Reliability-based Boundary Element models for plane linear elasticity

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Abstract. In Structural Engineering, it is paramount to find robust solutions that ensure structural safety without sacrificing cost-effectiveness. To achieve this goal, one strategy involves considering the uncertainties inherent in projects, taking into account the variability of design parameters, rather than relying solely on characteristic values provided by semi-probabilistic approaches. This probabilistic paradigm allows for estimating the probability of failure of structural elements and systems, providing robustness to design, contributing to decision-making process, and reducing risks associated with both overly conservative and expensive projects, and economical yet unsafe projects. This study aims to implement the coupling between mechanical and reliability models to analyze the structural behavior of plates in linear regime. Mechanical models based on the Boundary Element Method (BEM) and reliability models based on Monte Carlo Simulation (MCS) are developed and validated, by using Python language. The random variables considered may include mechanical parameters such as elastic modulus and Poisson's ratio, dimensions of structural elements, as well as applied loads. The objective is to evaluate the probability of occurrence of usual limit states in structural analysis, such as the violation of allowable displacement and stress values.

Keywords: Elastostatics; Boundary Element Method; Python; Structural reliability; Monte Carlo Simulation.

1 Introduction

Numerical simulations of engineering problems are widely employed in the analysis and design of structures, driven by continuous advances in hardware and software technology, enabling increasingly sophisticated simulations. Among the main methods to numerically represent physical problems are the Finite Element Method (FEM), the Finite Difference Method (FDM), and the Boundary Element Method (BEM), the latter emerging as a more recent approach. According to Ubessi [1], the characteristics that make BEM attractive for application in engineering problems stem from its mathematical foundation: the integral equations that result in the method are boundary-based, reducing one dimension in the discretization of the problem, and its mixed character, which considers both displacements and tractions on the boundary in its formulation.

Regarding the evaluation of structural safety, reliability-based modeling stands out. It considers the uncertainties inherent to the design variables and investigates their influence on the structural response, allowing to estimate the probability of occurrence of a specified failure mode. This paper addresses the probabilistic response of elastostatics problem by combining the well-known Monte Carlo Simulation (MCS) with mechanical modeling based on BEM.

2 Boundary Element Method applied to elastostatics

The general equilibrium equation of linear elastic problem, evaluated at a point q, is given in terms of the stress tensor σ_{kj} and the body forces vector b_k, as follows:

$$\sigma_{kj,j} + b_k = 0, \text{ in } \Omega. \tag{1}$$

The strain state (ε_{kj}) of this material point can be related to the displacement field as stated in eq. (2):

$$\varepsilon_{kj}(q) \ = \frac{1}{2}\left[u_{k,j}(q) + u_{,j,k}(q)\right]. \tag{2}$$

2.1 Boundary Integral Formulation

Rizzo [2] presented the formulation of the direct BEM for elasticity. Brebbia [3] illustrated the procedure for obtaining this formulation through the application of the weighted residual method. The integration of the product between eq. (1) and weight functions of the type u_k^* leads to eq. (3), as follows:

$$\int_{\Omega} (\sigma_{kj,j} + b_k) u_k^* d\Omega. \tag{3}$$

By integrating by parts the first term and grouping the boundary integrals on the right-hand side of the equation:

$$-\int_{\Omega} \sigma_{kj} u_{k,j}^* d\Omega \ + \int_{\Omega} b_k u_k^* d\Omega \ \ = -\int_{\Gamma} p_k u_k^* d\Gamma, \tag{4}$$

where the first term can be integrated by parts again, giving rise to eq. (5):

$$\int_{\Omega} u_k \sigma_{kj,j}^* d\Omega \ + \int_{\Omega} b_k u_k^* d\Omega = -\int_{\Gamma} p_k u_k^* d\Gamma + \int_{\Gamma} u_k p_k^* d\Gamma, \tag{5}$$

which corresponds to Betti's reciprocity theorem. Equation (5) must satisfy the following boundary conditions

$$u_k = \underline{u_k} \text{ in } \Gamma_1, \ p_k = \underline{p_k} \text{ in } \Gamma_2. \tag{6}$$

Dividing the boundary Γ into Γ_1 and Γ_2 to apply the respective boundary conditions, we obtain eq. (7):

$$\int_{\Omega} u_k \sigma_{kj,j}^* d\Omega \ + \int_{\Omega} b_k u_k^* d\Omega = -\int_{\Gamma_1} p_k u_k^* d\Gamma - \int_{\Gamma_2} \underline{p_k} u_k^* d\Gamma \ + \int_{\Gamma_1} \underline{u_k} p_k^* d\Gamma + \int_{\Gamma_2} u_k \ p_k^* d\Gamma. \tag{7}$$

Applying weight functions from fundamental solutions $[\ ]^*$, the domain integral simplifies to:

$$\int_{\Omega} u_k \sigma_{kj,j}^* d\Omega \ = \int_{\Omega} u_l \sigma_{lj,j}^* d\Omega = -\int_{\Omega} \Delta^i u_l e_l d\Omega = -u_l^i e_l, \tag{8}$$

which can be written more concisely without separating the unknowns and boundary conditions in eq. (9):

$$u_l^i \ + \int_{\Gamma} u_k p_{lk}^* d\Gamma \ = \int_{\Gamma} p_k u_{lk}^* d\Gamma + \int_{\Omega} b_k u_{lk}^* d\Omega. \tag{9}$$

Known as Somigliana's identity, this equation allows for calculation of displacements based on boundary values, domain forces, and fundamental solutions. Singularities in boundary must be isolated, leading to:

$$\int_{\Gamma} p_{lk}^* u_k d\Gamma = \lim_{e\to 0}\{\int_{\Gamma-\Gamma_e} p_{lk}^* u_k d\Gamma\} + \lim_{e\to 0}\{\int_{\Gamma_e} p_{lk}^* u_k d\Gamma\}. \tag{10}$$

It can be proven that, at points where the boundary is smooth, eq. (10) simplifies to eq. (11):

$$\lim_{e\to 0}\{\int_{\Gamma_e} p_{lk}^* u_k d\Gamma\} = \frac{1}{2}\delta_{lk}, \tag{11}$$

and all other integrals tend to zero. Thus, the left side of eq. (11) results in eq. (12), as follows:

$$\int_{\Gamma} p_{lk}^* u_k d\Gamma - \frac{1}{2}\delta_{lk} u_k^i = \int_{\Gamma} p_{lk}^* u_k d\Gamma - \frac{1}{2} u_l^i. \tag{12}$$

Denoting the free term as c_{lk}, eq. (9) can be written as eq. (13):

$$c^i_{lk}u^i_k + \int_\Gamma p^*_{lk}u_k d\Gamma = \int_\Gamma u^*_{lk}p_k d\Gamma + \int_\Omega b_k u^*_{lk} d\Omega. \tag{13}$$

At points where the boundary is not smooth (edges and vertices), the free term c_{lk} becomes dependent on formulations.

2.2 Numerical Implementation

Efficient implementation of BEM involves appropriate boundary discretization, suitable interpolation functions, and precise resolution of singular integrals. To calculate displacements and surface forces in two directions, global matrices G and H are formed, containing components of integrals associated with forces and displacements. These matrices form a system of linear equations that, when solved, provides unknown values at the boundaries and allows the determination of displacements and stresses at internal points, as described in equations (14) and (15):

$$[H]\{u\} = [G]\{p\}, \tag{14}$$

$$[A]\{X\} = \{B\}. \tag{15}$$

According to Vieira et al. [4], in the Boundary Element Method, displacements and stresses at internal points are obtained from boundary information, in terms of displacements and tractions. Equations (16) and (17) are used to calculate, respectively, the displacements and internal stresses:

$$u_i(s) = -\int_\Gamma p^*_{ij}(\mathrm{s},\mathrm{Q})u_j(Q)d\Gamma(Q) + \int_\Gamma u^*_{ij}(\mathrm{s},\mathrm{Q})p_j(Q)d\Gamma(Q) + \int_\Omega u^*_{ij}(s,q)b_j(q)d\Omega(q), \tag{16}$$

$$\sigma_{ij}(s) = -\int_\Gamma S^*_{ij}(\mathrm{s},\mathrm{Q})u_k(Q)d\Gamma(Q) + \int_\Gamma D^*_{ij}(\mathrm{s},\mathrm{Q})p_k(Q)d\Gamma(Q) + \int_\Omega D^*_{ij}(s,q)b_k(q)d\Omega(q). \tag{17}$$

These integrals give rise to the so-called BEM influence matrices, which are arranged in an algebraic system, that can be solved to obtain the outcomes of the method.

3 Structural Reliability Concepts

The failure mode to be investigated is represented by a limit state function $G(\boldsymbol{X})$, in which vector $\boldsymbol{X}$ contains n random variables (r.v) under consideration. This function determines regions whether a structure is safe or unsafe, this latter indicated by $G(\boldsymbol{X}) \leq 0$. The probability of a point belonging to failure region can be calculated by the n-dimensional integral of the joint probability density function of r.v. over this domain:

$$P_f = \int_{G(\boldsymbol{X})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \tag{18}$$

The evaluation of eq. (18) can become complex, considering the nature of the kernel $f_X(x)$, especially in problems with a large number of r.v. Thus, different reliability methods are proposed in the literature, as is the case of MCS. According to Melchers and Beck [5], the method exhaustively simulates N random events to estimate P_f, by generating random numbers to represent the distribution of variables and evaluating them in the limit state equation. Failure probability is estimated by the relation between failure events and the total number of events:

$$P_f = \frac{N_f}{N}. \tag{19}$$

The estimate of P_f via MCS is roughly random itself, depending on the sampling size of the r.v. Aiming to obtain uniform results across different runs requires estimating a minimum number of scenarios, taking into account the characteristic P'_f of the problem and the desired value for its coefficient of variation δ – usually taken as 5% – as follows:

$$N_{min} = \frac{1}{\delta^2}\frac{1-P'_f}{P'_f}. \tag{20}$$

4 Numerical Application

A benchmark example is addressed in this section, aiming to validate and apply MCS-BEM strategy proposed herein. Consider a 20 m side square plate with a 1 m central hole under tension (Fig. 1), in plane stress condition. It is applied an uniform tensile stress of 133 MPa. The material presents elastic modulus E=200×10^9 Pa and Poisson's ratio $v = 0.33$. The example was analyzed in the BEM Python-based routine developed. In order to conduct a convergence study, the response in terms of vertical stress σ_{22} at radius r is compared to the analytical solution (Kapturczak and Zieniuk [6]), given by eq. (21):

$$\sigma_{22} = \frac{p}{2}\left(2 + \frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right). \tag{21}$$

in which R stands for the hole radius and p refers to the stress applied to. The point with coordinates (1.1,0) is taken as reference for the validation. Going precisely to the analytical value, σ_{22} is 365,016,424.3 Pa at this point.

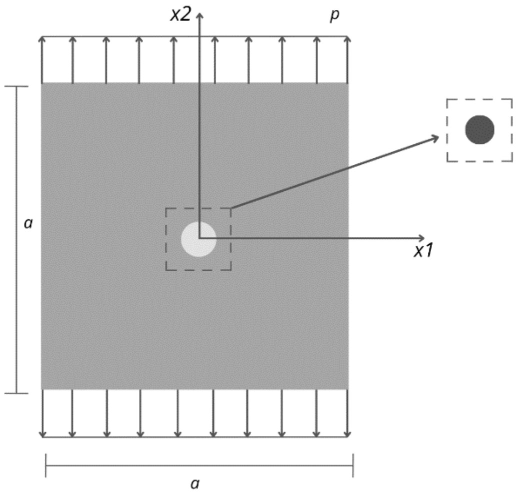

Figure 1. Square plate with circular hole under tension

From an initial mesh containing 8 boundary nodes, this discretization is enhanced to 94 nodes. Figure 2 illustrates that as discretization increases, the relative error to the analytical result decreases. It can be observed error values ranging from 1.27% up to 0.21%, this latter related to the finest 94-node mesh, which provide vertical stress of 365,016,403.2 Pa.

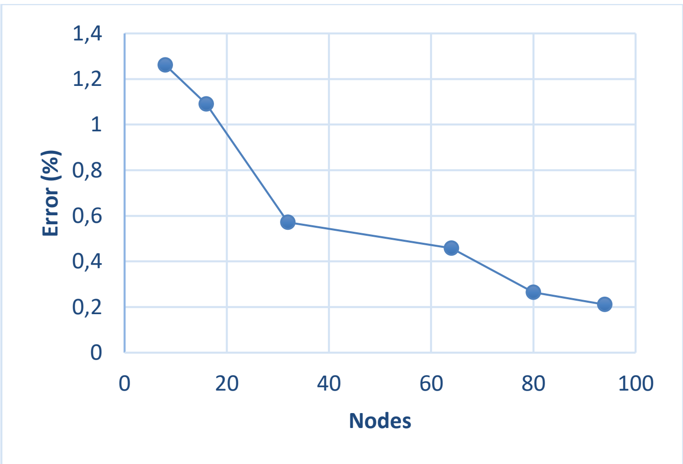

Figure 2. Error vs. Number of Nodes

By adopting a 32-node discretization, this problem is analyzed in the light of reliability analysis. In Table 1, it can be seen the statistical parameters adopted for the r.v. considered, the material yield limit and applied stress

(Hamilton [7]). The r.v. dispersion is described in terms of their coefficient of variation (COV), which relates the standard deviation to the mean value.

Table 1. Statistical description of the r.v. considered

Random Variable	Mean	COV	Distribution
f_y (MPa)	450	0.063	LogNormal
p (MPa)	133	0.05	Normal

The failure mode verified is related to the onset of the material yielding, as stated in the limit state function:

$$G(f_y, p) = f_y - \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2} \tag{22}$$

in which the second term on the right-hand side refers to the von Mises equivalent stress at the reference point, defined in terms of the principal stresses σ_1 and σ_2, which are outcomes of the BEM routine each of the thousands of times it is called. Subsequently, the r.v. associated to p is then considered.

Carrying out the Monte Carlo analysis, by generating 49,600 scenarios, a failure probability of 8.2 x 10^{-3} is estimated at a coefficient of variation of less than 5%, according to eq. (20).

5 Conclusions

Given the above, it can be concluded that the BEM implementation meets the proposed objective, as it presents the results of plate analysis, including displacements and stresses. Additionally, regarding the probabilistic analysis of the problem, it is possible to affirm that the mechanical-reliability coupling, by using MCS-BEM strategy, performed successfully. It must be noted the high computational cost associated, due to exhaustive calling to the numerical model.

Regarding reliability-based approaches, it stands out that there is no consensus on the admissible P_f values, lying in a case-driven analysis depending on the observed failure mode – whether it is an ultimate or serviceability limit state – and the risk tolerance defined by the analyst.

Future research can include the use of more sophisticated reliability methods, such as the First Order Reliability Method (FORM), besides the consideration of different failure modes.

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