Topology optimization of spatial frames

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Abstract. The pursuit of improving projects, such as optimizing resources and material usage, has led to an increased interest in structural optimization techniques. Among the various options available in the literature, the topology optimization stands out, allowing the design of structures with extreme behavior and significant material reduction, while adhering to constraints on the mechanical behavior of the structure. This study aims at implementing the topology optimization for spatial frames. The initial formulation is the traditional compliance minimization with volume constraint. Two approaches were studied: the use of optimality criteria and the use of the Augmented Lagrangian method. The mathematical formulation and the computer code were developed by the authors and are available on an open GitHub repository. The results show a considerable reduction of the objective function, while satisfying both the functional and the side constraints.

Keywords: Topology Optimization, Spatial frames, Compliance, Augmented Lagrangian.

1 Introduction

Optimization, in its most general sense, allows for the improvement and refinement of designs, aiming to achieve certain objectives, typically in the structural domain, such as volume or mass reduction while maintaining the necessary safety of the project. Structural optimization, therefore, addresses this need, striving to achieve these objectives.

To ensure that an engineering structural design is as optimized as possible in terms of the various design variables it may possess, the use of optimization is appropriate during the design phase. This way, it is possible for the structure to meet the specified objectives while adhering to the imposed functional and design constraints. In such cases, topology optimization stands out, allowing the structure to be formed from a fixed design region in space. This depends solely on the applied loads and supports [1].

As such, for the use of this approach, it is necessary the development and application of mathematical methods combined with computational means to solve optimization problems. These problems consist of an objective function, design variables and their constraints. Therefore, it is essential that the chosen approach is capable of integrating all constraints with the objective function, while varying the design variables.

The optimization of frames and trusses is of paramount importance, since these types of structures are widespread in structural design [1, 2]. This work addressed the design of spatial (3D) frames using topology optimization. The initial objective function is the static compliance with functional constraint in the volume of the design. Additional side constraints are applied to each design variable. The study follows the order where Section 2 provides an overview of topology optimization within the context of the present work. Section 3 addresses the solution of the problem at hand, while Section 4 discusses the studied examples. Section 5 concludes the study.

2 Topology Optimization

Topology optimization deals with the optimal material distribution within a fixed design domain [1]. Although the concept is commonly used to continuous problems, the first works in topology optimization were developed for trusses [1]. Indeed, most works addressing discrete structures, like trusses and frames, are devoted to planar (2D) problems, specially trusses. Nonetheless, most practical applications are subjected to (possible out of plane) bending and torsion, in addition to normal forces. Thus, the logical structural model to describe such structures is the spatial frame [3].

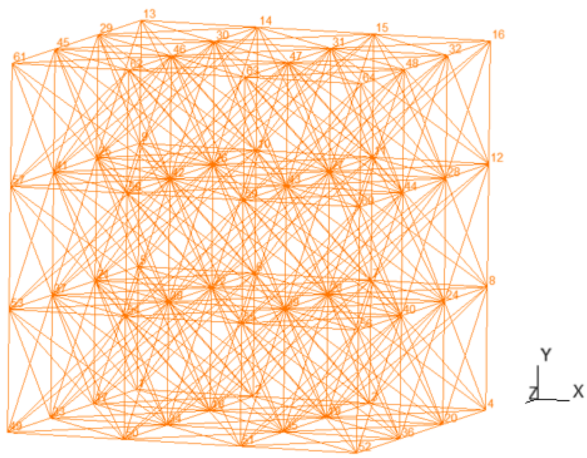

Figure 1. Ground Structure Ω with 467 elements

In this context, we assume that a fixed design domain Ω is discretized with a large number of finite elements Ω_e, also known as ground structure, as shown in Fig. 1

The stiffness matrix of each element is assumed to be directly proportional to the Young's Modulus, also known as the SIMP, or Solid Isotropic Material with Penalization, [1]

$$E_e(\rho_e) = \rho_e^p E^0 \tag{1}$$

in which E^0 is the Young's Modulus of the base material, ρ_e is the relative material density of element e, E_e is the effective Young's Module of element e and p is a positive penalty parameter. Thus, the stiffness matrix of element e is also proportional to ρ^P

$$\mathbf{K}_e(\rho_e) = \rho_e^p \mathbf{K}_e^0 \tag{2}$$

where $\mathbf{K}_e^0$ is the stiffness matrix evaluated using E^0.

The optimization problem is defined as

$$P \begin{cases} min\ C(\boldsymbol{\rho}) \\ S.t \\ \qquad \mathbf{K}(\boldsymbol{\rho})\mathbf{U}(\boldsymbol{\rho}) = \mathbf{F}\ , \\ \qquad V(\boldsymbol{\rho}) \leq \bar{V} \\ \qquad \mathbf{0} < \boldsymbol{\rho} \leq \mathbf{1} \end{cases} \tag{3}$$

where

$$C(\boldsymbol{\rho}) = \mathbf{U}^T(\boldsymbol{\rho})\mathbf{F} \tag{4}$$

is the static compliance,

$$V(\boldsymbol{\rho}) = \sum_{e=1}^{n} \rho_e v_e^0 \tag{5}$$

is the volume, $\mathbf{U}$ is the displacement vector, $\mathbf{F}$ is the force vector,

$$\mathbf{K}(\boldsymbol{\rho}) = \cup_{e=1}^{n} \rho_e^p \mathbf{K}_e^0 \tag{6}$$

is the global stiffness matrix, $\cup$ is the assembly operator and v_e^0 is the volume of element e. Derivative of the objective function with respect to relative density ρ_m is given by [1]

$$\frac{dC(\boldsymbol{\rho})}{d\rho_m} = -\mathbf{U}^T(\boldsymbol{\rho})\frac{d\mathbf{K}(\boldsymbol{\rho})}{d\rho_m}\mathbf{U}(\boldsymbol{\rho}) = -p\rho_m^{p-1}\mathbf{u}_m^T\mathbf{K}_m^0\mathbf{u}_m, \tag{7}$$

where $\mathbf{u}_m$ is the displacement vector of element m. The derivative of the volume with respect to ρ_m is

$$\frac{dV(\boldsymbol{\rho})}{d\rho_m} = \sum_{e=1}^{n_e} \frac{d\rho_e}{d\rho_m} v_e^0 = v_m^0. \tag{8}$$

3 Solution of the Optimization Problem

The optimization problem stated in eq. (3) can be reformulated as [4]

$$\min \mathcal{L}(\boldsymbol{\rho}, \mu) = \min C(\boldsymbol{\rho}) + \mu \left(V(\boldsymbol{\rho}) - \bar{V} \right) \tag{9}$$

where $\mathcal{L}$ is the Lagrangian function and $\mu \geq 0$ is the Karush-Kuhn-Tucker multiplier. Stationary condition $\boldsymbol{\nabla}\mathcal{L} = \mathbf{0}$ leads to

$$\frac{d\mathcal{L}}{d\rho_m} = \frac{dC}{d\rho_m} + \mu \frac{dV}{d\rho_m} = 0, \quad m = 1..n_e \tag{10}$$

and dividing by $\mu \frac{dV}{d\rho_m}$ gives

$$-\frac{\frac{dC}{d\rho_m}}{\mu \frac{dV}{d\rho_m}} = \beta = 1, \quad \forall m. \tag{11}$$

Using this useful relation, Bendsoe and Sigmund [1] proposed the update rule

$$\rho_m^{k+1} = \rho_m^k \beta^\eta, \tag{12}$$

where η is a relaxation constant (usually 0.5). This procedure, also known as optimality criteria, is simple to implement for this particular problem, since there is only one functional constraint and the derivatives of this constraint do not depend on $\boldsymbol{\rho}$. For more general problems, one must deduce and implement a completely different procedure. This procedure was the first approach studied in this research project.

Nonetheless, the objective of the research is to take into account a large number of functional constrains $g_j(\boldsymbol{\rho})$, like stress, natural frequencies and stability constraints, such that this approach is limited. Thus, the same optimization problem was addressed by means of the Augmented Lagrangian method [5]

$$\min L_A^k(\boldsymbol{\rho}) = C(\boldsymbol{\rho}) + \frac{c^k}{2} \sum_{j=1}^{m} \left\langle \frac{\mu_j^k}{c^k} + g_j(\boldsymbol{\rho}) \right\rangle^2, \tag{13}$$

where m is the number of functional constraints, $L_A^k(\mathbf{x})$ is the Augmented Lagrangian at an external iteration $k = 1..n_k$, n_k is the number of external iterations, c^k is the penalization term and μ_j^k are the multipliers for constraint j and iteration k. Operator $\langle a \rangle = \max(0, a)$ is used to account for the inequalities. This method is based in an approximation for the true Lagrangian Function, eq. (9), but the multipliers $\boldsymbol{\mu}$ are not variables of the problem. Instead, one starts with $\boldsymbol{\mu}^0 = \mathbf{0}$ and given c^0, such that the first external iteration $k = 0$, is a pure external penalization problem [4]. For the remaining external iterations, both $\boldsymbol{\mu}$ and c are updated as [5]

$$c^{k+1} = \gamma c^k \tag{14}$$

and

$$\mu_j^{k+1} = \langle \mu_j^k + c^k g_j(\boldsymbol{\rho}^k) \rangle. \tag{15}$$

The problem is considered solved when the optimality conditions $\mu_j^k g_j(\boldsymbol{\rho}^k) \leq \delta,\ \forall j$ are met, where δ is a tolerance. Thus, this approach was also implemented to solve the optimization problem stated by eq. (3).

The solution of eq. (13) for a given k is performed using the WallE solver [6], developed by the research group (https://github.com/CodeLenz/WallE.jl). The finite element solution and the optimization were implemented using the Julia language [7] and can be accessed at https://github.com/CodeLenz/Vigas3D. The visualization is performed using the gmsh free software [8].

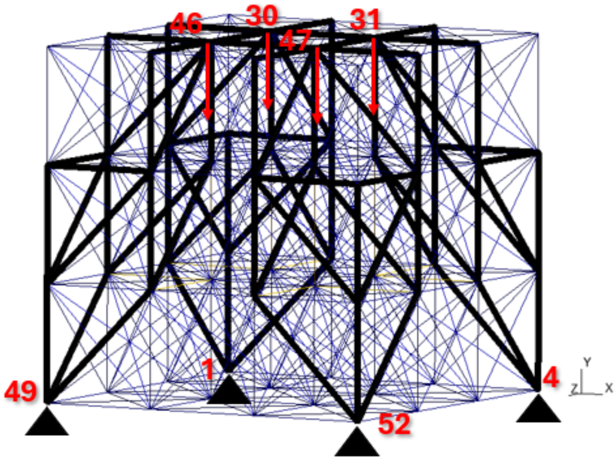

Figure 2. Simply supported structure under vertical load

4 Results

Two examples were used to study the implementation. The same material properties were used, with a Young's Modulus of $210GPa$ and Shear Modulus of $80GPa$. The geometric characteristics were also kept constant, with all elements having a circular cross-section with a radius of $57mm$. The initial domain for both examples is shown in Fig. 1. For both cases, the initial domain is a "cube" of $1 \times 1 \times 1$ meter, discretized as shown in Fig. 1, with 467 elements. The initial distribution for the design variables is assumed as constant for all elements, with the initial value equal to the volume fraction. SIMP exponent $p = 3$ is used in all examples.

Both solution methods were used with no modification on the final solution.

Example 1: Simply supported structure under vertical load. The first example consists of a simply support structure subjected to four point loads of 100N in the negative vertical direction (y) at nodes 30, 31, 46, and 47. The Dirichlet boundary conditions are null displacements in the x, y, and z directions at nodes 1, 4, 49, and 52 (bottom corner nodes). The minimum volume condition is 40% of the original volume. The initial penalization value, c^0, is set to 1.0. The visualization of the problem and the final result can be seen in Fig. 2. Full elements are shown in black and removed elements in blue. Initial compliance is 0.0060648Nm and final compliance is 0.00089169Nm.

Example 2: Simply supported structure subjected to torsion. The second example consists of a structure under point forces simulating torsion at the top of the structure. The applied load consist of two forces of $1500N$ in the x direction on nodes 16 and 61 and two forces of $2000N$ in the z direction on nodes 13 and 64. The essential boundary conditions were the same of the Example 1. The minimum volume constraint is 20% of the original volume. The initial penalization value, c^0, is set to 10.0 . The visualization of the problem and the final result can be seen in Fig.3. Initial compliance is 0.22646 Nm and final compliance is 0.01940 Nm. Full elements are shown in black and removed elements in blue.

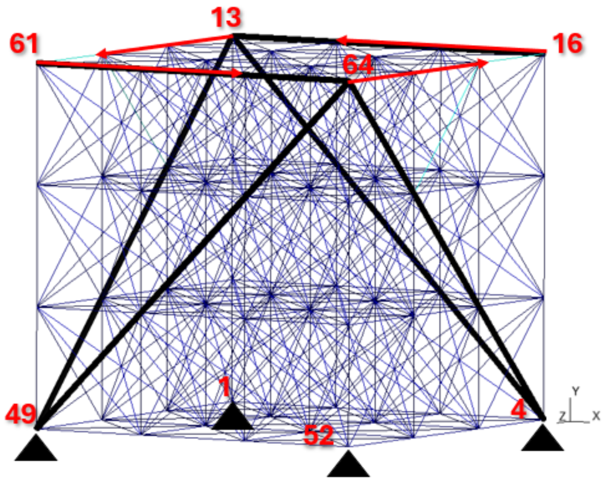

Figure 3. Simply supported structure exposed to torsion

5 Conclusions

In this study, we explored the topology optimization of a structure comprised of 3D frame elements. The classical minimum compliance problem with volume constraint is implemented with two different solution procedures: optimality criteria and the Augmented Lagrangian formulation. Two examples are studied, and the results show a significant minimization of the objective function, while respecting the functional constraint and side constraints. All results respect the optimality conditions and make physical sense (regarding the increase on the stiffness regarding the applied loads). The obtained results give confidence for the consideration of other functional constraints using the Augmented Lagrangian approach.

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