Reliability analysis of serviceability limit states of beams in a benchmark reinforced concrete building

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Abstract. The construction system of reinforced concrete entails inherent uncertainties concerning its execution method, as well as physical, chemical, and biological phenomena, along with the loads acting on buildings. Hence, understanding and characterizing random variables in designing reinforced concrete structures is pivotal for devising effective solutions that meet safety and performance requirements. In this context, limit state equations are employed for structural analysis and design, addressing different failure modes, while concepts of probability and statistics are utilized alongside reliability methods to ascertain the probability of failure and assess the structural integrity of the elements. Within this framework, this study endeavors to implement models for evaluating the reliability levels of reinforced concrete beams in a 2-storey building, considering the limit states of excessive deflection and crack width, and incorporating the effects of shrinkage and creep, according to NBR 6118:2023. The probabilistic assessment of the designed beams is performed by using the First Order Reliability Method (FORM), whose results are validated and compared with those obtained through the Monte Carlo simulation method. The outcomes indicate a conservative design of the beams for the failure modes addressed, as the reliability indices are in line with values stipulated by international standards in most cases analyzed. Although this, some critical beams subject to higher loads approached the allowable limits more closely, presenting a narrow safety margin. This study aims to contribute to integrating reliability analyses into reinforced concrete structural projects.

Keywords: Reinforced concrete beams, Serviceability limit states, Structural reliability, FORM.

1 Introduction

Accompanied by advances in structural materials and analytical models, there has been an increase in the complexity of structures, necessitating a more thorough assessment of their safety levels. Additionally, the variability of design variables, such as element dimensions, loads, and material strength, contribute to uncertainties in structural performance. These uncertainties can be incorporated by using the statistical description of variables within the framework of structural reliability theory. This approach allows the estimation of a structure's failure probability for specified limit states (Melchers and Beck [1]).

The serviceability limit states (SLS) relate to the ordinary use of the structure, such as excessive deflection, local damage, and excessive crack width. Its violation causes greater impacts on the maintenance and operation costs of structures than on their safety (Honfi [2]). In terms of uncertainty quantification and probabilistic analysis of SLS, several studies can be referenced, such as Stewart [3], who used Monte Carlo simulation to estimate deflections in reinforced concrete beams; Honfi and Mårtensson [4], who investigated the reliability of beams designed for serviceability limit states according to the Eurocode 2 [5]; and McLeod [6], who evaluated model uncertainty for crack width in reinforced concrete. In Brazilian literature, Coelho [7] analyzed the probability of failure in beams using Monte Carlo simulation, Lessa et al. [8] applied reliability concepts to short concrete corbels, Santiago et al. [9] calibrated safety coefficients in Brazilian standards for steel and concrete structures, and Lima et al. [10] verified the serviceability limit states in solid concrete slabs.

This paper deals with the reliability analysis of reinforced concrete beams for the SLS of excessive deflection and excessive crack width, taking into account the effects of concrete shrinkage and creep. The analysis employs the First Order Reliability Method (FORM), validated by Monte Carlo simulation.

2 Serviceability limit states of beams - NBR 6118:2023

The serviceability limit state of excessive crack width corresponds to the condition where cracks exhibit openings equal to the maximum limits specified by standards, potentially compromising the usability and durability of the concrete member. Meanwhile, the serviceability limit state of excessive deformation refers to the condition where deformations exceed the maximum limits defined by standards and acceptable for the normal use of the structure.

The verification of crack width can be carried out in accordance with the guidelines of NBR 6118:2023 [11], along with the formulation presented in Model Code 2010 [12]. The determination of the maximum crack width is calculated using the following expression:

$$\omega_{max} = 2.I_{s,max}.(\varepsilon_{sm} - \varepsilon_{cm} - \eta.\varepsilon_{cs}) \quad (1)$$

where $I_{s,max}$ is the length over which slippage occurs between the concrete and the steel, ε_{sm} and ε_{cm} are the average strains of the steel and the concrete over the length $I_{s,max}$, respectively, while ε_{cs} is the strain of the concrete due to shrinkage. The final shrinkage strain of the concrete $\varepsilon_{cs(\infty,t_0)}$ is calculated according to the prescriptions of NBR 6118:2023 [11], using the following expression:

$$\varepsilon_{cs_{(\infty,t_0)}} = \varepsilon_{cs\infty}[\beta_s(\infty) - \beta_s(t_0)] \quad (2)$$

where $\varepsilon_{cs\infty}$ is the final shrinkage value, and $\beta_s(t_0)$ or $\beta_s(t)$ are coefficients related to shrinkage at time t_0 or t, respectively. For more details, refer to NBR 6118:2023 [11].

For the calculation of the maximum deflection of reinforced concrete, the effects of creep and shrinkage are taken into account, as stated in eq. (1). The terms δ_1 and δ_2 refer to the deflection values calculated considering the uncracked and cracked conditions of the beam, respectively, as follows:

$$\delta_{max} = \xi\delta_2 + (1-\xi)\delta_1 \quad (3)$$

where ξ is the distribution coefficient accounting for the degree of cracking, calculated according to the following equation:

$$\xi = 1 - \beta_t.\left(\frac{M_{cr}}{M_a}\right)^2 \quad (4)$$

in which β_t is a coefficient that accounts for the duration of the load, M_{cr} is the cracking moment of the member, and M_a is the maximum bending moment along the span of the member. For the calculation of strains, the effective modulus of elasticity ($E_{c,ef}$), is used, being given by eq. (5),

$$E_{c,ef} = \frac{E_{cs}}{1+\varphi_{(\infty,t_0)}} \quad (5)$$

where E_{cs} is the secant modulus of elasticity and $\varphi_{(\infty,t_0)}$ is the creep coefficient of the concrete. This coefficient can be determined from the following expression:

$$\varphi_{(\infty,t_0)} = \varphi_a + \varphi_{f\infty}[\beta_f(\infty) - \beta_f(t_0)] + \varphi_{d_\infty}\beta_d \quad (6)$$

where φ_a is the coefficient of rapid creep, $\varphi_{f\infty}$ is the final value of the irreversible slow deformation coefficient of concrete, $\beta_f(\infty)$ or $\beta_f(t_0)$ is the coefficient related to irreversible slow deformation, φ_{d_∞} is the final value of the reversible slow deformation coefficient, and β_d is the coefficient related to reversible slow deformation after loading. For more information on the calculation of these coefficients, refer to NBR 6118:2023 [11].

3 Structural reliability analysis

Over the past decades, structural reliability has been widely employed in structural engineering to assess the safety levels of structures. Although there is not a consensus on the acceptable values of failure probability, some international organizations discuss the subject in different structural mechanics applications, such as the Joint Committee on Structural Safety (JCSS) and the European Committee for Standardization (CEN). To assess the probability of failure (P_f), it is necessary to define a performance function, statistically characterize the design variables and apply a reliability method. In the next sections, these procedures are briefly described.

3.1 Limit state equation and the probability of failure

Let $G(\boldsymbol{X})$ be the limit state function (lsf) representing the problem under analysis, where $\boldsymbol{X}$ is the vector of random variables (r.v.) of the chosen failure mode, arranged such that negative values represent failure events, i.e., $G(\boldsymbol{X}) \leq 0$. A basic form of a lsf is:

$$G(\boldsymbol{X}) = R(\boldsymbol{X}) - S(\boldsymbol{X}) \tag{7}$$

in which $R(\boldsymbol{X})$ is the strength term and $S(\boldsymbol{X})$ corresponds to the loading or load effect.

The structural reliability of a member is commonly quantified by its probability of failure P_f:

$$P_f = P(G(\boldsymbol{X}) \leq 0) = \int_{G(\boldsymbol{X})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x})\, dx \tag{8}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function (pdf) of the r.v. involved.

The solution of this integral may not be straightforward, especially given the nonlinear nature of the pdf, particularly when considering multiple random variables. The integral in eq. (8) can be computed by using simulation techniques, such as Monte Carlo (MC), or transformation methods, as is the case of the First Order Reliability Method (FORM).

The outcome of MC is an estimate of P_f, while FORM provides the reliability index β, and they can be related by using the inverse standard normal cumulative density function Φ (CDF):

$$\beta = -\Phi^{-1}\left(P_f\right). \tag{9}$$

3.2 First Order Reliability Method

The First Order Reliability Method (FORM) is widely used to evaluate the reliability of engineering systems, being efficient for problems with low failure probabilities and allowing for a rapid assessment of structural safety under different load conditions and uncertainties. This method involves transforming the r.v. in the limit state function into the standard normal space (with zero means and unit variances) by applying the normal tail approximation, and then calculating the reliability index, β. This index represents the minimum distance from the origin to the limit state surface in the standard normal space. The failure probability is then approximated using eq. (9). FORM employs the Nataf [13] transformation, along with Cholesky decomposition, to eliminate correlations between the random variables. These procedures are detailed in Melchers and Beck [1].

3.3 Monte Carlo simulation

The Monte Carlo simulation is carried out by generating N random scenarios, which are evaluated using the failure function $G(\boldsymbol{X})$. Scenarios where $G(\boldsymbol{X}) \leq 0$ are counted as failures (N_f), and the probability of failure is estimated by the relation $P_f = N_f/N$. This exhaustive generation requires sampling random values for the problem variables, respecting their probability distributions.

The accuracy of MC results depends on the number of scenarios used. For low P_f problems, executing MC can become impractical due to its high computational cost. Ang and Tang [14] provide a formula to compute a minimum number of realizations (N_{min}) to achieve a desired accuracy, by the following expression:

$$N_{min} = \frac{1}{COV^2} \cdot \frac{1-P'_f}{P'_f} \tag{10}$$

in which COV is the coefficient of variation of P_f and P'_f refers to its characteristic value for the problem under study.

4 Methodology

Regarding structural analysis and design, TQS software was used to launch, analyze, and design the structural elements of the benchmark RC building, extracting data on geometry, materials and applied loads. Then, failure functions and reliability analysis methods were implemented in Python. The values of P_f and β of the beams were calculated using FORM, and compared to MC simulation results. The safety levels of the beams were assessed by comparing the obtained reliability indices with the values recommended by international standards.

4.1 Case study RC building

It is addressed a residential architectural project of a 2-storey building with approximately 45 m² per floor. This model is adopted by the company TQS for training in their software. The building comprises three floors: ground, upper and roof. The foundation floor was created to lay the building foundations and simulate a real structural project, although these elements are not considered in the reliability analyses. In the upper floor is used a solid slab, with 9 cm thick and without level difference between them. For the roof, a lattice slab with a thickness of 12 cm was chosen. All slabs are supported by beams with dimensions of 14 x 30 cm (width x height), which are the focus of the study. Additionally, pillars with dimensions of 14 x 30 cm (width x height depending on their direction) were cast. A 30 MPa strength class concrete was used for the beams, slabs, foundations and columns, with a cover of 25 mm for the slabs and 30 mm for the beams and columns.

4.2 Statistical data

For the statistical characterization of deflection model errors, the methodology from Honfi and Mårtensson [4], based on section 7.4.3 of the Eurocode 2 [5], was adopted. For the maximum crack width model uncertainty, the McLeod [6] estimate was used, with a coefficient of variation of 38%. The strength, geometric and load variables related to concrete structures built in Brazil were collected in national studies by Santos, Stucchi and Beck [15] and Santiago et al. [9], aligned with data from Probabilistic Model Code [16], ensuring greater proximity to the building's location. Statistical parameters are presented in the following tables.

Table 1. Statistical description of model uncertainty r.v. for concrete elements

Design Variables	Mean	COV	Distribution	Unit	Source
Deflection strength (θ_R)	1.00	0.1	Log-normal	-	Honfi and Mårtensson [4]
Total deflection (θ_E)	1.00	0.1	Log-normal	-	Honfi and Mårtensson [4]
Crack width (θ_ω)	1.00	0.38	Log-normal	-	McLeod [6]

Table 2. Geometrical r.v. of concrete rectangular cross-sections, in terms of their nominal values $[\ ]_n$

Design Variables	Mean	COV	Distribution	Unit	Source
Height (h)	$1.00. h_n$	$4mm + 0.006. h_n$	Normal	cm	Santiago et al. [9]
Width (b)	$1.00. b_n$	$4mm + 0.006. b_n$	Normal	cm	Santiago et al. [9]
CG dist. from bar to lower fiber (d')	$1.00. d'_n$	0.27	Log-normal	cm	Santos, Stucchi and Beck [15]
Area of reinforcing bars (A_S)	$1.00. A_{Sn}$	0.02	Normal	cm²	Honfi and Mårtensson [4]

Table 3. Random variables associated to material strength and loads

Design Variables	Mean	COV	Distribution	Unit	Source
Dead loads (M_g)	$1.06. M_g$	0.12	Normal	KN.m	Santiago et al. [9]
Live loads (M_q)	$1.00. M_q$	0.40	Gumbel	KN.m	Santiago et al. [9]
Accidental loads (M_w)	$0.90. M_w$	0.34	Gumbel	KN.m	Santiago et al. [9]
Concrete compressive strength – C30 (f_c)	$1.22. f_{ck}$	0.15	Normal	MPa	Santiago et al. [9]
Modulus of elasticity of steel (E_S)	$1.00. E_{cs}$	0.04	Normal	MPa	Honfi and Mårtensson [4]

The mean is a measure of central tendency, also known as the expected value, first moment, or average of a random variable. The coefficient of variation (COV) is a standardized measure of dispersion in a probability distribution, defined as the ratio of the standard deviation to the mean.

4.3 Limit state equations

In this study, failure at the serviceability limit state of excessive deformation is characterized when the total deformation of the member exceeds the displacement limit established in table 13.3 of NBR 6118:2023 [11], with $\delta_{lim} = L/250$ based on the visual sensory acceptability criterion. The function chosen for this failure mode is the one described by Honfi [2], based on section 7.4.3 of the Eurocode 2 [5], as represented in the equation below:

$$G(\boldsymbol{X}_{Def}) = \theta_R \delta_{lim} - \theta_E \delta_{max} \tag{11}$$

where $\boldsymbol{X}_{Def} = [\theta_R, \theta_E, b, h, d', f_c, E_s, A_S, M_g, M_q, M_w]$ is the vector of r.v. associated with the assessment of excessive deformation, δ_{lim} is the displacement limit of the member and δ_{max} is the maximum deflection of the element for the reference period (50 years), calculated using eq. (3).

The model used as the limit state equation for crack width was presented by McLeod [6], based on the formulations in the Model Code 2010 [12]. It considers the effects of concrete shrinkage on the maximum crack width values. Failure at this limit state is characterized when the crack width exceeds the limit established by the standard. For this study, a crack width limit of $\omega_k = 0.3mm$ from NBR 6118:2023 [11] was adopted, corresponding to environmental exposure class III.

$$F(\boldsymbol{X}_{Fis}) = \omega_{lim} - \theta_\omega \omega_{max} \tag{11}$$

where $X_{Fis} = [\theta_\omega, b, h, d', f_c, E_s, A_S, M_g, M_q, M_w]$ is the vector of random variables associated with the assessment of crack width, ω_{lim} is the crack width limit of the member and ω_{max} is the maximum crack width of the element for the reference period (50 years), calculated using eq. (1).

5 Results and discussions

An initial validation of the FORM implementation for the problems discussed herein has been performed, adressing beams V204, V208 and V303, comparing the results with the values obtained from MC simulation. Thus, the reliability index (β) and P_f values were determined for both methods and compared in Tab. 4. It is observed a slight difference between β obtained by FORM and MC, not exceeding 0.17%.

Table 4. Comparison of the reliability results for beams V204, V208 and V303 using FORM and MC

SLS	Beam	FORM		MCS		
		β	P_f	$N_{mín}$ ($COV = 0.025$)	β	P_f
Excessive deflection	V204	2.581	4.93E-03	3.23E05	2.584	4.88E-03
	V303	3.824	6.55E-05	2.44E07	3.827	6.49E-05
Excessive crack width	V204	3.542	1.98E-04	8.08E06	3.548	1.94E-04
	V208	4.605	2.06E-06	7.77E08	4.609	2.02E-06

The nominal values of characteristic compressive strength of concrete ($f_{ck} = 30$MPa), elastic modulus of steel ($E_s = 210$GPa), width ($b = 14$cm) and height ($h = 30$cm) are adopted across all performance functions.

Regarding the various beams simulated, the ranges for the other variables are provided in the table below, separated between the tensile zone (+) and compressive zone (-) of the concrete.

Table 5. Range of values for the other random variables of the beams by floor

Floor	Zone	M_g (KN.cm)	M_q (KN.cm)	M_w (KN.cm)	A_s (cm²)	d' (cm)
Ground	(+)	58.84 – 725.69	0.13 – 0.92	9.80 – 58.84	1	3.9
	(-)	156.9 – 862.98	0.87 – 9.81	9.80 – 58.84	0.63 – 1.6	3.75 – 4
Upper	(+)	19.61 – 1225.84	0.86 – 235.36	58.84 – 225.55	1 – 2.4	3.9 – 4
	(-)	304.01 – 1235.64	19.61 – 215.75	58.84 – 225.55	1 – 2.4	3.9 – 4
Roof	(+)	1.53 – 794.34	0.23 – 245.17	19.61 – 98.07	1 – 1.6	3.9 – 4
	(-)	39.23 – 558.98	0.98 – 156.91	19.61 – 98.07	1 – 1.6	3.9 – 4

Thus, the reliability indices and failure probabilities via FORM by floor are presented in table below.

Table 6. Range of values for the serviceability limit state by floor using FORM

SLS	Floor	Zone	β	P_f
Excessive deformation	Ground	(+)	6.34 – 10.25	5.58E-25 – 1.09E-10
		(-)	3.21 – 10.17	1.34E-24 – 6.44E-04
	Upper	(+)	2.58 – 10.75	2.71E-27 – 4.93E-03
		(-)	2.74 – 8.98	1.25E-19 – 3.01E-03
	Roof	(+)	3.21 – 10.78	1.96E-27 – 6.63E-04
		(-)	5.12 – 9.80	5.53E-23 – 1.50E-07
Excessive crack width	Ground	(+)	5.43 – 9.65	2.29E-22 – 2.69E-08
		(-)	4.37 – 8.78	7.54E-19 – 6.21E-06
	Upper	(+)	3.54 – 8.32	4.31E-17 – 1.98E-04
		(-)	3.83 – 7.15	4.14E-13 – 6.31E-05
	Roof	(+)	5.33 - 9.45	1.55E-21 – 4.90E-08
		(-)	5.61 – 9.23	1.28E-20 – 9.56E-09

Most of the beams presented low P_f values, especially in the ground and roof floors, due to lower bending moments. The critical values – $\beta = 2.58$ and $P_f = 4.93\text{E}-03$ – have been observed in excessive deformation SLS for beam V204. The highest probabilities of failure occurred when the bending moments in the critical section were close to or exceeded the cracking moment values.

On the ground floor, the failure probability values for negative moments were higher than for positive moments, due to the low incidence of wind loads and the continuity of the beams, resulting in higher bending moments at the supports. In the roof floor, positive moments are associated to higher P_f values, influenced by the building configuration and the loads along the floor. The upper floor showed the highest probabilities of failure for excessive deformation and crack width, due to the greater quantity and intensity of vertical loads, especially accidental loads.

Concerning the occurrence of some high values of P_f it is important to remark that serviceability limit states are expected to occur more frequently than ultimate limit states, but do not lead to imminent risk of structural failure, affecting the structure service life and comfort for the user.

Acceptable failure probabilities can be found in documents from the Joint Committee on Structural Safety (JCSS), such as the Probabilistic Model Code [16], or in normative codes like Model Code 2010 [12]. The latter recommends a target reliability index of 3.0 for new structures, considering a one-year reference period and irreversible serviceability limit states. Comparing these values with those obtained in the study, it is noted that most of the beams met the minimum requirements, ensuring safety against excessive deflection and crack width. The design methodology of NBR 6118:2023 [11] showed satisfactory results regarding structural safety. However, the analyzed building presented a large number of beams with failure probability values below the recommended levels, indicating overdesign due to its small number of floors, low susceptibility to wind loads and reduced acting loads. The use of the limit states method (semi-probabilistic) by NBR 6118:2023 [11], with partial safety factors, contributes to this discrepancy.

6 Conclusions

Comparing structural reliability methods, the results of FORM and MC were similar, allowing for the verification of beam failure probabilities under the SLS addressed herein. Variations in failure probabilities were observed due to the relationship between the bending moment at the critical section and the cracking moment. More stressed beams showed higher failure probabilities, but the results aligned with target reliability indices from international standards, such as the Model Code 2010 [12] and the Probabilistic Model Code [16], validating NBR 6118:2023 [11] compliance. Additionally, it is recommended to study the reliability of other structural elements and SLS, such as decompression and excessive compression limits for prestressed concrete beams and slabs.

Finally, probabilistic evaluation provides a theoretical framework for understanding structural behavior during the design phase, serving as a powerful tool for engineers in decision-making. The results of this study are expected to contribute to further research on SLS and promote the adoption of probabilistic approaches in reinforced concrete structure design.

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