Topology Optimization for Local Fatigue Constraints: A Solution Using the Augmented Lagrangian Method

Carlos Eduardo Lino[1], Andre Luis Ferreira da Silva[1], Emílio Carlos Nelli Silva[1]

[1]*Department of Mechatronics and Mechanical System Engineering,*
Politechnique School of University of São Paulo - POLI USP,
Avenida Professor Mello de Morais, 2231, 05508-030 São Paulo, SP, Brazil
carlos.lino@usp.br, andre_fersi@usp.br, ecnsilva@usp.br

Abstract. Fatigue is a common failure mode for mechanical structures under load. While new materials can be developed to address fatigue, this approach can be expensive. Alternatively, optimizing the design using algorithms to determine the optimized material distribution can enhance fatigue life cost-effectively. This work proposes using topology optimization to design structures, aiming to extend their lifespan. The objective is to minimize volume while considering fatigue constraints. Existing literature often applies aggregate methods, typically used for stress constraints, to manage fatigue. However, conceptualizing fatigue as a localized phenomenon by employing the Modified Goodman method, in conjunction with a sensitivity factor for precise life estimation, offers a more sophisticated approach. Numerical examples substantiate the effectiveness of this methodology.

Keywords: Topology Optimization, Augmented Lagrangian, Fatigue Constraints, Modified Goodman.

1 Introduction

Structural optimization is the process of enhancing structural performance by minimizing material usage while satisfying established performance and requirements, Olhoff and Taylor [1]. It falls into three main categories, Parametric Optimization: Adjusts specific dimensions (e.g., thickness) of the structure. Shape Optimization: Seeks the optimal geometric shape of the structure. Topology Optimization (TO): Determines the optimized material distribution within a design space, including aspects like material quantity and spatial arrangement, Bendsøe and Sigmund [2]. This article is organized asfollows recent research on TO under fatigue constraints includes various methodologie. Sherif et al. [3] analyzed loading conditions using equivalent static loads. Holmberg et al. [4] proposed a probabilistic approach for fatigue-constrained TO, focusing on critical fatigue states. Jeong et al. [5] combined dynamic fatigue constraints with static failure considerations. Lee et al. [6] explored TO for random load fatigue failures in the frequency domain. Collet et al. [7] use SIMP to minimize weight while considering flexibility and high-cycle fatigue constraints. They apply the modified Goodman criterion and determine mean and alternating stresses using the Sines method, resulting in a topology with infinite life. Oest and Lund [8] present an optimization method that integrates fatigue analysis directly into the SIMP-based process for finite-life constraints. They impose a P-norm damage constraint and use the Sines criterion for fatigue damage, calculating accumulated damage with Palmgren-Miner's rule and an S-N curve. Zhang et al. [9], for non-proportional loads. Suresh et al. [10] created a continuous-time approach for diverse load histories. Chen et al. [11] focused on cumulative damage and its impact on design. Sherif et al. [3] apply equivalent static loads in topological optimization of dynamically loaded structures with fatigue constraints. They use the SIMP method for optimization via Tosca® and conduct fatigue analysis with FEMFAT®. Their approach proved effective for both theoretical and industrial scenarios. Holmberg et al. [17] explore topological optimization for mass reduction with static stress and high-cycle fatigue constraints. They separate fatigue analysis and optimization, using Palmgren-Miner's rule to convert fatigue constraints into stress constraints, and base sensitivity analysis on maximum principal stress. Jeong et al. [18] introduce a topological optimization method for static and fatigue constraints under constant and proportional loads, using SIMP to minimize structural volume. They assess static failures by yield stress and fatigue failures using the stress-life method, applying Goodman, Gerber, and Soderberg criteria. Nabaki et al. [19] adapt Holmberg et al. [17] method using BESO. They use Goodman's failure criterion instead of accumulated damage for calculating critical fatigue stress in volume minimization. Nabaki et al. [11] present a BESO method integrating the modified Goodman failure criterion directly into sensitivity analysis. Their approach minimizes flexibility with volume and high-cycle fatigue constraints, considering both flexibility and fatigue effects through the gradient of the Goodman criterion. This study proposes an advanced optimization strategy that use algorithm to enhance fatigue life while minimizing structural volume.

This approach emphasizes the design of structures to effectively withstand material fatigue under cyclic loading conditions. Integrating fatigue constraints is essential to ensure that the structure can endure repeated stress cycles without experiencing critical failures, thereby mitigating the risk of premature failure and enhancing overall durability.

The objective is to minimize volume while addressing fatigue constraints. The Augmented Lagrangian method manages constraints by combining them with penalty terms, iteratively updates parameters to achieve a feasible and optimized design. This ensures effective volume reduction while meetis fatigue requirements.

2 Theoretical formulation

This work considers the assumptions of isotropic material properties, small displacements and strains, linear elastic behavior, and plane stress conditions. The field equations of solid mechanics, formulate in their weak form, characterize the forward problem associated with the Topology Optimization problem (Zienkiewicz and Taylor [13], Bendsøe and Sigmund [2]):

$$a(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}) \tag{1}$$

where the *Energy bilinear form* and *Load linear form* are defined as:

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})d\Omega \tag{2}$$

$$L(\boldsymbol{v}) = \int_\Omega b_i\, v_i d\Omega + \int_{\Gamma t} t_i\, v_i ds \tag{3}$$

The variational formulation offers a robust framework for develops numerical methods, such as the Finite Element Method (FEM), which are well-suited for solving complex problems involves arbitrary geometries and material properties. The i^{th} components of the body force and surface force are represents by b_i and t_i, The i^{th} component of the virtual displacement vector is represents by v_i:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}(\boldsymbol{u}) \tag{4}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

the notation C_{ijkl} represents the components of the constitutive tensor $\boldsymbol{C}$. In this context, σ_{ij} denotes the components of the stress tensor, while ϵ_{ij} refers to the components of the linear strain tensor:

$$\boldsymbol{C} = \frac{E}{1+\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix} \tag{6}$$

In the simulation of isotropic materials, it can be beneficial to utilize the constitutive equation (4) in its matrix form, as expresses using Voigt notation, Lai et al. [21]. The constitutive equation can be expressed in the following form:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \boldsymbol{C}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} \tag{7}$$

The material model uses is the well-known SIMP (Solid Isotropic Material with Penalization) method, as describes by Bendsøe [14]. In this model, a pseudo-density variable multiplies the constitutive tensor, given by:

$$\sigma_{ij} = (\rho_{min} + (1 - \rho_{min})\hat{\rho}^{P})C_{ijkl}\epsilon_{kl}(\boldsymbol{u}) \tag{8}$$

in the SIMP (Solid Isotropic Material with Penalization) model, the minimum pseudo-density value is ρ_{min} , with ρ ranging from 0 and 1. The physical pseudo-density field is ρ, and P. Penalization P is treated as a design variable and optimized with pseudo-densities. To address issues such as mesh dependence and checkerboard patterns, regularization techniques are applied. This study uses the spatial filter proposed by Andreassen et al. [15] to ensure adherence to their mathematical framework.

The von Mises stress com be computerd as:

$$\sigma^{vm} = \sqrt{\sigma_i V_{ij} \sigma_j} \tag{9}$$

Where V_{ij} is a auxiliary matrix for computing the von Mises stress, that is:

$$\boldsymbol{V} = \begin{bmatrix} 1 & -0.5 & 0\\ -0.5 & 1 & 0\\ 0 & 0 & 3\end{bmatrix} \tag{10}$$

In topological optimization using density-based material interpolation models like SIMP, it utilizes the spatial filter from to tackle, mathematical framework, and follows the numerical formulation of Andreassen et al. [15] to minimize mesh and checkerboard patterns:

$$\tilde{\rho} = \frac{1}{\sum_{i \in N_e} H_{ei}} \sum_{i \in N_e} H_{ei}\rho \tag{11}$$

here, $\tilde{\rho}$ denotes the i^{th} component of the vector of filtered design variables, the index e represents the element e of the finite element mesh, and H_{ei} is a weight factor defined as follows:

$$H_{ei} = max(0,\, r_{min} - \Delta(e,i)) \tag{12}$$

The parameter r_{min} is the radius of the spatial filter, and $\Delta(e,i)$ is the distance between the centers of elements e and i. The spatial filter affects elements within a neighborhood where $\Delta(e,i)$is less than r_{min}.

To assist the pseudo-densities in attaining values of 0 and 1, which correspond to void and material within the domain, we employ the threshold projection method introduce by Xu et al. [20], formulate in the tanh function:

$$\hat{\rho} = \frac{\tanh(\beta\,\eta) + \tanh(\beta(\tilde{\rho} - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta(1 - \eta))} \tag{13}$$

where β denotes the filtered projection parameters and η represents the inflection parameter. In this context, β is considered a design variable and is optimized concurrently with ρ.

Optimization problem is to minimize total volume while finding an optimized structural configuration that satisfies constraints. The mathematical formulation for this problem is:

$$\begin{aligned} &\min_{\rho} && J = \frac{\int_{\Omega}\hat{\rho}\, d\Omega}{V} \\ &\textit{Such that} && F = a(\boldsymbol{u}, \boldsymbol{v}, \hat{\rho}) - L(\boldsymbol{v}) = 0 \\ & && G = D_e \leq 1 \\ & && \rho_{min} \leq \rho \leq 1 \end{aligned} \tag{14}$$

in this context, J represents the objective function, which corresponds to the structural volume. F denotes the forward problem, which includes the equations of solid mechanics. G is the local fatigue damage, and all others are box constraints. The topology optimization problem aims to minimize the structural volume while adheres to fatigue constraints:

$$\begin{aligned} \min_{\rho} \quad & L_a = J + \frac{r}{2}\left\langle \frac{\lambda}{r} + G \right\rangle^2 \\ \text{Such that} \quad & F = a(\boldsymbol{u}, \boldsymbol{v}, \hat{\rho}) - L(\boldsymbol{v}) = 0 \\ & \rho_{min} \leq \rho \leq 1 \end{aligned} \tag{15}$$

The parameters r and λ represent the penalization factor and the Lagrange multiplier, respectively, the r values are update after each solution of the external iteration, in accordance with the established methodology. To solve the optimization problem, the Augmented Lagrangian method is uses, as describe by Silva, G. A. da et al [16], where $\langle[\cdot]\rangle$ denotes the Macaulay brackets:

$$\langle[\cdot]\rangle = \begin{cases} 0, & [\cdot] < 0 \\ [\cdot], & [\cdot] \geq 0 \end{cases} \tag{16}$$

the update parameter γ adjusts the penalization parameter r with r_{max} representing the maximum allowable value for r.

$$r^{k+1} = min\left(\gamma r^k,\ \frac{r_{max}}{N_e}\right) \tag{17}$$

the update of the Lagrange multipliers λ is carries out as follows:

$$\lambda^{k+1} = \langle rG + \lambda^k \rangle \tag{18}$$

Cumulative fatigue damage is evaluates using Miner's rule, which sums the damage fractions from individual load cycles. Optimization is performs by plots points on the S-N curve using a double logarithmic scale, with the horizontal axis represents the number of cycles N and the vertical axis representing the stress amplitude. The Basquin equation, as Nabaki [11] describes, models the curve. The allowable number of cycles to failure, $N_i = 10^7$, calculates for a given stress amplitude:

$$\sigma_{nf} = \sigma_f (2N_i)^{b_f} \tag{19}$$

where the fatigue strength coefficient σ_f and fatigue strength exponent b_f are material-specific parameters. The fatigue limit σ_{nf}, corresponds to the allowable stress amplitude for a specified number of life cycles. The modified Goodman criterion is employs to compute the equivalent alternating stress σ_a by considers the minimum stress σ_{min} and maximum stress σ_{max} the mean stress σ_m. Fatigue calculations based on von Mises stresses, as outlines by Nabaki [11].

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} \tag{20}$$

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \tag{21}$$

$$D_e = \frac{\sigma_a}{\sigma_{nf}} + \frac{\sigma_m}{\sigma_{ut}} \tag{22}$$

here D_e represents the maximum allowable damage to the structure according to the modified Goodman criterion, and σ_{ut} denotes the ultimate tensile stress. This study utilizes a gradient-based algorithm to tackle the optimization problem and computates of sensitivities within the Augmented Lagrangian framework.The open-source FEniCS platform, which uses the FEM to solve the problem, employs computations. Sensitivities derive using an automatic differentiation algorithm available in Dolfin-adjoint.

The problem is addresses using the Augmented Lagrangian method implemented within the FEniCS Project software framework. Sensitivity analysis is performs with the aid of automatic differentiation techniques provided by Dolfin-Adjoint, while the optimization process is executes using the L-BFGS-B algorithm available in the SciPy library. The optimization workflow is depicts in Fig. 1.

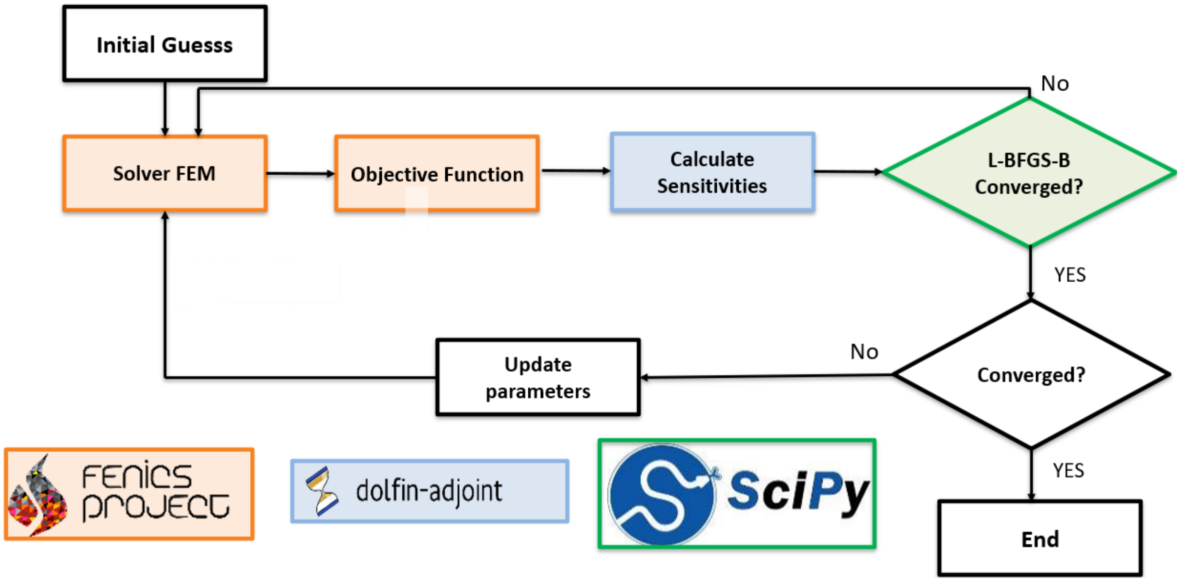

Figure 1. Topology optimization flowchart

3 Results

The L-bracket represented in Fig. 2, this configuration represents a prevalent example in topology optimization problems with fatigue constraints and stress constraints, largely due to the stress concentration observed at the corner of the L-beam. Topology optimization employs to minimize the volume while satisfying a fatigue constraint. The selected failure criterion must remain below 1. Material properties Young's modulos E = 210 GPa and Poisson's ratio ν = 0.28, number of cycles to failure, $N_i = 10^7$, $\sigma_f = 593$ MPa, $b_f =$ -0.086, σ_{ut} = 358 MPa. Within the domain, two loads varying between -100 N and 100 N apply to four elements at the tip of the L-bracket, as indicated by the arrows in the Fig. 2. The domain was discretizes using a mesh consisting of 24 000 square elements with unit side lengths.

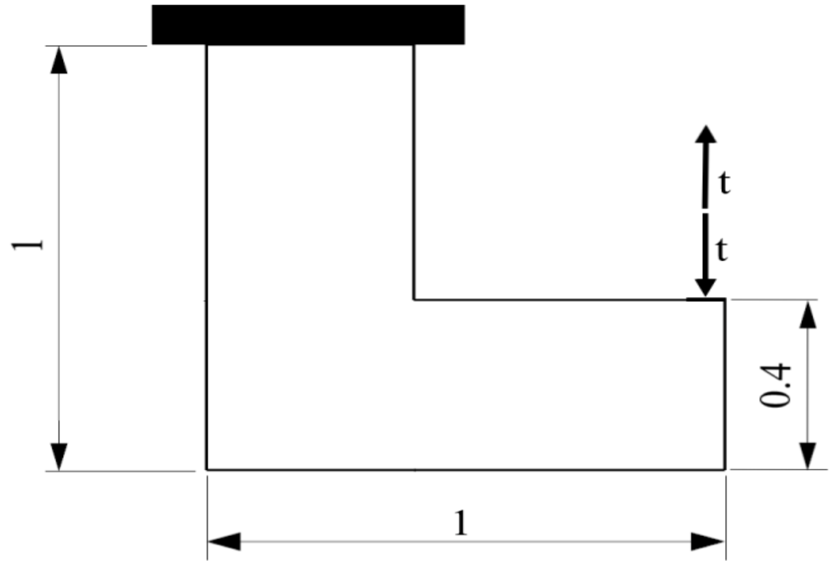

Figure 2. L-bracket domain

Fatigue analysis undertakes to determine the critical stress levels that could induce fatigue failure. This analysis, applied to the structure undergoing TO, depicted in Fig. 3, which shows the reduction in volume objective function corresponding to the and the associated stress relief. Furthermore, Fig. 4 displays the distribution of the fatigue constraints throughout the optimized domain.

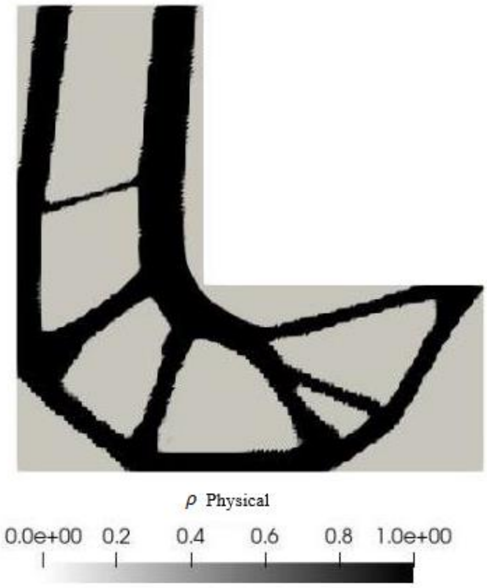

Figure 3. ρ Physical Field

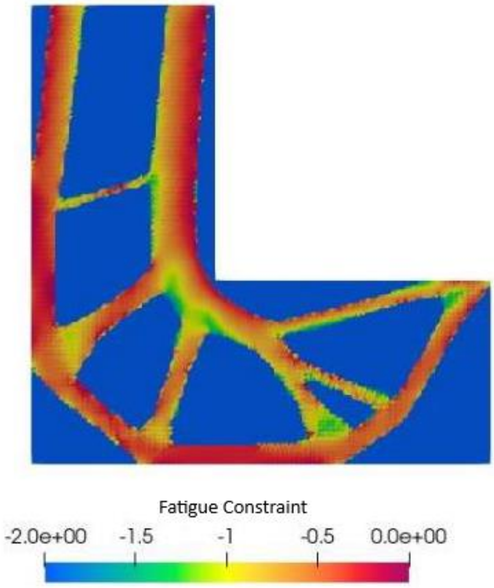

Figure 4. Fatigue Constraint Field

4 Conclusions

The optimization problem used a local fatigue constraint within the Augmented Lagrange framework to ensure adherence to the constraint. Additionally, a topology optimization approach was employed to optimize the volume while accounting for fatigue constraints. The employed procedure demonstrated its efficacy in producing fatigue-resistant geometries. However, it is essential to emphasize that additional criteria should be evaluated, and further adjustments are necessary to achieve improved outcomes. The incorporation of the penalty as a design variable field in the Modified Goodman method was proposed, demonstrating its efficacy in achieving the anticipated optimization within the projected number of material cycles.

Acknowledgements. We gratefully acknowledge the support of the RCGI – Research Centre for Greenhouse Gas Innovation (23.1.8493.1.9), hosted by the University of São Paulo (USP) and sponsored by FAPESP – São Paulo Research Foundation (2020/15230-5) and Shell Brasil, as well as the strategic importance of the support given by ANP (Brazil's National Oil, Natural Gas and Biofuels Agency) through the R&DI levy regulation. The third author thanks the financial support of CNPq (National Council for Research and Development) under grant 304508/2023-3

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