Comparison between different sets of interpolation functions for Timoshenko frame elements

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Abstract. The objective of this paper is to discuss, in the context of a second-order geometric non-linear analysis, the differences in results considering different sets of interpolation functions for frame elements. The usual way to obtain interpolation functions for Timoshenko frame elements is using cubic Hermitian polynomials since they are the solution for the fourth-order differential equation which represents the bending behavior of the infinitesimal element. When dealing with more complex problems (geometrical and/or physical nonlinearities), the usual strategy is to subdivide the elements, which circumvents the limitation of the interpolation functions. Since discretization can sometimes be unwanted, especially for undergraduate students who still do not grasp this concept, a solution which overcomes this is interesting from a didactic point of view. This work proposes the use of different sets of shape functions to interpolate displacements, rotations and bending moments along the element's length, to account for the nonlinearities that arise from the change in geometry during loading. Shape functions obtained directly from the solution of the differential equation of an axially loaded deformed infinitesimal element and traditional Hermitian polynomials are used. Comparisons were made against analytical and numerical solutions using the two-cycle approach. Initial results indicate the ability of the formulation to capture the nonlinear behavior without the need to over discretize the domain.

Keywords: Timoshenko beam theory, Frame finite elements, Hermitian polynomials, Geometric nonlinearity, Differential equations.

1 Introduction

One of the great advances in engineering in the last century was the use of the Finite Element Method (FEM) to solve complex problems, representing displacements, stresses, strains more accurately. Computational advancements have enabled finite element analysis to be increasingly utilized in common applications. A key aspect in conducting such analyses is the discretization of the domain. According to Filho [1], discretization involves dividing the structure into smaller parts connected at discrete points, functioning akin to assembling elements.

One of the primary approaches used to depict the development of frame finite elements is using cubic (Hermitian) shape functions, calculated considering the undeformed configuration of the infinitesimal element under analysis. In this case, small geometric non-linearities are entirely disregarded, leading to cubic functions.

However, for problems in which geometric non-linearity cannot be ignored, the discretization of the structure becomes increasingly intense to capture the effects of non-linearity. This concept of discretization is typically taught at the postgraduate level, often proving complex for undergraduate students. Many authors seek to address problems by considering the equilibrium of infinitesimal elements in the deformed configuration rather than the undeformed configuration. Consequently, shape functions are derived precisely from the solutions to these differential equations of equilibrium, rendering the discretization of the structure no longer indispensable.

2 Solution of the differential equation of frames

In this section, the equilibrium conditions of an infinitesimal element considering the deformed condition will be developed. Figure 1 represents the free-body diagram of an element subjected to transverse and axial loads, which is used for the development of the equations.

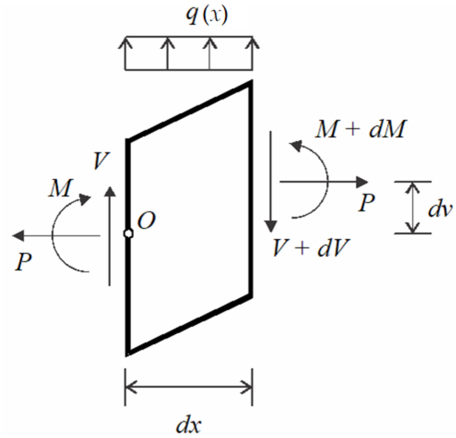

Figure 1. Infinitesimal beam element in equilibrium [2]

According to Martha and Burgos [2], using equilibrium imposition, the following equations are obtained:

$$\sum F_y \rightarrow \frac{dV}{dx} = q(x)$$
$$\sum M_O \rightarrow \; dM - (V + dV)dx - Pdv + q(x)\frac{dx^2}{2} = 0 \rightarrow \; \frac{dM}{dx} - P\frac{dv}{dx} = V \, , \tag{1}$$

where *v(x)* is the transverse displacement, *q(x)* is the distributed vertical load, *V(x)* is the vertical component of the force in the cross-section, *P* is the horizontal component of the force in the cross-section, and *M(x)* is the bending moment at the cross-section. It is interesting to notice that one of the consequences of using the deformed configuration is that the transversal load *V*(*x*) is no longer the derivative of the bending moment. This is because in the deformed configuration, the shear force and the transversal force are not the same. If P=0, then the traditional equations are restored.

Considering that M(x) = EI dθ/dx, substituting it into Eq. (1) and taking its derivative with respect to *x*:

$$EI\frac{d^3\theta}{dx^3} - \frac{dV}{dx} - P\frac{d^2v}{dx^2} = 0 \, , \tag{2}$$

where *E* is the modulus of elasticity of the material, *I* is the moment of inertia of the cross-sectional area, and *θ* is the rotation of the cross-section. Substituting the expression for the derivative of the vertical force dV/dx = q(x):

$$EI\frac{d^3\theta}{dx^3} - P\frac{d^2v}{dx^2} = q(x) \, . \tag{3}$$

Considering Euler-Bernoulli beam theory, the rotation of the cross-section can be described as the derivative of the transverse displacement, that is, θ = dv/dx. This results in:

$$\frac{d^4v}{dx^4} - \frac{P}{EI}\frac{d^2v}{dx^2} = \frac{q(x)}{EI} \, . \tag{4}$$

According to Martha and Burgos [2], this equation represents the Euler-Bernoulli formulation considering a constant axial load, neglecting shear deformations. Solving this equation determines the transverse displacement v(x), and the rotation is determined by its derivative (θ(x) = dv/dx).

However, in the case of the Timoshenko beam theory, the rotation of the section is not determined by the derivative of the transverse displacement, because of the consideration of shear distortion, as illustrated in Figure 2. The following equations show how this new information can be included.

$$\gamma(x)=\frac{dv(x)}{dx}-\theta(x)$$
$$Q(x)=-\chi GA\gamma(x)=\chi GA\left(\theta(x)-\frac{dv(x)}{dx}\right), \tag{5}$$

where G is the shear modulus of the material, A is the cross-sectional area, χ is the factor defining the effective area relative to shear of the cross-section, $Q(x)$ is the shear force, and $\gamma(x)$ is the shear distortion.

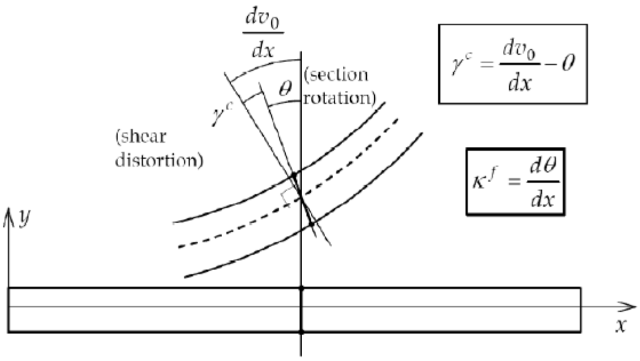

Figure 2. Cross-section rotation and shear strain in Timoshenko beam theory [2]

Considering shear distortion and applying it to the differential equation leads to:

$$EI\left(1+\frac{P}{\chi GA}\right)\frac{d^3\theta(x)}{dx^3}-P\frac{d\theta(x)}{dx}=q(x)\,. \tag{6}$$

Relating v(x) and θ(x) from Eq. (1):

$$\frac{dv(x)}{dx}=\theta(x)-\Omega L^2\frac{d^2\theta(x)}{dx^2},\quad \Omega=\frac{EI}{\chi GAL^2}\,. \tag{7}$$

Manipulating equation (7) to account for tensile axial load, the following equation is obtained:

$$\frac{d^3\theta(x)}{dx^3}-\frac{\mu^2}{\left(1+\Omega\mu^2L^2\right)}\frac{d\theta(x)}{dx}=\frac{q(x)}{EI\left(1+\Omega\mu^2L^2\right)},\quad \mu^2=\frac{P}{EI}\,. \tag{8}$$

The solution is normally obtained using the homogeneous equation:

$$\frac{d^3\theta(x)}{dx^3}-\Lambda^2\frac{d\theta(x)}{dx}=0,\quad \Lambda=\frac{\mu}{\sqrt{1+\Omega L^2\mu^2}},$$
$$\theta(x)=A\sinh(\Lambda x)+B\cosh(\Lambda x)+C. \tag{9}$$

Deriving the equation for θ(x) and writing hyperbolic function expressions:

$$v(x)=\left(1-\Omega\Lambda^2L^2\right)\left[C_1\sinh(\Lambda x)+C_2\cosh(\Lambda x)\right]+Cx+D,$$
$$\theta(x)=\Lambda C_1\cosh(\Lambda x)+\Lambda C_2\sinh(\Lambda x)+C. \tag{10}$$

Performing the same analysis for compressive axial loads results in:

$$v(x) = \left(1+\Omega\Lambda^2L^2\right)\left[C_1\sin(\Lambda x)+C_2\cos(\Lambda x)\right]+Cx+D,$$
$$\theta(x) = \Lambda C_1\cos(\Lambda x)-\Lambda C_2\sin(\Lambda x)+C, \quad \mu^2=-\frac{P}{EI}, \quad \Lambda=\frac{\mu}{\sqrt{1-\Omega L^2\mu^2}}. \tag{11}$$

3 Tangent stiffness matrix considering shear deformation

The shape functions of a matrix are used to interpolate nodal values within the element, obtaining displacements and rotations, according to Silva [3].

Considering the equations for u(x), v(x) and θ(x) with shear strain distortion and applying the matrix form:

$$\begin{aligned} u(x) &= N_1^u(x)d_1+N_4^u(x)d_4 \\ v(x) &= N_2^v(x)d_2+N_3^v(x)d_3+N_5^v(x)d_5+N_6^v(x)d_6 \\ \theta(x) &= N_2^\theta(x)d_2+N_3^\theta(x)d_3+N_5^\theta(x)d_5+N_6^\theta(x)d_6 \end{aligned} \rightarrow \begin{Bmatrix} u(x) \\ v(x) \\ \theta(x) \end{Bmatrix} = [N]\{d\}. \tag{12}$$

Using these shape functions and an energy-based formulation, it is possible to arrive at the following stiffness coefficients for tensile axial load.

$$k_1=\frac{12EI}{L^3}\frac{(\Lambda L)^3\sinh(\Lambda L)}{12\left(1-\Omega\Lambda^2L^2\right)D}, \quad k_2=\frac{6EI}{L^2}\frac{(\Lambda L)^2\left[\cosh(\Lambda L)-1\right]}{6D},$$
$$k_3=\frac{4EI}{L}\frac{\Lambda L\left[\Lambda L\cosh(\Lambda L)-\left(1-\Omega\Lambda^2L^2\right)\sinh(\Lambda L)\right]}{4D},$$
$$k_4=\frac{2EI}{L}\frac{\Lambda L\left[\left(1-\Omega\Lambda^2L^2\right)\sinh(\Lambda L)-\Lambda L\right]}{2D}, \tag{13}$$
$$D=\left(1-\Omega\Lambda^2L^2\right)\left[2-2\cosh(\Lambda L)\right]+\Lambda L\sinh(\Lambda L).$$

Rewriting equations (22) with trigonometric coefficients for compressive axial load:

$$k_1=\frac{12EI}{L^3}\frac{(\Lambda L)^3\sin(\Lambda L)}{12\left(1+\Omega\Lambda^2L^2\right)D}, \quad k_2=\frac{6EI}{L^2}\frac{(\Lambda L)^2\left[1-\cos(\Lambda L)\right]}{6D},$$
$$k_3=\frac{4EI}{L}\frac{\Lambda L\left[\left(1+\Omega\Lambda^2L^2\right)\sin(\Lambda L)-\Lambda L\cos(\Lambda L)\right]}{4D},$$
$$k_4=\frac{2EI}{L}\frac{\Lambda L\left[\Lambda L-\left(1+\Omega\Lambda^2L^2\right)\sin(\Lambda L)\right]}{2D}, \tag{14}$$
$$D=\left(1+\Omega\Lambda^2L^2\right)\left[2-2\cos(\Lambda L)\right]-\Lambda L\sin(\Lambda L).$$

The stiffness matrix of the frame element is as follows:

$$K_t=\begin{bmatrix} \frac{EA+P}{L} & 0 & 0 & \frac{-EA-P}{L} & 0 & 0 \\ 0 & k_1 & k_2 & 0 & -k_1 & k_2 \\ 0 & k_2 & k_3 & 0 & -k_2 & k_4 \\ \frac{-EA-P}{L} & 0 & 0 & \frac{EA+P}{L} & 0 & 0 \\ 0 & -k_1 & -k_2 & 0 & k_1 & -k_2 \\ 0 & k_2 & k_4 & 0 & -k_2 & k_3 \end{bmatrix}. \tag{15}$$

4 Nonlinear analysis

Nonlinear analysis is becoming increasingly essential in solving various problems involving structural engineering. According to Souza and Júnior [4], when dealing with large deformations/displacements, the deformed configuration of the structure differs significantly from the undeformed configuration, making linear analysis inappropriate. The nonlinear analysis method used in this article will be the two cycle iterative method, which is briefly explained in the following section.

4.1 Two cycle iterative method

This method, developed by Chen & Lui [5], uses the equilibrium equation (24) in two stages, where *{F}* is the force vector, $\{U\}$ is the displacement vector, and $\{K\}$ is the stiffness matrix:

$$\{F\} = [K]\{U\}. \tag{16}$$

According to Silva et al. [6], the method begins with the first cycle, which involves a linear analysis of the structure using the stiffness matrix K, without considering nonlinearity. This first cycle is used to capture the values of the axial loads in each element. In the second cycle, the consideration of the geometric stiffness matrix of the structure is added. Therefore, starting from the updated stiffness matrix, it is possible to perform the cycle again, this time considering nonlinearity. Thus, according to Silva [7] the step-by-step process for the method of two iterative cycles involves conducting a linear analysis of the structure using the elastic stiffness matrix. With the forces obtained, calculate the tangent stiffness matrix, which accounts for the effects of the elastic stiffness matrix combined with the geometric stiffness matrix. In this paper, the tangent matrix is obtained directly and not by summing a separate geometric matrix. Finally, recalculate the forces in the elements using the last obtained matrix. This process is iterative, and the forces are applied only once in each cycle.

5 Numerical examples

In this article, two structural models will be analyzed: a fixed-free column and a doubly supported column with opposing moments. A second-order analysis using the two-cycle iterative method in *Matlab* will be compared with the second-order analysis performed by the nonlinear analysis software *Mastan2* [8] which uses the Predictor-Corrector method for the nonlinear iterative analysis. The shape functions used in Mastan2 are the traditional Hermitian polynomials considering Timoshenko beam theory. These functions don't present shear locking since they are the cubic solution of the beam without axial load. However, for problems with geometric nonlinearity, discretization is necessary. The models in *Mastan2* were discretized using 1 and 5 elements per bar for comparison.

5.1 Fixed-Free Column

According to Figure 3, the first model analyzed is a column with a fixed base and free top, subjected to a compressive load *P* and a transverse load *αP*. The lateral load is usually small (alpha is a small parameter) since its acts only to impose an initial displacement, as if it was an imperfection. The parameters used for this analysis were: L=6 m; E=10^8 kN/m²; A=1×10^{-4} m^2; χ=5/6; I=1×10^{-5} m⁴; α=0.01.

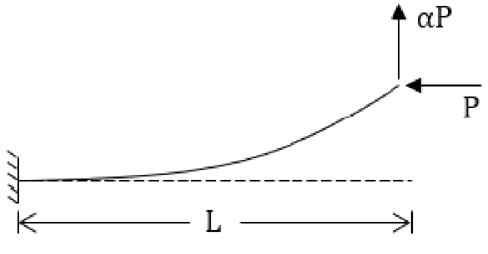

Figure 3. Fixed-Free Column

The displacement data at the free end of the bar was recorded using only 1 element of the presented formulation. Results were compared with the ones obtained from Mastan2 software using 1 and 5 elements discretization.

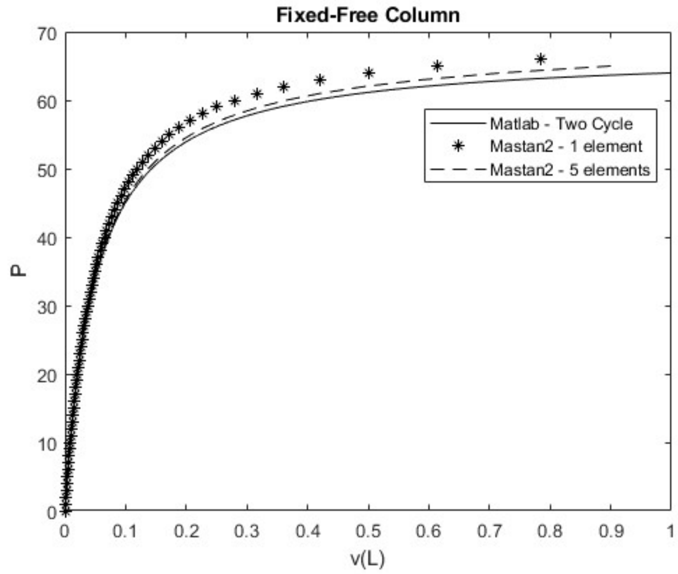

Figure 4. Equilibrium path for fixed-free column

Observing Figure 4, it is evident that the results produced by *Matlab* using the two-cycle iterative method closely align with those from *Mastan2* with 5 elements, which utilizes the predictor-corrector method for second-order analysis. In this model, the discretization showed minor differences at the beginning of loading (P < 40), but larger discrepancies emerged for P > 50. This highlights the importance of careful consideration by the designer when choosing the element discretization which is something difficult for undergraduate students.

5.2 Simply supported Column

The second model analyzed is a simply supported column, according to Figure 5, subjected to a compressive load *P* and two moments with opposite directions at the ends. The moments have the same purpose as the lateral load in the previous example. Properties of the bar: $L = 6\ m$; $E = 10^8\ kN/m^2$; $A = 1x10^{-4}\ m^2$; $\chi = 5/6$; $I = 1x10^{-5}\ m^4$; $M = 0.01PL$.

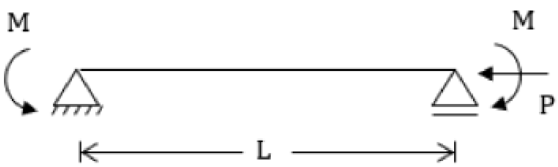

Figure 5. Simply supported column with opposite moments at the ends [3]

Rotation data at the end of the bar was extracted for the presented formulation and compared with 1 and 5 element discretization from Mastan2 software.

Observing the results shown in Figure 6, it can be observed that the results using *Mastan2* with 1 element is significantly different from the ones generated by the two-cycle iterative method using analytical shape functions. It is also clear that the two-cycle approach can only be performed up to the critical load, after which the displacements become too large, and the differential equation loses its validity.

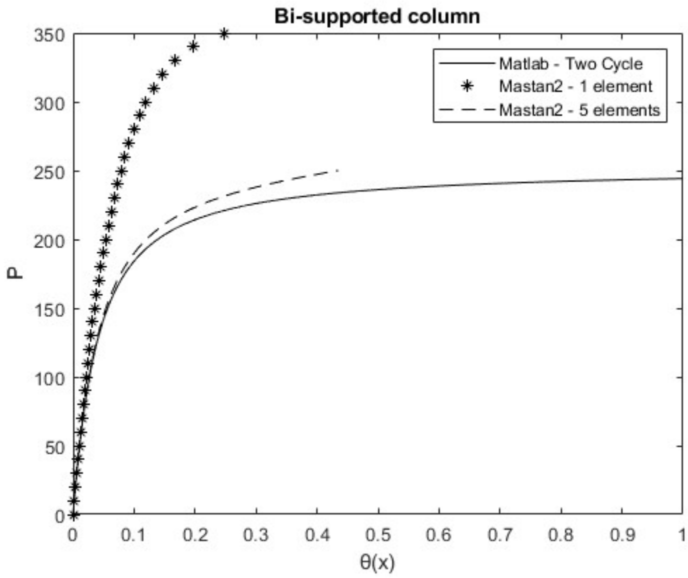

Figure 6. Equilibrium path for bi-supported column

6 Conclusions

In this study, it is evident how crucial it is to develop analysis methods that are independent of structure discretization, especially considering that many students and even young professionals lack the necessary experience in element subdivision and iterative nonlinear procedures, which is essential for satisfactory results.

The fixed-free column model yielded results from *Mastan2* with 5 elements were close to those obtained by *Matlab* using the two-cycle iterative method but showed a more pronounced difference compared to *Mastan2* with 1 element. In the simply supported column model, the difference between *Mastan2* with 5 elements and *Matlab* was similar, but there was a significant discrepancy compared to *Mastan2* with 1 element, resulting in much higher results than expected. This highlights the importance of correct element discretization when using Hermitian shape functions. All examples with the proposed methodology used only 1 element per bar and the two-cycle strategy in terms of incremental-iterative algorithm, without the need for load steps for achieving good results.

The two-cycle iterative method considering shear deformation proved effective, producing results close to *Mastan2* with 5 elements in both models, regardless of discretization. Several future suggestions are promising, such as conducting similar analyses that incorporate effects of elastic foundation and variations in cross-sectional areas, building upon the findings of this study.

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