A compliance-based topology optimization approach using conservative convex separable approximations and PSB Hessian estimation

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Abstract. In compliance-based topology optimization, methodologies for updating design variables, such as the widely-used Optimality Criteria (OC), play a pivotal role in efficiently addressing problems with monotonically decreasing objectives subject to certain design constraints. However, despite its effectiveness, the OC method is based on first-order optimality conditions and does not incorporate second-order terms. To harness the benefits of second-order terms, in this work, we use conservative convex separable approximations (CCSA) of the objective function, alongside the Powell symmetric Broyden (PSB) update for estimating the diagonal terms of the Hessian matrix. Three variations are considered: quadratic, logarithmic, and square-root approximations. This approach aims to enhance the efficiency and effectiveness of compliance minimization in topology optimization problems subject to a volume constraint. To demonstrate the efficacy of the aforementioned methodology, a case study is presented and evaluated using the CCSA update scheme. The obtained results and performance metrics are compared to those of the OC method, providing evidence of the advantages offered by the proposed approach.

Keywords: Compliance-based topology optimization, Conservative convex separable approximation, PSB Hessian estimation.

1 Introduction

The traditional Optimality Criteria (OC) update scheme [1, 2] is well-suited for problems with monotonically decreasing objective functions and simple constraints. It is commonly applied to compliance minimization problems subjected to a maximum allowable volume of material. This method is renowned for its excellent convergence properties, computational efficiency, and ability to handle large-scale optimization problems [3].

The OC method incorporates the objective and constraint sensitivities in its update schema, but does not consider second-order derivatives. In this way, this study proposes the use of conservative convex separable approximations (CCSA) [4] of the objective function, considering different approximation functions such as quadratic, logarithmic, and square-root forms. Unlike the OC method, which employs a separable linear approximation and results in a multiplicative update term, the CCSA update scheme operates additively. This distinction can lead to divergent behaviors in density recovery during the iterative process of topology optimization. Additionally, this work incorporates the Powell symmetric Broyden (PSB) quasi-Newton update [5, 6] to estimate the diagonal terms of the second-order Hessian matrix [7], enhancing the efficiency of the proposed methodology. The PSB update is particularly advantageous because it maintains the symmetry and positive definiteness of the Hessian approximation, which is crucial for ensuring stable and efficient convergence.

The focus of this paper is on applying this novel CCSA-PSB update scheme to solve compliance-based problems. We integrate the CCSA-PSB update scheme into the PolyTop [8] framework to investigate its performance on compliance-based optimization problems. PolyTop is a versatile and widely-used framework for topology optimization. Our approach involves a analysis of computational efficiency, and solution quality of the proposed method. By comparing the results obtained using the CCSA-PSB update scheme with those from traditional OC method, we aim to demonstrate the potential advantages of our approach.

2 Topology optimization problem

The compliance-based topology optimization problem is described using a nested formulation:

$$\begin{aligned}
\min_{\boldsymbol{z}} \quad & f = \boldsymbol{F}^T \boldsymbol{u} \\
\text{s.t.} \quad & g = \frac{\displaystyle\sum_{e=1}^{N} \phi_V\left(\hat{z}_e\right) V_e}{\displaystyle\sum_{e=1}^{N} V_e} - \overline{v} \leq 0 \\
& z_{\min} \leq \boldsymbol{z} \leq z_{\max} \\
\text{with} \quad & \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}
\end{aligned} \tag{1}$$

where the design variables $\boldsymbol{z}$ consist of generalized densities assigned to each one of the N elements.

The objective of the optimization problem is to minimize the system's compliance f. A maximum volume constraint g is considered, where V_e is the original volume of element e, $\overline{v}$ is the maximum volume fraction required, and ϕ_V is the material interpolation function for volume, which is applied to the elemental value $\hat{z}_e$ of element e. Bound constraints are also used, where $z_{\min}$ and $z_{\max}$ are the lower and the upper bounds for the design variables, respectively.

The displacement vector $\boldsymbol{u}$ is determined to ensure the equilibrium of the system given the external force vector $\boldsymbol{F}$ and the global stiffness matrix $\boldsymbol{K}$, which is defined accumulating each element stiffness contribution and considering the material interpolation function for stiffness ϕ_E

$$\boldsymbol{K} = \sum_{e=1}^{N} \phi_E\left(\hat{z}_e\right) \boldsymbol{K}_e^0 \tag{2}$$

where $\boldsymbol{K}_e^0$ is the local stiffness matrix of element e.

A linear density filter is applied to ensure smoothness of the design field. In this way, a sparse filter matrix $\boldsymbol{\chi}$ maps design to elemental values based on a filtering radius r. The terms of the filter matrix are defined as:

$$\chi_{ij} = \frac{\max\left\{0, V_j\left(1 - \dfrac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|}{r}\right)\right\}}{\sum_{j \in S_i} V_j\left(1 - \dfrac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|}{r}\right)} \tag{3}$$

where $\boldsymbol{x}_i$ is the centroid position of element i and S_i is the set of indices of j elements whose centroid falls within radius r of the centroid of element i. Therefore, a set of elemental values $\hat{\boldsymbol{z}}$ is defined as filtered densities, $\hat{\boldsymbol{z}} = \boldsymbol{\chi}\boldsymbol{z}$.

As the material interpolation, SIMP [9] is used. This method applies transformations into elemental values to better describe stiffness and volume factors, using the functions ϕ_E and ϕ_V, respectively. SIMP is a widely adopted method in which material densities are continuously varied between design variables lower bound value (void) and upper bound value (solid),

$$\phi_E\left(\hat{z}_e\right) = \varepsilon + (1 - \varepsilon)\, \hat{z}_e^p \tag{4}$$

$$\phi_V\left(\hat{z}_e\right) = \hat{z}_e \tag{5}$$

where p is a penalization parameter and ε represents the Ersatz parameter.

3 Update scheme

Some conservative convex separable approximations (CCSA) were proposed by Svanberg [4]. In this work, quadratic, logarithmic and square root approximations are considered, defined as:

$$\tilde{f}_{\text{QUAD}}\left(\boldsymbol{z}\right) = f\left(\boldsymbol{z}^{(k)}\right) + \boldsymbol{\nabla} f\left(\boldsymbol{z}^{(k)}\right)^{T}\left(\boldsymbol{z} - \boldsymbol{z}^{(k)}\right) + \frac{1}{2}\sum_{e=1}^{N}\rho_e\left(\frac{z_e - z_e^{(k)}}{\sigma}\right)^{2} \tag{6}$$

$$\tilde{f}_{\text{LOG}}\left(\boldsymbol{z}\right) = f\left(\boldsymbol{z}^{(k)}\right) + \boldsymbol{\nabla} f\left(\boldsymbol{z}^{(k)}\right)^{T}\left(\boldsymbol{z} - \boldsymbol{z}^{(k)}\right) - \frac{1}{2}\sum_{e=1}^{N}\rho_e \ln\left[1 - \left(\frac{z_e - z_e^{(k)}}{\sigma}\right)^{2}\right] \tag{7}$$

$$\tilde{f}_{\text{SQRT}}\left(\boldsymbol{z}\right) = f\left(\boldsymbol{z}^{(k)}\right) + \boldsymbol{\nabla} f\left(\boldsymbol{z}^{(k)}\right)^{T}\left(\boldsymbol{z} - \boldsymbol{z}^{(k)}\right) + \sum_{e=1}^{N}\rho_e\left[1 - \sqrt{1 - \left(\frac{z_e - z_e^{(k)}}{\sigma}\right)^{2}}\right] \tag{8}$$

where ρ_e and σ are positive parameters of the approximation.

Note that for logarithmic and square root approximations, the update of design variables is subjected to specific limitations, $\left\| z_e - z_e^{(k)} \right\| < \sigma$ for logarithmic approximation and $\left\| z_e - z_e^{(k)} \right\| \leq \sigma$ for square root approximation, whereas no limits are imposed for the quadratic approximation.

It is also noted that the when evaluating the Hessian of the quadratic, logarithmic and square root CCSA functions (see equations (6), (7) and (8)) at $z^{(k)}$, they all yield the same result:

$$\tilde{h}_e\left(z^{(k)}\right) = \left.\frac{\partial^2 \tilde{f}}{\partial z_e^2}\right|_{\boldsymbol{z}=z^{(k)}} = \frac{\rho_e}{\sigma^2} \tag{9}$$

and once the Hessian is estimated, the parameter ρ_e can be determined in terms of the limiting factor σ.

In this way, a PSB Hessian estimation approach is considered based on the methodology proposed by Giraldo-Londoño and Paulino [7] using a diagonal approximation based on the Powell symmetric Broyden (PSB) quasi-Newton update. The Hessian can be estimated in a iterative manner,

$$\tilde{\boldsymbol{h}}\left(\boldsymbol{z}^{(k)}\right) \approx \tilde{\boldsymbol{h}}\left(\boldsymbol{z}^{(k\,-\,1)}\right) + \frac{\boldsymbol{c}^{(k)T}\boldsymbol{s}^{(k)} - \tilde{\boldsymbol{h}}\left(\boldsymbol{z}^{(k\,-\,1)}\right)^{T}\boldsymbol{s}^{(k)2}}{\left(\boldsymbol{s}^{(k)2}\right)^{T}\boldsymbol{s}^{(k)2}}\boldsymbol{s}^{(k)2} \tag{10}$$

where $\boldsymbol{c}^{(k)}$ and $\boldsymbol{s}^{(k)}$ are the difference between sensitivities and the difference between design variables from iteration k to iteration $k+1$,

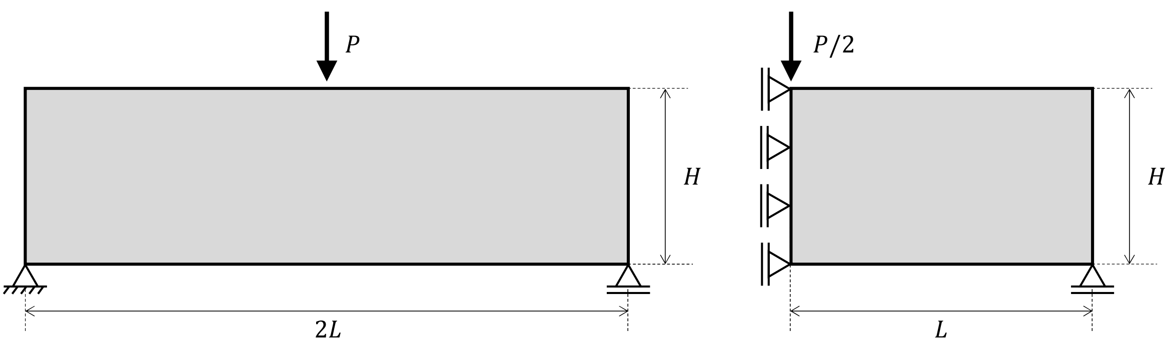
$$\boldsymbol{c}^{(k)} = \left.\frac{\partial \tilde{f}}{\boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{z}^{(k)}} - \left.\frac{\partial \tilde{f}}{\boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{z}^{(k\,-\,1)}} \tag{11}$$

$$\boldsymbol{s}^{(k)} = \boldsymbol{z}^{(k)} - \boldsymbol{z}^{(k\,-\,1)} \tag{12}$$

Initial values of sensitivity and Hessian approximation are respectively considered as zero and a small value (e.g., 10^{-3}) for element contributions.

4 MBB beam problem

In order to illustrate the proposed methodology, a MBB beam problem [10] is presented. The domain is defined by a rectangular region of length $2L$ and height H (refer to Figure 1a). A vertical load of intensity P is applied at the central top position. Due to symmetry of the problem, only half of the domain is analyzed (refer to Figure 1b).

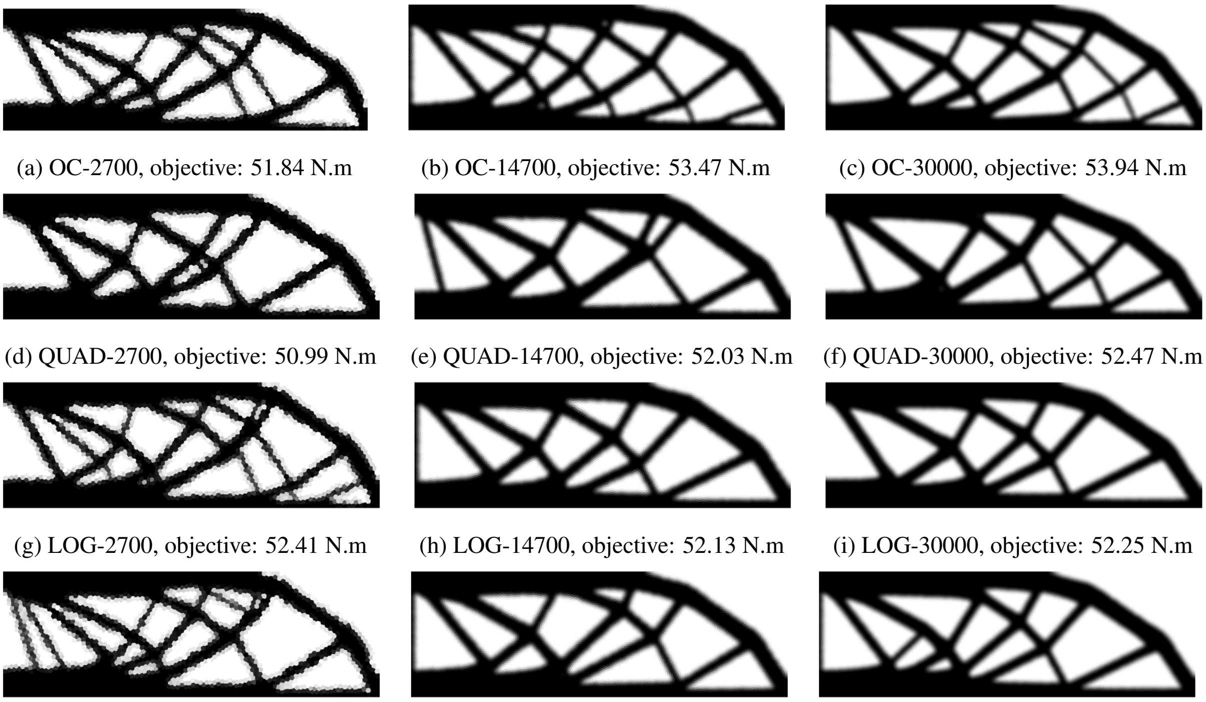


(a) (b)

Figure 1. MBB beam problem, loading and support conditions: (a) full domain; (b) half of the domain considered due to problem symmetry.

The properties of the MBB beam optimization problem are defined as: domain length L = 3 m, domain height H = 1 m, maximum volume fraction $\overline{v}$ = 0.5, load intensity P = 1 N, Young's modulus of the material E_0 = 1.0, Poisson's raio of the material ν = 0.3, filtering radius r = 0.04, Ersatz parameter ε = 10^{-4}, CCSA move parameter σ = 0.2. The initial guess consists of all design variables set to the maximum volume fraction value. A continuation-based penalization scheme is employed and the penalty parameters used for SIMP method are $\{1, 1.5, 2, 2.5, 3, 3.5, 4\}$. The analysis also assumes plane stress conditions.

The optimization problem is evaluated for three different levels of mesh refinement, considering 2,700, 14,700 and 30,000 polygonal finite elements. Figure 2 presents the final topologies obtained for each evaluated model.



(a) OC-2700, objective: 51.84 N.m (b) OC-14700, objective: 53.47 N.m (c) OC-30000, objective: 53.94 N.m

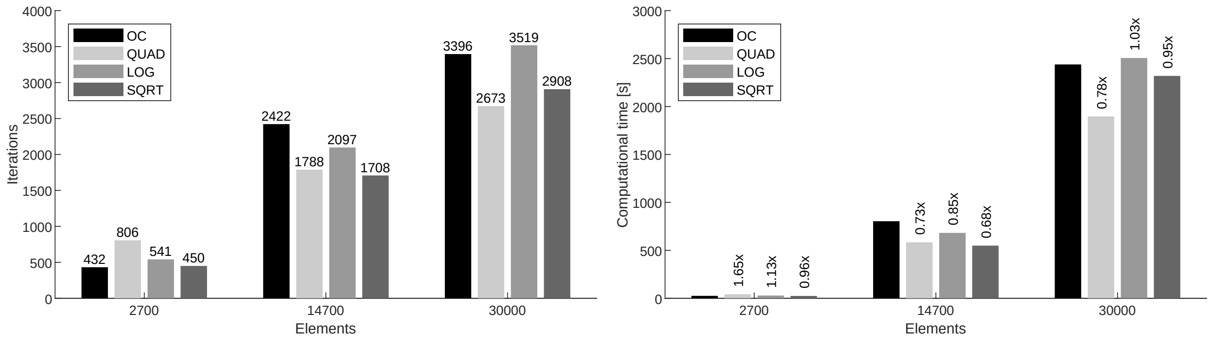
(d) QUAD-2700, objective: 50.99 N.m (e) QUAD-14700, objective: 52.03 N.m (f) QUAD-30000, objective: 52.47 N.m

(g) LOG-2700, objective: 52.41 N.m (h) LOG-14700, objective: 52.13 N.m (i) LOG-30000, objective: 52.25 N.m

(j) SQRT-2700, objective: 51.11 N.m (k) SQRT-14700, objective: 52.67 N.m (l) SQRT-30000, objective: 52.69 N.m

Figure 2. Final tolopogies for OC, CCSA-PSB quadratic (QUAD), CCSA-PSB logarithmic (LOG) and CCSA-PSB square root (SQRT) approximations, considering 2700, 14700 and 30000 elements.

Figure 3a displays the number of iterations required for the convergence of each evaluated model. Figure 3b illustrates the computational time expended by each evaluated model, along with a relative comparison between the CCSA-PSB approximations and the OC update scheme.



(a) (b)

Figure 3. Comparing metrics obtained from OC with CCSA-PSB update scheme: (a) iterations required for convergence; (b) measured computational time.

The final topologies gain resolution as the number of elements increases. For models with 2,700 and 14,700 elements, the smallest compliance values are obtained using the CCSA-PSB quadratic approximation, with values of 50.99 N.m and 52.03 N.m, respectively. For models with 30,000 elements, the smallest objective function value is achieved using the CCSA-PSB logarithmic approximation, at 52.25 N.m. Across all element counts, the highest objective function values are generally produced by the OC update scheme, except for the 2700-element model, where the CCSA-PSB logarithmic approximation yields a final value of 52.41 N.m.

The iterations required for convergence of the evaluated models were assessed using the continuation penalization scheme. For models with 2,700 elements, the OC update scheme required fewer iterations and less computational time than the CCSA-PSB approximations. For models with 14,700 elements, the OC update scheme demanded 2,422 iterations, whereas the CCSA-PSB approximations required 1,788 iterations for the quadratic approximation, 2,097 iterations for the logarithmic approximation, and 1,708 iterations for the square-root approximation. In terms of computational time, all CCSA-PSB approximations also outperformed the OC update scheme. Finally, for models with 30,000 elements, only the logarithmic approximation of the CCSA-PSB update scheme, with 3,519 iterations and a computational time of 2,506.02 seconds, performed worse than the OC update scheme, which required 3,396 iterations and 2,438.98 seconds of computational time.

5 Conclusions

In this study, we have introduced and evaluated a methodology for compliance-based topology optimization, leveraging conservative convex separable approximations (CCSA) and the Powell symmetric Broyden (PSB) quasi-Newton update scheme. By considering quadratic, logarithmic, and square-root approximations, we aimed to enhance the efficiency and effectiveness of compliance-based minimization problems subjected to a maximum volume constraint.

Our findings indicate that the proposed CCSA-PSB update scheme generally outperforms the traditional Optimality Criteria (OC) method, particularly in terms of convergence behavior and computational efficiency. Specifically, for models with 2,700 and 14,700 elements, the CCSA-PSB quadratic approximation achieved the smallest compliance values, while for models with 30,000 elements, the logarithmic approximation yielded the smallest objective function value. Across all element counts, the CCSA-PSB approximations typically required fewer iterations and less computational time compared to the OC method, except for the largest model where the logarithmic approximation performed slightly worse in terms of computational time.

Our results suggest that the CCSA-PSB methodology offers a robust and computationally efficient alternative to traditional OC-based approaches for the presented example. Future work may focus on further refining the CCSA-PSB approximations and exploring their applicability to a wider range of optimization problems, including non-monotonic objective functions and more complex constraints. The insights gained from this study pave the way for more efficient and effective optimization techniques in structural design and other engineering applications.

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