rad, k_{3t} = k_{4t} = 5\,.\,10^5\ kN/m$$

State E0:

$$E_0 = \begin{bmatrix} \theta.\left(\frac{-E.I}{L_1}\right) - 40 \\ -\delta.\left(\frac{3.E.I}{L_3^2}\right) \\ 0 \\ 100 - \delta.\left(\frac{3.E.I}{L_3^2}\right) \end{bmatrix} = \begin{bmatrix} -3{,}588\,.\,10^4\ kN.m \\ -3{,}36\,.\,10^4\ kN.m \\ 0\ kN \\ -8{,}3\,.\,10^3\ kN.m \end{bmatrix}$$

Displacements, with the initial approximation $d = [0]$

$$d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

d_i = displacement in position "*i*"

1° Displacement:

$$Res_{1.it1} = K.\begin{bmatrix} \frac{E_{01,1}}{K_{1,1}} \\ 0 \\ 0\ m \\ 0\ m \end{bmatrix} + E_0 = K.\begin{bmatrix} 0.011\ rad \\ 0\ rad \\ 0\ m \\ 0\ m \end{bmatrix} + \begin{bmatrix} -3{,}588\,.\,10^4\ kN.m \\ -3{,}36\,.\,10^4\ kN.m \\ 0\ kN \\ -8{,}3\,.\,10^3\ kN.m \end{bmatrix} = \begin{bmatrix} 0\ kN.m \\ -2{,}04\,.\,10^4\ kN.m \\ 1{,}32\,.\,10^4\ kN \\ -2{,}15\,.\,10^4\ kN \end{bmatrix}$$

2° Displacement:

$$Res_{2.it1} = K.\begin{bmatrix} 0\ rad \\ \frac{Res_{1.it1}}{K_{2,2}} \\ 0\ m \\ 0\ m \end{bmatrix} + Res_{1.it1} = K.\begin{bmatrix} 0\ rad \\ 0.005\ rad \\ 0\ m \\ 0\ m \end{bmatrix} + \begin{bmatrix} 0\ kN.m \\ -2{,}04\,.\,10^4\ kN.m \\ 1{,}32\,.\,10^4\ kN \\ -2{,}15\,.\,10^4\ kN \end{bmatrix} = \begin{bmatrix} 5{,}76\,.\,10^3\ kN.m \\ 0\ kN.m \\ 1{,}90\,.\,10^4\ kN \\ -2{,}56\,.\,10^4\ kN \end{bmatrix}$$

The continuous residue process for displaceabilities d_3 and d_4, completing the 1st iteration, required 5 iterations to find the approximate solution, as shown in Table 1

Table 1. Example results of Gauss Seidel approximation.

Iteration	Displacement			
	d_1 (10^{-2}rad)	d_2 (10^{-2}rad)	d_3 (cm)	d_4 (cm)
1	1.1	5.00E-01	-1.5	1.01E-01
2	7.33E-01	4.11E-01	-4.32E-01	6.41E-01
3	2.44E-01	1.83E-01	5.93E-04	3.25E-01
4	5.20E-02	5.11E-02	1.05E-01	1.40E-01
5	6.87E-03	2.05E-04	9.04E-02	4.64E-02
Total	2.1359	1.1453	-1.7360	1.2534

In Table 1, it is possible to observe that many iterations were necessary. But, the Gauss-Seidel approximation didn't require matrix inversion. To solve this example with Classic Cross method, would need to be used in three states to find the support reactions and solve a system of equations, which would be more laborious as the number of transverse displaceabilities increases.

7 Conclusion

Considering the information presented, concluded the Cross Method had great importance after his publication. But lost, the reasons are limitations with transversal displacements, although you can use indirect solutions, but the advantages of the method are lost. The suggested solution was the Cross Method by Gauss-Seidel, which uses a Gauss-Seidel approximation to find the solution to the equilibrium equations.

With example, it was observed that the Gauss-Seidel approximation was able to solve structures with transverse displaceabilities, without increasing the number of iterations and using simple calculations. And stablishing the Classical Cross Method as a particular case of the Cross Method by Gauss-Seidel, through their physical interpretations.

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