A positional FEM approach for free vibrations of orthotropic laminated shells

Vinícius B. Souza[1], Humberto B. Coda[1]

[1]*Dept. of Structural Engineering, São Carlos School of Engineering, University of São Paulo*
Av. Trabalhador São Carlense, 400, 13566-590, São Paulo/São Carlos, Brazil
vbarros@usp.br, hbcoda@sc.usp.br

Abstract. This paper presents a positional formulation of the Finite Element Method (FEM) for free vibration analysis of laminated shells with cross-ply and angle-ply fibers. The starting point of this formulation is the use of positions and generalized vectors as degrees of freedom, i.e., three coordinates and three vector components per node in the 3D space. The formulation is more general than Reissner-Mindlin by introducing an additional parameter which allows linear strain in the generalized vector direction, avoiding locking effects. The material layers are orthotropic according to the Saint-Venant Kirchhoff law, in which the constitutive tensor regarding the shell reference axes is obtained with a second-order tensor transformation matrix. The numerical solution combines Hammer quadrature to integrate the shell surface with Gauss-Legendre along the thickness, while the free vibration problem is solved as a standard eigenvalue and eigenvector problem. Two numerical examples are simulated, a toroidal and conical laminated shells, and results are compared against other theories and commercial software.

Keywords: composite material, laminated orthotropic shells, free vibration, finite element method.

1 Introduction

Laminated shell structures are widely used in engineering fields, such as aerospace, automotive, vessel, mechanical, biomechanical, civil industries and others. In the last decades, the demand for these structures is mainly correlated to their higher strength-to-weight and stiffness-to-weight ratios if compared to conventional materials [1,2]. In addition, laminated composites exhibit long fatigue life, resistance to corrosion and flexible design. However, the growing use of such structures requires accurate tools in order to achieve a deeper knowledge regarding design, modeling and computing of shell structures [3].

Different shell theories have been proposed over the years, which can be classified in three distinct groups [4]: the 3D elasticity, Equivalent Single Layer (ESL) and Layerwise (LW) theories. The mathematical complexity involving the equations based on 3D theories leads to the development of several 2D theories in the last two groups. In general, due to the reduced thickness dimension, laminated shell theories are approximated by hypotheses and assumptions which reduce a 3D continuum problem into a 2D problem [1,5]. According to Tornabene [4], the most popular numerical tool to carry out dynamic analyses of anisotropic shell structure is the Finite Element Method, which is based on a variational or weak formulation of the system of governing equations. The present paper also uses the FEM, although we apply a positional Lagrangian description to describe the shell reference surface, and generalized vectors to map any point in the analysis domain.

Because of this description, the provided formulation is fully nonlinear geometrically-exact. By using the so-called positional FEM, we aim to solve the dynamic free vibration problem of orthotropic laminated shells, disregarding effects of eventual elastic support. Free vibration modal analysis seeks to provide dynamic properties of linear structures, such as natural frequencies and model shapes, which are fundamental to predict resonance frequencies in structural design.

The free vibrations of doubly-curved laminated composite shells was computationally addressed by Viola, Tornabene and Fantuzzi [3], and Tornabene, Viola and Fantuzzi [6] using a general 2D higher-order LW and ESL theories, respectively. In both papers, the mechanical model is based on a unified approach named Carrera Unified Formulation (CUF). Wu et al. [7] also presents a unified formulation for linear vibrations of moderately thick shells of revolution made of orthotropic layers. A modified variational principle together with a partitioning technique was employed to derive the formulation based on the First-Order Shear Deformation Theory (FSDT).

Using modified high-order bases of a hierarchical FEM, a curved composite laminated shell element was developed by Wu, Xing and Liu [8]. This authors employed the LW theory with orthogonal polynomials (linear per layer) to provide an accurate and flexible model. Although the LW theory is recognizably more accurate than ESL theories, and yields results comparable to 3D elasticity [9], it leads to a high computational cost since the variables are correlated to the number of layers.

The theory used herein fits in the FSDT (Reissner-Mindlin-like). Regarding computational effort, this theory stands out due to the low cost required, since the number of degrees of freedom is the same irrespective of the amount of layers in the shell structure. In addition, we use a triangular degenerate shell element of cubic order to approximate the reference surface, whereas the initial shell thickness stands as a geometric parameter. Two examples are discussed and compared against results available in the specialized literature, from which the accuracy of the present formulation is checked.

2 Positional Finite Element Method

The positional FEM formulation was originally conceived by Bonet et al. [10] for fluid-supported membrane analysis and later on extended by Coda [11] for solid mechanics applications. For further details, one may consult the reference [12].

2.1 Kinematical approximation and positional mapping

The present kinematics is described by the mapping functions f_i^0 and f_i^1, which provide the initial (x_i) and current (y_i) coordinates of the continuum, respectively, where index i represents the vector component associated with the finite element node. Considering the shell's (or plate) reference surface in the stiffness center of the cross section, the mapping functions at initial and current configurations are, respectively:

$$f_i^{0k}(\xi_1,\xi_2,\xi_3) = \phi_\alpha(\xi_1,\xi_2)X_{\alpha i} + \left(d^k + \frac{h_0^k}{2}\xi_3^k\right)\phi_z(\xi_1,\xi_2)V_{zi}\,, \tag{1}$$

$$f_i^{1k}(\xi_1,\xi_2,\xi_3) = \phi_\alpha(\xi_1,\xi_2)Y_{\alpha i} + \left[\left(d^k + \frac{h_0^k}{2}\xi_3^k\right) + \phi_\gamma(\xi_1,\xi_2)A_\gamma\left(d^k + \frac{h_0^k}{2}\xi_3^k\right)^2\right]\phi_z(\xi_1,\xi_2)G_{zi}\,, \tag{2}$$

in which k represents a layer of thickness h_0^k; d^k is the distance from the mid surface of layer k to the reference surface, $X_{\alpha i}$ is the i-th initial coordinate of node α, and $Y_{\alpha i}$ is the i-th current coordinate of node α, whose value is unknown. $\vec{V}$ is the generalized vector in the initial configuration (unitary and normal to the reference surface) and $\vec{G}$ is the generalized vector in the current configuration (without restrictions). Finally, nodal parameter A_γ introduces a linear strain in the current configuration. The Lagrangian shape function ϕ_α related to node α is evaluated in dimensionless coordinates ξ_1 and ξ_2 from the reference surface mapping. The isoparametric domain is completed with dimensionless coordinate ξ_3^k along the thickness.

Total deformation function $\vec{f}$ that describes the configuration changing is given, for each layer, by:

$$\vec{f}^k = \vec{f}^{1k} \circ (\vec{f}^{0k})^{-1}\,. \tag{3}$$

And the gradient of Equation (3) can be written as follows:

$$\mathbf{A}^k = \nabla(\vec{f}^k) = \mathbf{A}^{1k}\cdot(\mathbf{A}^{0k})^{-1}\,, \tag{4}$$

in which:

$$\mathrm{A}_{ij}^{0k} = \frac{\partial f_i^{0k}}{\partial \xi_j} \text{ and } \mathrm{A}_{ij}^{1k} = \frac{\partial f_i^{1k}}{\partial \xi_j}. \tag{5}$$

Therefore, the positional formulation has six known nodal values in the initial configuration (stored in $\vec{X} = \{X_1, X_2, X_3, V_1, V_2, V_3\}^t$) and seven unknown degrees of freedom per node in the current configuration (stored in $\vec{Y} = \{Y_1, Y_2, Y_3, G_1, G_2, G_3, T\}^t$).

2.2 Constitutive equation

A material is called orthotropic when three mutually orthogonal planes of material symmetry exist [13]. For such materials, the Saint Venant-Kirchhoff constitutive model is written as:

$$\mathbf{S} = \mathfrak{C}_0 : \mathbf{E}, \tag{6}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, $\mathbf{E}$ is the Green-Lagrange strain tensor, and $\mathfrak{C}_0$ is the stiffness tensor, written in Voigt notation as:

$$\mathfrak{C}_0 = \begin{bmatrix} \frac{E_1(1-\nu_{23}\nu_{32})}{\Delta} & \frac{E_1(\nu_{21}-\nu_{31}\nu_{23})}{\Delta} & \frac{E_1(\nu_{31}-\nu_{21}\nu_{32})}{\Delta} & 0 & 0 & 0 \\ & \frac{E_2(1-\nu_{31}\nu_{13})}{\Delta} & \frac{E_2(\nu_{32}-\nu_{31}\nu_{12})}{\Delta} & 0 & 0 & 0 \\ & & \frac{E_3(1-\nu_{12}\nu_{21})}{\Delta} & 0 & 0 & 0 \\ & sym. & & 2G_{12} & 0 & 0 \\ & & & & 2G_{13} & 0 \\ & & & & & 2G_{23} \end{bmatrix}, \tag{7}$$

in which E_i stands the Young's moduli in *i*-th material direction, G_{ij} stands the shear moduli in the *ij* plane, ν_{ij} is Poisson's ratio, and $\Delta = 1-\nu_{12}\nu_{21}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-2\nu_{21}\nu_{32}\nu_{13}$.

In order to represent cross-ply and angle-ply laminated schemes, we introduce an orthogonal coordinate system (called global, $\bar{x}-\bar{y}-\bar{z}$) at each integration point of the shell's reference surface, from which the reference direction is defined (layers at 0°). If an arbitrary layer is oriented through an angle φ about the transverse $\bar{z}$-axis, the elastic constitutive tensor is transformed to the global coordinate system according to the following tensor transformation law:

$$\bar{\mathfrak{C}} = \mathbf{T}^{-1} \cdot \mathfrak{C}_0 \cdot \mathbf{T}, \tag{8}$$

where $\bar{\mathfrak{C}}$ is the transformed stiffness matrix (non-symmetric) and

$$\mathbf{T} = \begin{bmatrix} m^2 & n^2 & 0 & 2mn & 0 & 0 \\ n^2 & m^2 & 0 & -2mn & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -nm & nm & 0 & m^2-n^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & -n & m \end{bmatrix}, \tag{9}$$

whose $m = \cos\varphi$ and $n = \sin\varphi$.

The final stress-strain relation takes the form:

$$\begin{Bmatrix} \bar{S}_{11} \\ \bar{S}_{22} \\ \bar{S}_{33} \\ \bar{S}_{12} \\ \bar{S}_{13} \\ \bar{S}_{23} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 2\bar{C}_{14} & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 2\bar{C}_{24} & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{24} & 2\bar{C}_{34} & 0 & 0 \\ \bar{C}_{14} & \bar{C}_{24} & \bar{C}_{34} & 2\bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\bar{C}_{55} & 2\bar{C}_{56} \\ 0 & 0 & 0 & 0 & 2\bar{C}_{56} & 2\bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \\ \bar{E}_{12} \\ \bar{E}_{13} \\ \bar{E}_{23} \end{Bmatrix}. \tag{10}$$

2.3 Free vibration solution

The free vibration studies the behavior of a conservative (undamped) system in the absence of external loads. To access this problem, we apply the equation of motion in the positional FEM taking in account an infinitesimal perturbation ($\delta\vec{Y}$) in the body through the initial configuration, i.e.:

$$\mathbf{H}_0 \cdot \delta\vec{Y} + \mathbf{M}\cdot \delta\ddot{\vec{Y}}(\vec{Y},t) = \vec{0}, \tag{11}$$

where $\mathbf{H}_0$ is the initial Hessian matrix, $\mathbf{M}$ is the mass matrix, and t time.

For a linear analysis, the displacements are considered small. Hence, the previous equation can be rewritten in terms of displacements (u):

$$\mathbf{H}_0 \cdot \vec{u} + \mathbf{M}\cdot \ddot{\vec{u}} = \vec{0}. \tag{12}$$

Equation (12) can be solved as a standard eigenvalue and eigenvector problem in the following form:

$$(\mathbf{H}_0 - \omega_n^2 \cdot \mathbf{M})\cdot \vec{\chi} = \vec{0}, \tag{13}$$

in which $\vec{\chi}$ stands the mode shape vector, and ω_n is the natural frequency in rad/s. Alternatively, we can measure the natural frequency in Hertz (f_n), as follows:

$$f_n = \frac{\omega_n}{2\pi}. \tag{14}$$

3 Numerical applications

The toroidal shell of Figure 1a is defined by the following parameters: $R_c = 12$ m, $R = 3$ m, $h_0 = 0.6$ m with $\alpha_1 \in [0°, 120°]$ and $\alpha_2 \in [-80°, 80°]$. This shell is clamped on the right edge and made with three layers of same thickness according to the cross-ply scheme 0°/90°/0°, in which 0° and 90° are the transverse and longitudinal directions, respectively. On the other side, a conical shell (Fig. 1b) is bonded by two equal layers of orthotropic material with an angle-ply (-45°/45°) stacking sequence, $R = 2.5$ m, $L = 3$ m, $h_0 = 0.3$ m and $\theta = 60°$. Two boundary conditions are applied: clamped at the bottom and free at the top (C-F), and free at the bottom and clamped at the top (F-C).

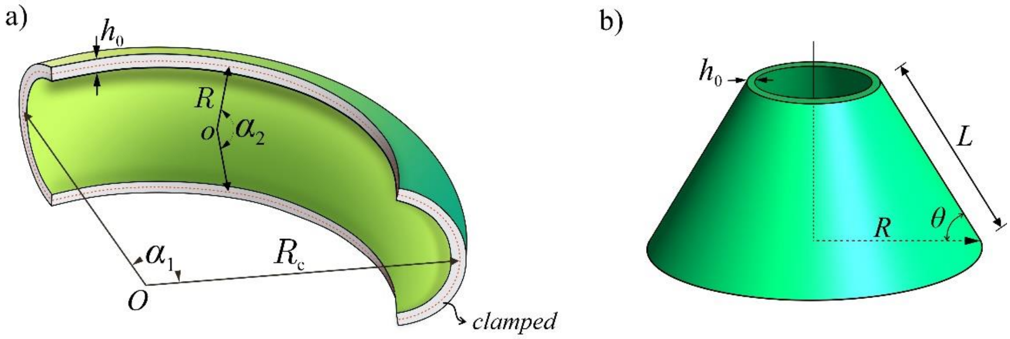

Figure 1. Geometry of the a) toroidal and b) conical shells

For both structures, the material properties of each layer are the following $E_1 = 137.9$ GPa, $E_2 = E_3 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = \nu_{13} = 0.25$, $\nu_{23} = 0.49$, $\rho = 1450$ kg/m³.

The mesh to discretize the mid surface of the toroid contains 13x31x2 finite elements (3760 nodes), while in the conical shell we use a coarse mesh of 6x30x2 finite elements (1710 nodes). It is worth mentioning that the A parameter was constrained in the cone analysis. Table 1 and Table 2 compare the first ten natural frequencies provided by the present formulation with solutions from other models, including the layerwise theory, FSDT, Second-Order Shear Deformation theory (SSDT), Third-Order Shear Deformation Theory (TSDT) and 3D finite element solution using Abaqus [3,6,7,8].

The natural frequencies provided by the Positional FEM show good agreement with the reference models. Although we note more uniform results in Table 1 (toroidal shell) than Table 2 (conical shell), the divergences are less than 5.0% for both structures when compared to the FSDT. The first nine model shapes for the toroidal shell are gathered in Fig. 1. The corresponding model shapes for C-F conical shell are summarized in Fig. 2. As it can be seen, the conical shell presents some symmetrical modes, which is not the case with the toroid. According to Viola, Tornabene and Fantuzzi [3], it appears that complete shell of revolution present some symmetrical modes due to the symmetry of the problem considered in the 3D space.

Table 1. First ten natural frequencies for laminated toroidal shell (0°/90°/0°)

f_n (Hz)	Present	FSDT[1] [3]	SSDT [3]	TSDT [3]	Abaqus[2] [3]
f_1	1.88990	1.88171	1.88317	1.88368	1.8831
f_2	4.39492	4.36432	4.36315	4.36225	4.3686
f_3	9.64528	9.60576	9.60508	9.60783	9.6053
f_4	14.80129	14.74315	14.73846	14.72986	14.7150
f_5	25.30271	25.21035	25.20773	25.20750	25.1970
f_6	39.26028	39.10891	39.09887	39.07198	38.9880
f_7	41.57737	41.47292	41.46660	41.45758	41.4560
f_8	52.08586	51.98249	51.97638	51.95684	51.9510
f_9	60.78184	60.66366	60.65618	60.63816	60.6540
f_{10}	64.19467	64.10283	64.07188	64.03832	63.5520

[1]Shear correction factor assumed equal to 5/6. Data originally from M. S. Qatu (2004), "Vibration of laminated shells and plates", Elsevier, 2004;

[2]Mesh with 40x80x12 Brick 3D finite elements (20 nodes per element).

Table 2. First ten natural frequencies for laminated conical shell (-45°/45°)

f_n	Clamped – Free (C-F)				Free – Clamped (F-C)		
(Hz)	Present	FSDT [7]	LW [8]	Abaqus[1] [3]	Present	FSDT [7]	Abaqus[2] [6]
f_1	180.44152	181.574	179.689	176.190	62.65573	63.852	61.350
f_2	180.44174	181.574	179.689	176.190	62.65577	63.852	61.350
f_3	260.09184	252.191	252.674	247.710	96.43623	94.025	95.600
f_4	260.09220	252.191	252.674	247.710	96.43640	94.025	95.600
f_5	269.33997	267.253	269.356	267.390	102.04558	103.857	98.150
f_6	269.34025	267.253	269.356	267.390	102.04569	103.857	98.150
f_7	332.98981	326.712	333.139	328.760	173.44286	171.978	165.680
f_8	361.12065	350.024	347.768	341.430	173.44297	173.941	165.680
f_9	361.12084	350.024	347.768	341.430	182.25890	173.941	181.150
f_{10}	395.24553	396.493	393.227	386.060	221.67055	217.695	219.910

[1]Mesh with 40x80x12 Brick 3D finite elements (20 nodes per element);

[2]Solved with $\nu_{12} = \nu_{13} = 0.30$ and a mesh with 60x100x12 Brick 3D finite elements (20 nodes per element).

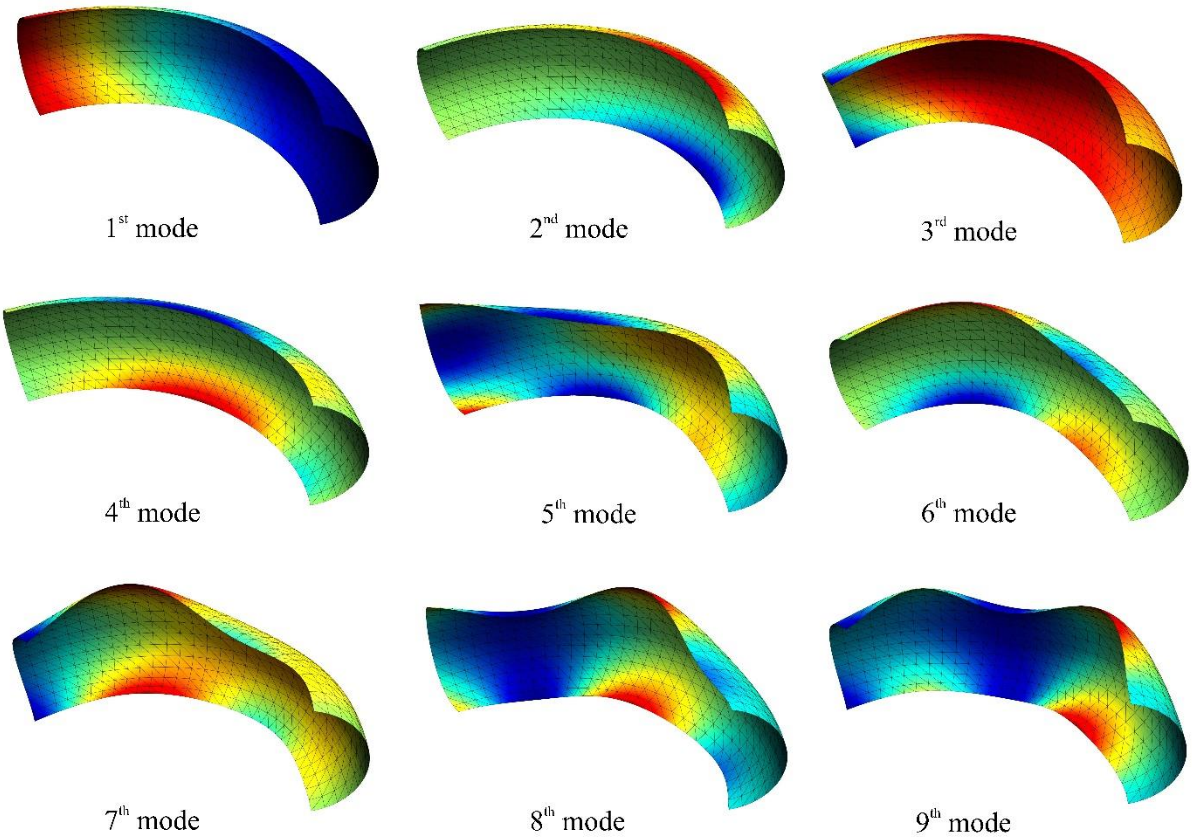

Figure 1. First nine mode shapes for laminated toroidal shell (0°/90°/0°)

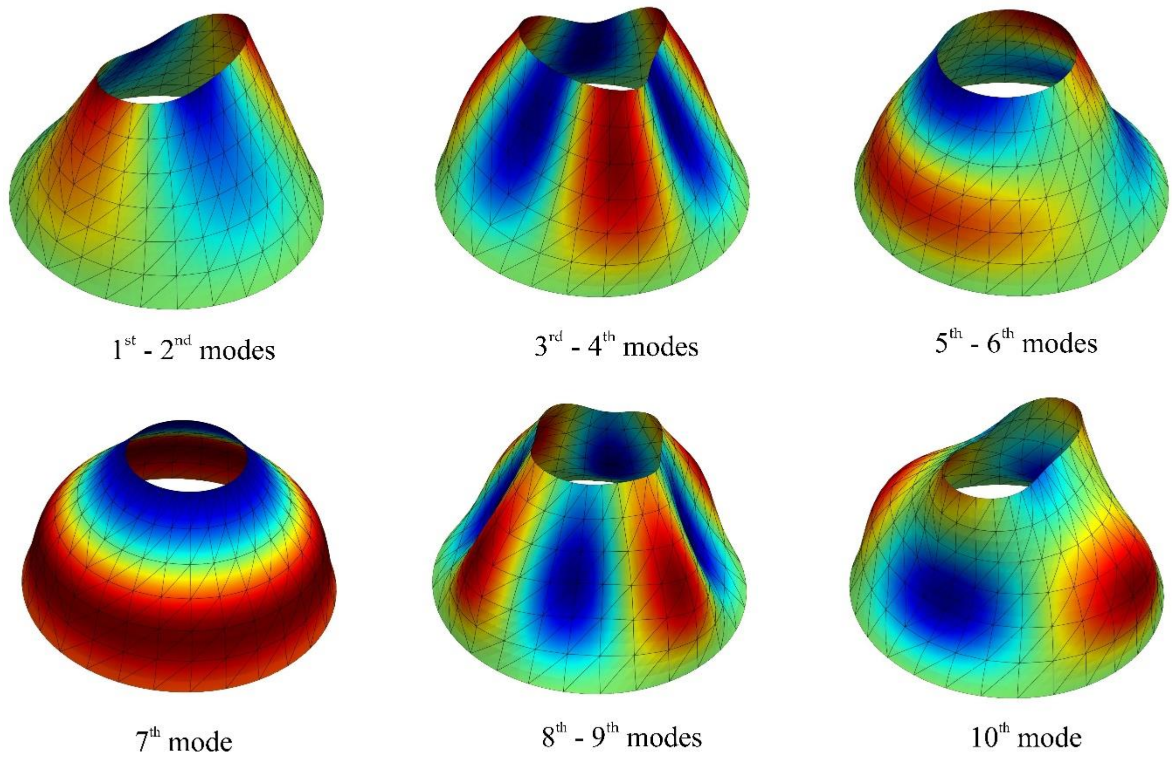

Figure 2. First ten mode shapes for laminated conical shell (-45°/45°) with C-F boundary conditions

4 Conclusions

Based on the dynamic positional FEM formulation, the free vibration analysis of single and doubly-curved laminated shells has been numerically solved. For both structures made of cross-ply and angle-ply stacking sequences, the natural frequencies calculated with this formulation are in good agreement with the benchmark models. The presented FSDT was able to provide results of same level of accuracy of ones obtained for the layerwise, higher-order theories and finite element software. The modal shapes graphically presented, although not compared, also seem consistent with those depicted in papers of the reference.

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