Geometric nonlinear analysis of planar frames constituted of nonprismatic Timoshenko-like elements referred to their noncentroidal axis

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Abstract. This paper presents a geometrically nonlinear formulation for analyzing two-dimensional frames modeled with Timoshenko-like elements. These elements are defined by a noncentroidal referece axis, which may be any arbitrary straight-line segments intersecting the cross-sections at any position. The kinematics of deformation and cross-sectional constitutive relationships are referenced to the chosen axis, and the location of the cross-sectional centroid need not to be known. The interaction between axial and flexural effects is consistently accounted for in the formulation. The methodology utilizes a flexibility-based approach, based on the Principle of Virtual Forces to determine the structural property coefficients. The accuracy and validity of the proposed formulation are demonstrated through comparisons with three-dimensional (3D) finite element models generated using ANSYS. The study investigates the nonlinear equilibrium paths, highlighting the capabilities of the proposed methodology in capturing the non trivial behavior of nonprismatic structures with curvilinear centroids.

Keywords: Geometrically nonlinear formulation, Timoshenko frame elements, Noncentroidal axis, Principle of Virtual Forces.

1 Introduction

Since the nineteenth century, the concept of nonprismatic structures has been widespread in both research and practice. The significant material and cost savings, along with the optimized strength and stiffness offered by nonprismatic members, have led many authors to propose different analysis methods [1–7]. However, this type of structure may possess curvilinear centroidal axis, which increases the difficult of modeling elements using formulations based on the centroidal axis.

The proposed approach starts with the consideration of the interaction between normal and flexural effects in the kinematics of deformation within the Timoshenko Beam Theory (TBT). The material is assumed to be isotropic and the cross sections may be heterogeneous. In the end, the structural property coefficients are derived based on the Principle of Virtual Forces (PVF) [8, 9]. Herein, the Unit Load Method is used. Moreover, a variational approach is employed to find out expressions of the nonlinear increments in the internal forces due to the deformed configuration of the finite element [10]. Notice that since the proposed formulation is a flexibility-type method, one avoids solving complex systems of Ordinary Differential Equations (ODEs).

Boundary integrals are used to compute the cross-sectional rigidities. Notice that is of no relevance the location of the cross-sectional centroid. Furthermore, polynomials of different orders are used to interpolate the rigidity values along the element axis. The formulation allows one to determine exat - within the restrictions of the TBT - structural-property matrices. The involved integrals are evaluated via regular low-order Gauss-Legendre quadrature.

A metal nonprismatic monosymmetric I-beam with heterogeneous cross section made of structural steel (in the web) and aluminum alloy (in the flanges) was modeled to validate the proposed formulation. The geometry configuration was chosen so that the beam's centroidal axis is curved. To verify the efficiency and robustness of our formulation, the results were compared with the ones obtained from a highly refined 3D finite-element model analyzed using the ANSYS package.

2 Timoshenko Beam Theory

This section presents a formulation based on the work by de Araújo and Mageveske [7] and Mageveske [11]. The formulation process begin by deriving the kinematics of deformation for a Timoshenko beam element referred to an arbitrary noncentroidal axis. This formulation explicitly accounts for the coupling between normal and flexural effects. Subsequently, cross-sectional constitutive relations are stated. Finally, based on the PVF, we obtain expressions for structural-property coefficients for an in-plane shear-deformable Timoshenko frame element.

2.1 Kinematics of deformation

The in-plane displacements increments, $u_i,\ i=1,2$, and corresponding linear and nonlinear strains, e_{ij} and η_{ij}, of a shear-deformable model may be described by:

$$
\begin{gathered}
u_1({}^1x_1,{}^1\mathbf{x}) = u({}^1x_1) - {}^1x_2 \cdot \beta({}^1x_1),\ u_2({}^1x_1,{}^1\mathbf{x}) = w({}^1x_1),\\
{}_1e_{11}({}^1x_1,{}^1\mathbf{x}) = u'({}^1x_1) - {}^1x_2 \cdot \beta'({}^1x_1),\ {}_1e_{12}({}^1x_1) = \tfrac{{}_1\gamma({}^1x_1,{}^1\mathbf{x})}{2},\\
{}_1\eta_{11}({}^1x_1,{}^1\mathbf{x}) = \tfrac{1}{2}\cdot\left[({}_1u')^2 - 2\cdot{}^1x_2\cdot{}_1u'\cdot{}_1\beta' + {}^1x_2^2\cdot({}_1\beta')^2 + ({}_1w')^2\right],\ \text{and}\\
{}_1\eta_{12}({}^1x_1,{}^1\mathbf{x}) = \tfrac{1}{2}\cdot\left(-{}_1u'\cdot\beta + {}^1x_2\cdot\beta\cdot{}_1\beta'\right).
\end{gathered} \tag{1}
$$

Note that $[\bullet]' = \partial[\bullet]/\partial^1x_1$. Moreover, the coordinate system is described by $\{({}^1x_1,{}^1\mathbf{x}) \in \Omega({}^1x_1)|{}^1\mathbf{x} = ({}^1x_2,{}^1x_3)\}$, where ${}^1x_1$ indicates the position of the cross section along the axis. By means of Timoshenko Beam hypotheses, the deformation of $\Omega({}^1x_1)$ is represented by the displacements increments $u({}^1x_1)$, $w({}^1x_1)$, and $\beta({}^1x_1)$. In addition, ${}_1\gamma({}^1x_1,{}^1\mathbf{x})$ represents the average constant shearing strain of $\Omega({}^1x_1)$. Owning to the kinematics relationships in Equation (1), the heterogeneous cross-sectional constitutive relations for an isotropic material may expressed as follows:

$$
\begin{gathered}
\sigma_{11}({}^1x_1,{}^1\mathbf{x}) = E({}^1x_1,{}^1\mathbf{x})\cdot[u'({}^1x_1) - {}^1x_2\cdot\beta'({}^1x_1)],\ \text{and}\\
\tau_{12}({}^1x_1,{}^1\mathbf{x}) = k_2({}^1x_1)\cdot G({}^1x_1,{}^1\mathbf{x})\cdot w_s'({}^1x_1),
\end{gathered} \tag{2}
$$

where $k_2({}^1x_1)$ is the shear correction factor in ${}^1x_2$ direction, $E({}^1x_1,{}^1\mathbf{x})$ and $G({}^1x_1,{}^1\mathbf{x})$ are the material elasticity longitudinal and transverse modulus. Notice that $\sigma_{11}({}^1x_1,{}^1\mathbf{x})$ and $\tau_{12}({}^1x_1,{}^1\mathbf{x})$ denote the normal and shear stresses. Thus, expressions for $N_1({}^1x_1)$ (normal force), $M_3({}^1x_1)$ (bending moment), and $Q_2({}^1x_1)$ (shear force) may be obtained as follows:

$$
\begin{gathered}
N_1({}^1x_1) = \int_{\Omega({}^1x_1)} \sigma_{11}({}^1x_1,{}^1\mathbf{x})\, d\Omega = k_a({}^1x_1)\cdot u'({}^1x1) - k_{st3}({}^1x_1)\cdot\beta'({}^1x1),\\
M_3({}^1x_1) = \int_{\Omega({}^1x_1)} \sigma_{11}({}^1x_1,{}^1\mathbf{x})\cdot{}^1x_2\, d\Omega = -k_{st3}({}^1x_1)\cdot u'({}^1x1) + k_{b3}({}^1x_1)\cdot\beta'({}^1x1),\\
Q_2({}^1x_1) = \int_{\Omega({}^1x_1)} \tau_{12}({}^1x_1,{}^1\mathbf{x})\, d\Omega = k_{s2}({}^1x_1)\cdot w_s'({}^1x_1),
\end{gathered} \tag{3}
$$

where $k_a({}^1x_1)$, $k_{s2}({}^1x_1)$, $k_{b3}({}^1x_1)$, and $k_{st3}({}^1x_1)$ represent the axial, shear, flexural, and statical cross-sectional rigidity values. Notably, $k_{st3}({}^1x_1)$ couples the axial and flexural effects due to the consideration of noncentroidal axis. If the axis position coincides with the cross-sectional centroid, the two effects decouple since $k_{st3}({}^1x_1)$ becomes zero. Finally, since the previous relations stated by Equations (3) are invertible, its possible to obtain the expressions for the cross-sectional kinematics relations:

$$
\begin{gathered}
u'({}^1x_1) = \tfrac{1}{|\mathbf{K}|}[k_{b3}({}^1x_1)\cdot N_1({}^1x_1) + k_{st3}({}^1x_1)\cdot M_3({}^1x_1)],\ \text{and}\\
\beta'({}^1x_1) = \tfrac{1}{|\mathbf{K}|}[k_{st3}({}^1x_1)\cdot N_1({}^1x_1) + k_a({}^1x_1)\cdot M_3({}^1x_1)],\ \text{with}\ |\mathbf{K}| = k_a({}^1x_1)\cdot k_{b3}({}^1x_1) - k_{st3}^2({}^1x_1).
\end{gathered} \tag{4}
$$

2.2 Structural property coefficients

For the nonprismatic finite element referred to an arbitrary noncentroidal axis, the Unit Load Method gives:

$$\bar{f}_j{\cdot}u_{ji} = \int_0^{{}^1l} \bar{N}_{1j}({}^1x_1)\, du_i({}^1x_1) + \int_0^{{}^1l} \bar{M}_{3j}({}^1x_1)\, d\beta_i({}^1x_1) + \int_0^{{}^1l} \bar{Q}_{2j}({}^1x_1)\, dw_{si}({}^1x_1) + u_{jq}. \tag{5}$$

As $\bar{f}_{ij} = 1$, Equation (5) expresses the nodal displacements $u_{ji}, j = 1, \cdots, 6$ due to the deformation mode $i, i = 1, \cdots, 6$, where $\bar{N}_{1j} = N_1(\bar{f}_j)$, $\bar{M}_{3j} = M_3(\bar{f}_j)$, and $\bar{Q}_{2j} = Q_2(\bar{f}_j)$ are the generalized stresses. The external loads ${}^1q_1, {}^1q_2$, and ${}^1q_3$ acting on the element induces nodal displacements u_{jq}. Thus, the linear expressions for the PVF based on the Updated Lagrangian formulation may be written as follows:

$$u_{ji} = \int_0^{{}^1l} \frac{\tilde{N}_{1j}({}^1x_1){\cdot}N_{1i}({}^1x_1)}{|\mathbf{K}|}\, d^1x_1 + \int_0^{{}^1l} \frac{\tilde{M}_{3j}({}^1x_1){\cdot}M_{3i}({}^1x_1)}{|\mathbf{K}|}\, d^1x_1 + \int_0^{{}^1l} \frac{\bar{Q}_{2j}({}^1x_1){\cdot}Q_{2i}({}^1x_1)}{k_{s2}}\, d^1x_1 + u_{jq}, \tag{6}$$

where

$$\begin{aligned}\tilde{N}_{1j}({}^1x_1) &= k_{st3}({}^1x_1){\cdot}\bar{M}_{3j}({}^1x_1) + k_{b3}({}^1x_1){\cdot}\bar{N}_{1j}({}^1x_1), \text{ and}\\ \tilde{M}_{3j}({}^1x_1) &= k_a({}^1x_1){\cdot}\bar{M}_{3j}({}^1x_1) + k_{st3}({}^1x_1){\cdot}\bar{N}_{1j}({}^1x_1).\end{aligned} \tag{7}$$

De Araújo and Ribeiro [3] and de Araújo et al. [5] presented a similar expression for the PVF. The proposed formulation differs from those formulations due to the term $k_{st3}({}^1x_1)$, also presented by de Araújo and Mageveske [7]. When $k_{st3}({}^1x_1)$ equals zero (centroidal axes), Equation (6) reduces to the formulation presented by De Araújo and Ribeiro [3] and de Araújo et al. [5]. In this context, the previous integral expression can be rewritten in the following algebraic form:

$$\sum_{k=k_0}^{k_1} a_{jk}{\cdot}{}_1f_{ki} = u_{ji} - \sum_{k=k_0}^{k_1} g_{jki}{\cdot}{}^1f_k - u_{jq},\ k_0 = [(m-1){\cdot}\text{ndofn}]{+}1,\ k_1 = m{\cdot}\text{ndofn}. \tag{8}$$

where, $j, i = 1, 2, 3\,(m = 1)$, $j, i = 4, 5, 6\,(m = 2)$, $i = q$, and 'ndofn' is the number of the degree of freedom. To obtain the expression for the elastic stiffness coefficients ${}_1k_{kj} = {}_1f_{ki}$ the nonlinear and element loading terms should be neglected, i.e. $\sum_{k=k_0}^{k_1} g_{jki}{\cdot}{}^1f_k = u_{jq} = 0$. It results then:

$$\begin{aligned}
a_{11} &= a_{44} = \int_0^{{}^1l} \tfrac{{}_1k_b({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\ \ a_{12} = a_{21} = \int_0^{{}^1l} -\tfrac{{}^1x_1\cdot{}_1k_{st}({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\\
a_{13} &= a_{31} = a_{46} = a_{64} = \int_0^{{}^1l} \tfrac{{}_1k_{st}({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\ \ a_{22} = \int_0^{{}^1l} \left[\tfrac{{}^1x_1^2\cdot{}_1k_a({}^1x_1)}{{}_1|\mathbf{K}|} + \tfrac{1}{{}_1k_s({}^1x_1)}\right] d^1x_1,\\
a_{23} &= a_{32} = \int_0^{{}^1l} -\tfrac{{}^1x_1\cdot{}_1k_a({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\ \ a_{33} = a_{66} = \int_0^{{}^1l} \tfrac{{}_1k_a({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\\
a_{45} &= a_{54} = \int_0^{{}^1l} \tfrac{(l-{}^1x_1)\cdot{}_1k_{st}({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1,\ \ a_{55} = \int_0^{{}^1l} \left[\tfrac{(l-{}^1x_1)^2\cdot{}_1k_a({}^1x_1)}{{}_1|\mathbf{K}|} + \tfrac{1}{{}_1k_s({}^1x_1)}\right] d^1x_1, \text{ and}\\
a_{56} &= a_{65} = \int_0^{{}^1l} \tfrac{(l-{}^1x_1)\cdot{}_1k_a({}^1x_1)}{{}_1|\mathbf{K}|}\, d^1x_1.
\end{aligned} \tag{9}$$

Notice that all integrals in Equation (9) may be evaluated by a low-order Gauss-Legendre quadrature. Moreover, rigidity approximation are given by a polynomial interpolation of $n{+}1$ orders defined by $p(x) = \sum_0^{n+1} \psi_n{\cdot}x^n$, where ψ_n are the interpolation coefficients. With the well-defined expressions for the a_{ji} terms, in Equation (8), the nonlinear terms g_{jki} may be obtained by taking $u_{ji} = u_{jq} = 0$. A similar process used before to obtain the flexibility coefficients may be employed to obtain the nonlinear terms. Furthermore, as pointed out by Mageveske [11] based on de Araújo et al. [5], a variational approach is used to obtain the additional nonlinear terms.

By definition, geometric stiffness results exclusively from changes in the equilibrium configuration of the structural system. This concept means no increment in the external work, i.e., ${}_1^2W_{ext} -_1^1 W_{ext} = 0$. Thus, the relations stated by Equation (4) may be introduced in the linear expression of the Principle of Virtual Displacements [10] as follows:

$$ {}^2_1W_{ext} - {}^1_1W_{ext} = \int_{{}^1\Omega} {}_1C_{mnkl} \cdot {}_1\varepsilon_{kl} \cdot \delta({}_1\varepsilon_{mn}) d^1\Omega + \int_{{}^1\Omega} {}^1\tau_{mn} \cdot \delta({}_1\eta_{mn}) d^1\Omega = 0 \tag{10} $$

where ${}_1C_{mnkl}$ is the constitutive matrix. Replacing the variables provided by the set of kinematic expressions derived in Equation 10, we can obtain the following relation:

$$ \begin{aligned} {}^2_1W_{ext} - {}^1_1W_{ext} \;=\; & \int_0^{{}^1l} \left[{}_1k_a \cdot {}_1u' - {}_1k_{st} \cdot {}_1\beta' + {}^1N \cdot {}_1u' + {}^1M \cdot {}_1\beta' - {}^1Q \cdot {}_1\beta \right] \cdot \delta({}_1u')\, d^1x_1 \\ + & \int_0^{{}^1l} \left[-{}_1k_{st} \cdot {}_1u' + {}_1k_b \cdot {}_1\beta' + {}^1M \cdot {}_1u' + {}^1P \cdot {}_1\beta' + {}^1R \cdot {}_1\beta \right] \cdot \delta({}_1\beta')\, d^1x_1 \\ + & \int_0^{{}^1l} \left[{}^1N \cdot {}_1w' - {}^1Q \cdot {}_1u' + {}^1R \cdot {}_1\beta' \right] \cdot \delta({}_1\beta)\, d^1x_1 \\ + & \int_0^{{}^1l} \left[{}_1k_s \cdot ({}_1w' - {}_1\beta) + {}^1N \cdot {}_1w' \right] \cdot \delta({}_1w' - {}_1\beta)\, d^1x_1 \\ = & \int_0^{{}^1l} \left[{}_1N^{(g)} + {}^1N \cdot {}_1u' + {}^1M \cdot {}_1\beta' - {}^1Q \cdot {}_1\beta \right] \cdot \delta({}_1u')\, d^1x_1 \\ + & \int_0^{{}^1l} \left[{}_1M^{(g)} + {}^1M \cdot {}_1u' + {}^1P \cdot {}_1\beta' - {}^1N \cdot \tilde{w} + {}^1Q \cdot \tilde{u} - {}^1R \cdot c_\beta \right] \cdot \delta({}_1\beta')\, d^1x_1 \\ + & \int \left[{}_1Q^{(g)} + {}^1N \cdot {}_1w' \right] \cdot \delta({}_1w' - {}_1\beta)\, d^1x_1 + \left[{}^1N \cdot \tilde{w} - {}^1Q \cdot \tilde{u} + {}^1R \cdot \tilde{\beta} \right] \cdot \delta({}_1\beta) \Big|_0^{{}^1l} = 0. \end{aligned} \tag{11} $$

where ${}^1P$ and ${}^1R$ are the high-order stress resultants, and

$$ \begin{aligned} & {}_1N^{(g)} \equiv {}_1N^{(g)}({}^1x_1) = -{}^1N \cdot {}_1u' + {}^1Q \cdot {}_1\beta - {}^1M \cdot {}_1\beta', \\ & {}_1M^{(g)} \equiv {}_1M^{(g)}({}^1x_1) = {}^1N \cdot ({}_1w + c_w) - {}^1Q \cdot ({}_1u + c_u) - {}^1M \cdot {}_1u' - {}^1P \cdot {}_1\beta' + {}^1R \cdot c_\beta, \text{ and} \\ & {}_1Q^{(g)} \equiv {}_1Q^{(g)}({}^1x_1) = -{}^1N \cdot {}_1w'. \end{aligned} \tag{12} $$

The constants c_u, c_w, and c_β are provided by the boundary conditions of the finite element in ${}^1x_1 = 0$ and ${}^1x_1 = {}^1l$. Finally, ${}_1N^{(g)}$, ${}_1Q^{(g)}$, and ${}_1M^{(g)}$ expressions may be included in the Equation (11), which gives the following term for taking into account geometric nonlinear effects:

$$ \begin{aligned} g_{jki} \;=\; & \int_0^{{}^1l} \left[\frac{{}_1\tilde{N}_j \cdot (-{}^1N \cdot {}_1u' + {}^1Q \cdot {}_1\beta - {}^1M \cdot {}_1\beta')}{{}_1|\mathbf{K}|} \right] d^1x_1 \\ + & \int_0^{{}^1l} \left[\frac{{}_1\tilde{M}_j \cdot ({}^1N \cdot ({}_1w + c_w) - {}^1M \cdot {}_1u'}{{}_1|\mathbf{K}|} \right] d^1x_1 \\ + & \int_0^{{}^1l} \left[\frac{{}_1\bar{Q}_j \cdot (-{}^1N \cdot {}_1w')}{{}_1k_s} \right] d^1x_1, \end{aligned} \tag{13} $$

Once the g_{jki} terms are known, one may obtain the geometric stiffness coefficients by solving Equation (8) as follows:

$$\sum_{k=1}^{3} a_{jk} \cdot {}_1 f_{ki} \;=\; -\begin{Bmatrix} g_{11i} \\ g_{21i} \\ g_{31i} \end{Bmatrix} \cdot {}^1 f_1 - \begin{Bmatrix} g_{12i} \\ g_{22i} \\ g_{32i} \end{Bmatrix} \cdot {}^1 f_2 - \begin{Bmatrix} g_{13i} \\ g_{23i} \\ g_{33i} \end{Bmatrix} \cdot {}^1 f_3, \; i = 1, 2, 3, \text{ and}$$

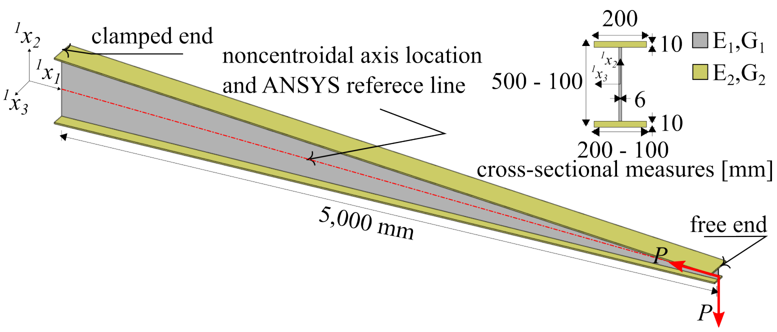
$$\sum_{k=4}^{6} a_{jk} \cdot {}_1 f_{ki} \;=\; -\begin{Bmatrix} g_{44i} \\ g_{54i} \\ g_{64i} \end{Bmatrix} \cdot {}^1 f_4 - \begin{Bmatrix} g_{45i} \\ g_{55i} \\ g_{56i} \end{Bmatrix} \cdot {}^1 f_5 - \begin{Bmatrix} g_{46i} \\ g_{56i} \\ g_{66i} \end{Bmatrix} \cdot {}^1 f_6, \; i = 4, 5, 6. \tag{14}$$

2.3 Nonlinear solver

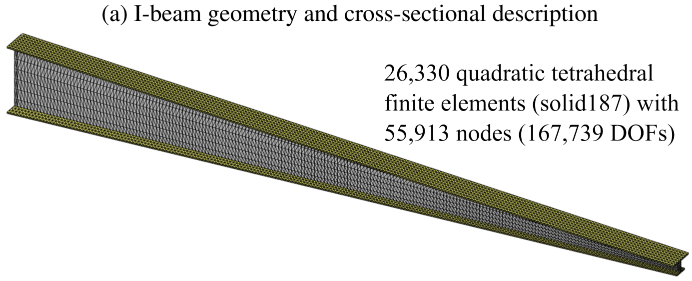
A nonlinear geometric analysis is sensitive and highly dependent on the accurate modeling of the structural system. In this study, the Generalized Displacement Control Method (GDCM) with a load increment based on the stiffness parameter (GSP) was adopted, as presented by Yang and Shieh [12]. Yang and Kuo [10] provide a comprehensive and detailed algorithm for implement this approach.

3 Monosymmetric heterogeneous I-beam with variable web and bottom flange

In this section is analyzed a nonprismatic, monosymmetric I-beam with the web made of structural steel (material 1) and flanges made of an aluminum alloy (material 2) - Figure 1a. Material 1 has Young's modulus $E_1 = 200$ GPa and a shear modulus $G_1 = 76.9$ GPa, while material 2 has $E_2 = 71$ GPa and $G_2 = 26.7$ GPa. The cross-sectional geometry is assumed to vary linearly, with corresponding measure presented in Figure 1a. The obtained results were validated using highly accurate 3D numerical simulation conducted with the commercial finite element package ANSYS - Figure 1b. Furthermore, the well-known arc-length method was employed within ANSYS, utilizing a minimum arc-length of 10^{-7} and a maximum arc-length of 1 with 100 load steps.



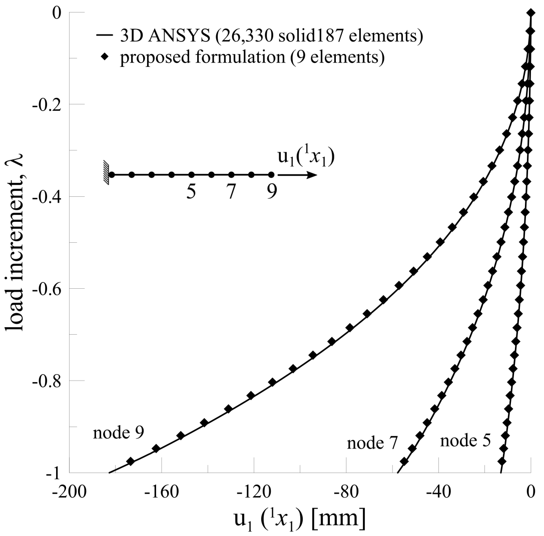
(a) I-beam geometry and cross-sectional description



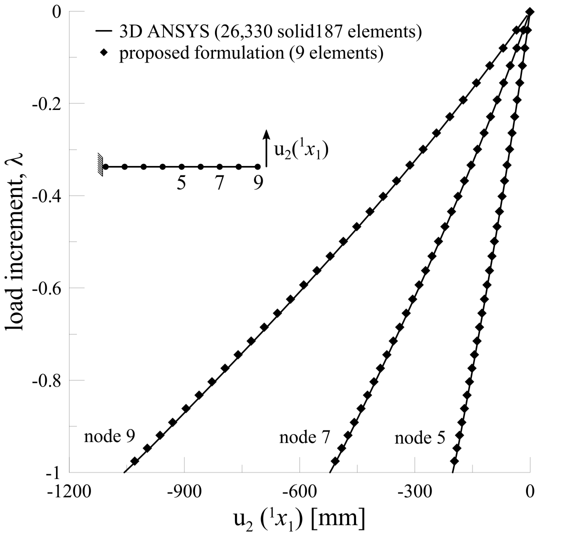
(b) 3D ANSYS solid 187 finite element mesh

Figure 1. Monosymmetric I-beam with heterogeneous cross section

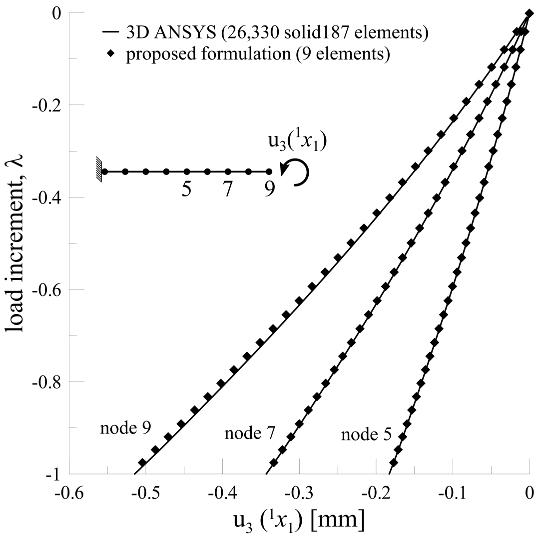
The axis location remains at the same position at half the height of the beam. The loading case adopted in the study was $P = 2.4{\cdot}10^5$ N and $M = 12.93{\cdot}P$ Nmm as can be seen in the illustration of the finite element with 9 nodes in Figure 3. The nonlinear problem was solved with the following parameters: initial load λ_0^0=10^{-3} and force and displacement tolerance equal to 10^{-3}. The number of loads steps was 1,237 with 3-4 iterations per load step. Finally, Figure 2a, 2b, and 2c portraits the results, where can be observed excellent agreement with 3D ANSYS results.



(a) $\lambda \times u_1(^1x_1)$



(b) $\lambda \times u_2(^1x_1)$



(c) $\lambda \times u_3(^1x_1)$

Figure 2. Equilibrium trajectories

4 Conclusions

This paper proposed an extension of the method previously investigated by De Araújo and Ribeiro [3], incorporating the coupling between axial and flexural effects. Structural-property coefficients were derived based on the PVF in the form of the Unit Load Method. To validate the robustness of the proposed formulation, a monosymetric, heterogeneous I-beam with a curvilinear centroidal axis was analyzed. The results obtained from this study demonstrated excellent agreement with 3D ANSYS simulations.

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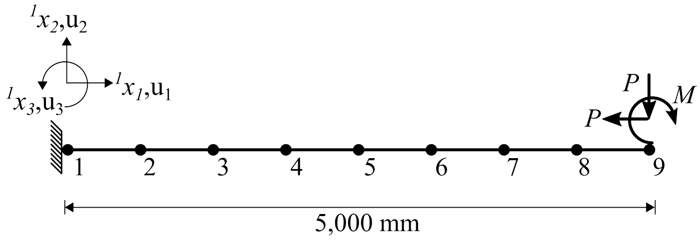

Figure 3. 1D Finite element mesh with boundary conditions

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