Application of Q4 Strain Gradient Notation Finite Element Method (SGN-FEM) in Level Set Method (LSM) Optimization

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Abstract. This paper presents the application of the Q4 Strain Gradient Notation Finite Element Method (SGN-FEM) in topology optimization studies using the Level Set Method (LSM) to minimize the compliance of two-dimensional linear elastic structures. SGN-FEM employs physically interpretable polynomials to develop finite elements and eliminate parasitic shear sources, thereby preventing shear locking, establishing itself as a locking-free element. LSM is known for being a boundary-based structural optimization method that provides a smooth structural boundary throughout the optimization process. This study investigates the differences obtained using SGN-FEM in optimization problems performed using LSM. The LSM model used is the parameterized level set method, as defined by [1]. The model is validated by comparing results with other studies performed using isoparametric finite elements. The results show that in problems like the cantilever beam or Messerschmitt-Bölkow-Blohm (MBB) beam, using a mesh of 1 element for each unit of measure, the SGN-FEM can converge in fewer iterations, such as 93 iterations for the cantilever beam and 100 iterations for the MBB beam, compared to 100 and 103 iterations using isoparametric elements. In the examples addressed, the SGN-FEM is able to obtain compliance values nearly equivalent to the results of the original study, demonstrating it is a viable alternative for optimization studies using LSM.

Keywords: Strain Gradient Finite Element; Shear locking; Level Set Method; Topology Optimization; Compliance Minimization.

1 Introduction

The basic theory for implementing topology optimization in materials design are presented in works such as [2], [3] and [4]. Other studies have been published, addressing various formulations of materials optimization problems. Among these formulations, there are studies that focus on optimizing compliance using the finite element method linked to optimization methodologies such as the level set method (LSM).

Within the finite element methodologies there is the Strain Gradient Notation Finite Element Method (SGN-FEM). SGN-FEM employs physically interpretable polynomials in the development of finite elements, allowing the precise identification and subsequent elimination of parasitic shear sources that cause shear locking. The element is corrected a priori during development by removing the spurious terms from the shear strain polynomials that way, it is possible to obtain a locking-free element [5].

The Level Set Method (LSM) is an optimization technique widely employed in topology optimization to minimize the compliance of structures [6]. The model produced using the level adjustment method can be easily manufactured, as the maximum local curvature can be controlled to produce a smooth surface. This is achieved by mapping the defined level field to the FEM model through the approximate Heaviside function projection, which increases the accuracy of the analysis and reduces intermediate densities around the boundary areas.

While numerous studies explore various approaches to calculate level set optimization, fewer investigate the contribution of finite element techniques to this analysis. This paper addresses this gap by presenting the application of Q4 SGN-FEM in optimization studies using the LSM to minimize the compliance of a two-dimensional linear elastic structure. The differences resulting from using the SGN-FEM model are examined. The parametrized level set method, as defined by Wei et al. [1] is employed alongside SGN-FEM for this study. The model is validated through comparison with other studies, demonstrating that SGN-FEM is a viable alternative for conducting optimization studies using the level set method.

2 Key Concepts and Theories

The Level Set Method (LSM) is well applied in structural optimization by providing precise boundary and geometry information [7]. This makes it effective for solving boundary and geometry-related problems. LSM's implicit representation allows an easy handling of complex shape and topology changes, such as boundary splitting and merging. Additionally, LSM can be integrated into CAD systems, increasing its practical usefulness. The complete explanation of the LSM applied here, specifically using the radial basis function, can be found in [1]. It will be addressed superficially.

In the level set method, dynamic structural interfaces are implicitly represented as the zero level set of a higher-dimensional level set function ϕ(x, t), typically defined as follows.

$$\begin{aligned} \phi(x,t) &> 0 \ \forall x \in \Omega \setminus \partial\Omega \\ \phi(x,t) &= 0 \ \forall x \in \partial\Omega \\ \phi(x,t) &< 0 \ \forall x \in D \setminus \Omega \end{aligned} \tag{1}$$

Let $x \in D \subset \mathbb{R}^2$ denote any point within the complete design domain D, where ∂Ω is the boundary of the solid domain Ω, as shown in Figure 1 for the 2-D, it is possible to be applicated in the optimization problem.

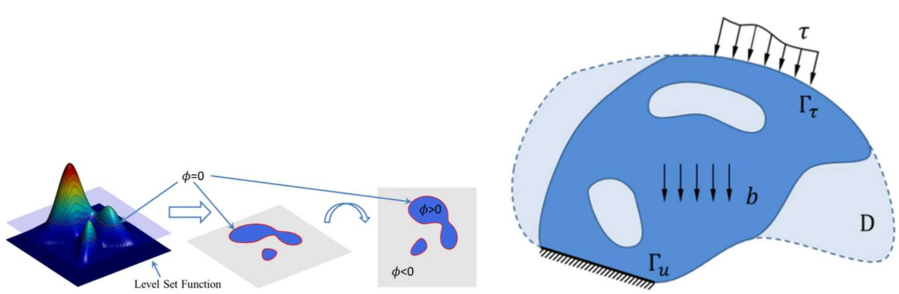


Figure 1. Representation of LSM and optimization problem definition [1]

The following evolution equation is used to update the level set function in the conventional level set method:

$$\begin{gathered} V_n = V.\left(\frac{\nabla\phi}{|\nabla\phi|}\right) \\ \frac{\partial\phi}{\partial t} - V_n|\nabla\phi| = 0 \end{gathered} \tag{2}$$

Here, ∇(·) represents the gradient of a scalar function, *t* denotes pseudo time representing the evolution of the level set function, and Vn=Vn(x,t) denotes the normal velocity directed outward, determined based on the shape derivative of an optimization problem.

The level set method involves solving a Hamilton-Jacobi Partial Differential Equation (PDE) on a fixed Cartesian grid using upwind differencing schemes and re-initialization to maintain a stable signed distance function. Numerical stability requires small time steps satisfying the CFL condition, limiting step size relative to grid spacing and velocity magnitude [1]. The method has limitations in creating new voids within material domains but is effective in handling topological changes like merging voids, benefiting from strategies such as initial void placement or nucleation techniques.

The approach developed by [8] and [9] integrates Radial Basis Functions (RBFs) into the level set method for shape and topology optimization. Representing the level set function as a linear combination of RBFs and coefficients simplifies evolution to updating coefficients rather than the function itself. RBFs, relying solely on spatial coordinates, facilitate smooth function evolution without needing re-initialization during optimization. Natural velocity extension reduces reliance on initial designs, aiding in creating new voids within material domains. Challenges include potential convergence issues due to high absolute values in level set functions, mitigated by approximate re-initialization and delta functions to regulate function values.

The parameterized level set function $\phi(x,t)$ is expressed as an interpolation using MultiQuadric (MQ) splines centered at fixed knots, and is formulated as:

$$\phi(x,t) = \sum_{i=1}^{n} \alpha_i(t) g_i(x) + p(x,t) \tag{3}$$

Here, $\alpha_i(t)$ denotes the time-dependent expansion coefficient of the MQ spline located at the *i-th* knot, while *p(x,t)* although optional for some RBFs, represents a linear polynomial in *x* and *t* ensuring the linear and constant components of $\phi(x,t)$ maintaining solution positivity [10]. For a comprehensive description of the Level Set Method (LSM) using Radial Basis Functions (RBFs), refer to the original article [1]

The compliance minimization problem aims to optimize a linear elastic structure under static loads while adhering to a material volume constraint. Using the level set method, the objective is to minimize the functional $J(u,\phi)$, which evaluates strain energy:

$$\text{Minimize}_{\phi} \quad J(u,\phi) = \int_D (\epsilon(u) : C : \epsilon(u)) H(\phi) d\Omega \tag{4}$$

Where $J(\phi)$ is the objective function, u the displacement field, ϵ denotes the strain tensor, C the elasticity tensor, and $H(\phi)$ is the Heaviside function determining material presence. The formulation includes a Lagrange multiplier term to enforce the desired volume fraction. Additional details on numerical implementation and optimization methods can be found in the referenced literature.

In strain gradient notation technique, SGN-FEM incorporates a physically interpretable notation into a displacement-based finite element method. Although SGN-FEM is not new, its literature is limited and the number of users is still small. The formulation procedure starts with displacement approximation functions written in terms of physically interpretable coefficients. These coefficients, which include rigid body motions, strains, and strain derivatives (generally called strain gradients), have been obtained for three-dimensional displacement polynomials and are tabulated for prompt use [11].

The four-node plane element (Q4) in strain gradient notation has been formally presented [11].

To make the coefficients physically interpretable, rigid body motions are expressed as translational displacements (u_{rb} and v_{rb}) and rotation (r_{rb}), while strains are expressed as derivatives of displacements. The coefficients are evaluated as follows: $a_0=(u_{rb})_0$, $b_0=(v_{rb})_0$, $a_1=(\varepsilon_x)_0$, $b_1=(\gamma_{xy}/2+r_{rb})_0$, $a_2=(\gamma_{xy}/2-r_{rb})_0$ and $b_2=(\varepsilon_y)_0$. Second order coefficient are: $a_3=(\varepsilon_{x,y})_0$ and $b_3=(\varepsilon_{y,x})_0$. Thus, the polynomials for the Q4 element become:

$$u(x,y) = (u_{rb})_0 + (\varepsilon_x)_0 x + (\gamma_{xy}/2 - r_{rb})_0 y + (\varepsilon_{x,y})_0 xy \tag{5}$$

$$v(x,y) = (v_{rb})_0 + (\gamma_{xy}/2 + r_{rb})_0 x + (\varepsilon_y)_0 y + (\varepsilon_{y,x})_0 xy \tag{6}$$

The strain components, expressed as derivatives of displacements, are:

$$\varepsilon_x = (\varepsilon_x)_0 + (\varepsilon_{x,y})_0 y \quad (7)$$

$$\varepsilon_y = (\varepsilon_y)_0 + (\varepsilon_{y,x})_0 x \quad (8)$$

$$\gamma_{xy} = (\gamma_{xy})_0 + (\varepsilon_{x,y})_0 x + (\varepsilon_{y,x})_0 y \quad (9)$$

The terms $(\varepsilon_{x,y})_0$ e $(\varepsilon_{y,x})_0$ represent flexural quantities. These are identified as spurious (parasitic shear terms) that can cause shear locking. They can be eliminated a priori from eq. (9) to definitely remove shear locking from the finite element model. The complete formulation of Q4 SGN-FEM can be found in [12].

3 Numerical Experiments and Results

Two numerical experiments originally proposed in [1] are conducted to evaluate the application of SGN-FEM in LSM using radial basis functions for optimization problems. The first experiment involves a cantilever beam with a load applied at the midpoint of the free end, as illustrated in Figure 2. Optimization was performed on five different mesh configurations using both the strain gradient element and the standard notation element, allowing for a comparative analysis. The number of iterations required for each mesh was also evaluated. The results are presented in Table 1 and 2.

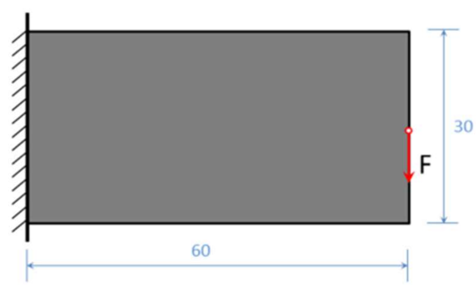


Figure 2. Example 1: Cantilever Beam. [1]

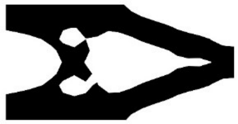
Table 1. Example Cantilever Beam: SGN-FEM

Nº Iterations	NX	NY	Compliance	Volume
200	20	10	625.049.375.796	0.501497
200	30	15	614.468.027.085	0.502373
93	60	30	600.072.056.979	0.499798
68	90	45	601.925.907.498	0.500471
80	120	60	602.263.936.474	0.500201

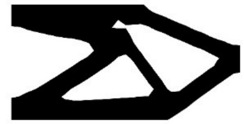
Table 2. Example Cantilever Beam: Standard Finite Elements

Nº Iterations	NX	NY	Compliance	Volume
200	20	10	620.652.744.768	0.499830
200	30	15	609.419.075.440	0.499325
100	60	30	598.576.476.999	0.499627
75	90	45	601.084.846.827	0.500335
100	120	60	600.480.712.560	0.500209

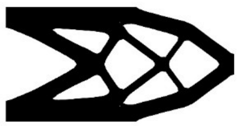
It can be noted that SGN-FEM converges with the same number of iterations as standard finite elements for the 20x10 and 30x15 meshes. For the 60x30, 90x45, and 120x60 meshes, SGN-FEM requires fewer iterations to converge. This demonstrates the efficiency of SGN-FEM in reducing the computational effort while maintaining good compliance results and meeting the volume requirements presented by [1] in the original work. Figure 3 presents the results obtained for SGN-FEM in each of the meshes.



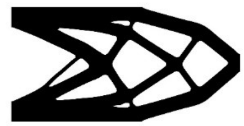
a) Mesh: 20x10



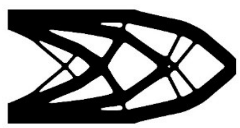
b) Mesh: 30x15



c) Mesh: 60x30



d) Mesh: 90x45



e) Mesh: 120x60

Figure 3. Compliance minimization SGN-FEM

For the second experiment, a simply supported beam with a load applied at the center of the beam, as illustrated in Figure 4, was analyzed. Optimization was performed on five different mesh configurations using both the strain gradient element and the standard notation element, allowing for a comparative analysis similar to the first experiment. The number of iterations required for each mesh was evaluated, and the results are presented in Table 3 and 4.

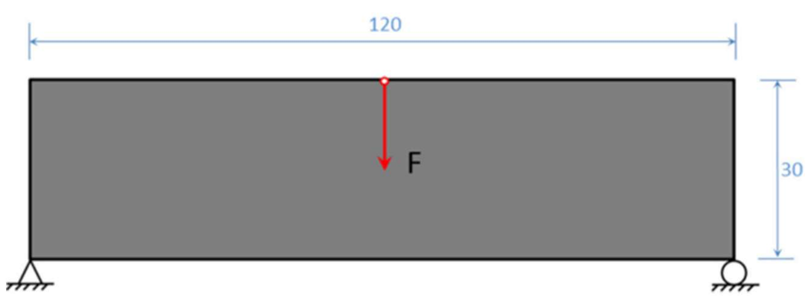


Figure 4. Example 2: MBB Beam.

Table 3. Example MBB Beam: SGN-FEM

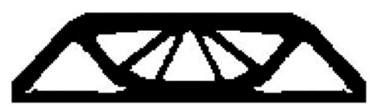
Nº Iterations	**NX**	**NY**	**Compliance**	**Volume**
116	40	10	359.366.587.473	0.499932
154	80	20	365.713.721.453	0.499796
100	120	30	364.568.480.027	0.499869
83	160	40	368.232.802.047	0.499570
70	200	50	373.391.610.616	0.499561

Table 4. Example MBB Beam: Standard Finite Elements

Nº Iterations	**NX**	**NY**	**Compliance**	**Volume**
84	40	10	354.551.582.420	0.500045
131	80	20	358.921.226.667	0.499759
103	120	30	361.854.917.949	0.500083
114	160	40	364.862.716.670	0.499910
74	200	50	368.556.307.117	0.499725

The SGN-FEM method demonstrates clear advantages over standard finite elements in the MBB beam example. Specifically, SGN-FEM achieves convergence with fewer iterations in 3 out of the 5 cases studied. For instance, in the original example case proposed by [1], with NX=120 and NY=30, the SGN element converges in 100 iterations compared to 103 iterations for standard finite elements. This indicates faster convergence and greater computational efficiency at high mesh densities. The results obtained using SGN-FEM are illustrated in Figure 5.

a) Mesh: 20x10



b) Mesh: 30x15

c) Mesh: 60x30

d) Mesh: 90x45

e) Mesh: 120x60

4 Conclusions

The application of the Q4 Strain Gradient Notation Finite Element Method (SGN-FEM) in Level Set Optimization using Radial Basis Functions demonstrates promising results when compared to standard finite elements. SGN-FEM is a locking-free method, effectively addressing issues commonly affecting standard finite elements. Being a locking-free element, SGN-FEM can be particularly advantageous in cases like the original example where, with NX=120 and NY=30, SGN-FEM requires 100 iterations while standard finite elements require 103 iterations. Despite the slight increase in compliance, SGN-FEM maintains a comparable volume fraction, ensuring material efficiency.

These findings highlight the potential of SGN-FEM for optimizing structural designs more effectively than traditional finite element methods. While SGN-FEM shows a tendency to require fewer iterations to converge in several cases, suggesting potential computational advantages, further exploration and validation are warranted. Overall, SGN-FEM presents itself as a viable alternative for topology optimization studies using the Level Set Method, offering an avenue for more efficient and effective structural design optimization.

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