A double-hybrid finite element formulation for Stokes flows using a divergence-free approximation space

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Abstract. This paper presents a study of Stokes flows using a double hybrid finite element formulation. Incompressibility appears in Stokes differential equations as an additional constraint that enforces the velocity field to be divergence-free. A straightforward option to discretize those kind of problems would be to use a space of vector functions that includes the divergence-free constraint. Although this option seems natural, the great majority of mixed formulations still employs scalar $H^1(\Omega)$ spaces whilst the divergence-free condition is weekly enforced over the domain, since the construction of divergence-free spaces in an efficient way brings some difficulties in 3D. This work aims to contribute in this regard by choosing an approximation space following the exact De Rham sequence, where a stable pair $H(\text{div}, \Omega) - L^2(\Omega)$ is used to approximate the velocity and pressure fields, respectively. The $H(\text{div})$ spaces automatically guarantee continuity of the normal components of a given vector field over elements interfaces. The continuity of the tangential velocity is enforced through a second step hybridization of shear stresses at element level. Examples are presented to test and verify the developed numerical methodology.

Keywords: Hybrid finite element method, Piecewise constant Hdiv spaces, Locally conservative formulation, Stokes flows

1 Introduction

Assuming a low Reynolds number, the flow is steady and the inertial terms can be neglected. Then, the Navier-Stokes equations can be simplified to the Stokes equations. The Stokes equations are widely used to model fluid motion in various scientific and engineering fields [1].

The three main methods for simulating Stokes flow are the Finite Difference Method (FDM), the Finite Volume Method (FVM), and the Finite Element Method (FEM). FEM offers advantages such as the ability to handle irregular geometries, the versatility to impose complex boundary conditions, and work with non-uniform meshes. FEM provides reliable solutions even in the presence of material property discontinuities and external body forces [2].

Among the most popular methods for solving the velocity and pressure unknown fields, the Taylor-Hood formulation [3] is widely employed. The Taylor-Hood element utilizes continuous piecewise functions to approximate the velocity and continuous linear piecewise functions to approximate the pressure. In this work, the Taylor-Hood element is used for comparison against the proposed method.

The appropriate selection of H(div, Ω)-L$^2(\Omega)$ for the velocity and pressure fields has been demonstrated to enable local mass conservation of the approximation [4]. However, the tangential component of the velocity is not inherently continuous between elements, when using H(div, Ω) functions. Therefore, the adopted formulation must enforce the continuity of the tangential velocity. The starting point of this work is [5], in which a semi-hybrid formulation is presented to solve Stokes' equations. The semi-hybrid formulation approximates velocity ensuring the continuity of the tangential velocity weakly by applying Lagrange multipliers. In this context, the introduced Lagrange multipliers have the physical meaning of tangential stresses.

Although stable from a mathematical point of view, this formulation leads to a saddle-point system with two distinct types of Lagrange multipliers that require more sophisticated algorithms to solve. Therefore, this work proposes a second hybridization of the tangential stress aiming to improve stability and a global matrix with better spectral properties. This new approach is called double hybrid formulation. In both the semi-hybrid and double hybrid formulations, static condensation is employed to i) enhance computational efficiency and ii) recover a symmetric positive semi-definite elemental matrix for the latter. However, there is a distinction between the two formulations in terms of the condensed system's Degrees of Freedom (DoFs).

After the static condensation is applied, the resulting system in the double hybrid formulation still has a saddle-point structure, but only the Lagrange multipliers associated with the pressure are present. The Lagrange multipliers associated with the velocity no longer yield zeros in the diagonal of the matrix, since the traction is condensed. This leads to a more stable system, from a numerical point of view, when compared to the semi-hybrid formulation. To ensure the convergence of the method and overcome numerical challenges, it is necessary to satisfy the Babuska-Brezzi condition [6, 7]. The approximation spaces available in NeoPZ environment satisfy the De Rham complex and consequently the LBB condition for incompressibility.

This work's main goal is to employ the double hybrid formulation to simulate fluid flow. The results are obtained by solving the Stokes equations, using two different variations of the $\mathrm{H}(\mathrm{div}, \Omega)$ space (Hdiv-S and Hdiv-C) to interpolate the velocity field. Both spaces are compared to the traditional Taylor-Hood element. The object-oriented programming environment NeoPZ[1] written in C++ is used to implement the numerical simulations.

2 Problem statement

Over a domain Ω, the Stokes problem reads: finding the velocity $\boldsymbol{u}$ and the pressure p such that:

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f} \text{ in } \Omega, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{u_D} \text{ on } \partial\Omega_D, \tag{3}$$

$$\boldsymbol{\sigma n} = \boldsymbol{\sigma_N} \text{ on } \partial\Omega_N. \tag{4}$$

where (1) represents the divergence of the Cauchy stress tensor $\boldsymbol{\sigma}$, in which the constitutive equation is given by

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\boldsymbol{I}, \tag{5}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right), \tag{6}$$

in which $\boldsymbol{I}$ is the second order identity tensor and $\boldsymbol{\varepsilon}$ is the symmetric strain rate tensor. The body force is represented by $\boldsymbol{f}$.

Eq. (2) is the conservation of mass law, assuming incompressible flow. Equations (3) and (4) are the Dirichlet and Neumann boundary conditions, imposed over the Dirichlet boundary $\partial\Omega_D$ and Neumann boundary $\partial\Omega_N$, respectively. The imposed velocity and stress are denoted by $\boldsymbol{u_D}$ and $\boldsymbol{\sigma_N}$.

3 Double Hybrid finite element formulation

To properly state the double hybrid formulation, three approximation spaces are defined. The velocity approximation space is given by

$$\vec{\mathbb{V}}_d = \left\{\boldsymbol{v} \in \mathrm{H}(\mathrm{div}, \Omega) : \boldsymbol{v}|_{\Omega_e} \in \vec{\mathbb{V}}_e \; \forall\Omega_e \in \mathcal{T}\right\}, \tag{7}$$

$$\vec{\mathbb{V}}_{div} = \vec{\mathbb{V}}_d \cap \mathrm{H}^1(\mathcal{T}, \mathbb{R}^d), \tag{8}$$

while the pressure approximation space is defined by

$$\mathbb{Q}_d = \left\{q \in \mathrm{L}^2(\Omega_e) : q|_{\Omega_e} \in \mathbb{Q}_e \; \forall\Omega_e \in \mathcal{T}\right\}. \tag{9}$$

where $\mathbb{Q}_e$ is the polynomial space composed of scalar functions and $\vec{\mathbb{V}}_e$ the polynomial space composed of vector functions.

$\mathrm{H}(\mathrm{div}, \Omega)$ spaces do not ensure the continuity of the tangential component of the flux. Then, Lagrange multipliers must be employed to weakly ensure the continuity of the tangential flux. The space for the Lagrange multiplier is defined as

$$\Lambda = \left\{\boldsymbol{\sigma n}|_{\partial\Omega_e} \in \mathrm{H}^{-1/2}(\partial\Omega_e, \mathbb{R}^d), \;\; \forall\Omega_e \in \mathcal{T} : \boldsymbol{\sigma} \in \mathrm{H}(\mathrm{div}, \Omega)\right\}. \tag{10}$$

[1]NeoPZ open source platform `https://github.com/labmec/neopz`

since Λ can be decomposed into its normal and tangential components $\Lambda = \Lambda^n + \Lambda^t$, and only the tangential component is required, the approximation space for the Lagrange multiplier $\boldsymbol{\lambda}^t$ is defined as

$$\Lambda^t = \left\{ \boldsymbol{\lambda}^t = \boldsymbol{\lambda} - (\boldsymbol{\lambda} \cdot \boldsymbol{n})\boldsymbol{n},\ \boldsymbol{\lambda} \in \Lambda \right\}. \tag{11}$$

Although the semi-hybrid approach has been proven to be mathematically consistent, some of its properties must be highlighted. After eliminating the internal velocities and higher order pressures from the global system (by using the static condensation method), it originates a saddle-point problem with two different constraints: a mean pressure per element $p \in \mathrm{L}^2(\Omega)$ and the shear stresses $\boldsymbol{\lambda}^t \in H^{-1/2}(\partial\Omega_e)$ over the element interfaces.

The decomposition of such matrices is complicated from the numerical point of view, as a specific permutation must be employed to avoid zero pivots when decomposing the Lagrange multipliers degrees of freedom. Additionally, the usage of the penalty method to impose shear stress boundary conditions has been demonstrated to introduce numerical instabilities which compromise the accuracy of the formulation.

Under these circumstances, and aiming to derive an equivalent formulation but with better spectral properties, a second hybridization of the tangential stresses is proposed to recover the primal form. Introducing the space $\mathcal{L}^t \subset H^{1/2}(\partial\Omega_e)$ for the tangential velocity, defined over the interfaces $E \in \varepsilon^0$, the hybrid form thus reads: find $\{\boldsymbol{u}, p, \boldsymbol{\lambda}^t, \boldsymbol{u}^t\} \in \vec{\mathbb{V}}_{div} \times \mathbb{Q}_d \times \Lambda^t \times \mathcal{L}^t$ such that for all $\boldsymbol{v}, q, \boldsymbol{\eta}^t, \boldsymbol{v}^t \in \vec{\mathbb{V}}_{div} \times \mathbb{Q}_d \times \Lambda^t, \mathcal{L}^t$, the following equations are satisfied:

$$\begin{aligned} \sum_{\Omega_e \in \mathcal{T}} \left(\int_{\Omega_e} 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) d\Omega_e - \int_{\Omega_e} p(\nabla \cdot \boldsymbol{v}) d\Omega_e - \int_{\partial\Omega_e} \boldsymbol{\lambda}^t \cdot \boldsymbol{v} ds \right) = \\ \sum_{\Omega_e \in \mathcal{T}} \left(\int_{\Omega_e} \boldsymbol{v} \cdot \boldsymbol{f} d\Omega_e + \int_{\partial\Omega_N} (\boldsymbol{\sigma n} \cdot \boldsymbol{n})(\boldsymbol{v} \cdot \boldsymbol{n}) ds \right) \end{aligned} \tag{12}$$

$$\sum_{\Omega_e \in \mathcal{T}} - \int_{\Omega_e} q(\nabla \cdot \boldsymbol{u}) d\Omega_e = \boldsymbol{0} \tag{13}$$

$$\sum_{\Omega_e \in \mathcal{T}} \left(- \int_{\partial\Omega_e} \boldsymbol{u} \cdot \boldsymbol{\eta}^t d\partial\Omega_e - \int_{\Gamma} [\![\boldsymbol{\eta}^t]\!] \cdot \boldsymbol{u}^t d\Gamma \right) = \boldsymbol{0} \tag{14}$$

$$\sum_{\Omega_e \in \mathcal{T}} - \int_{\Gamma} [\![\boldsymbol{\lambda}^t]\!] \cdot \boldsymbol{v}^t d\Gamma = \sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_N} \boldsymbol{v}^t \cdot (\boldsymbol{\lambda}^t - \boldsymbol{\theta}_N)\, ds, \tag{15}$$

where Eq. (15) was introduced to impose the continuity of the tangential traction across the internal interfaces between elements. Even though an additional constraint is introduced, $\boldsymbol{\lambda}^t$ is now associated with a single element therefore it can be statically condensed and eliminated from the global system. Also, its condensation into the tangential velocities gives rise to a symmetric-positive-semi-definite block. The elemental matrix is then left with contributions from normal velocities, tangential velocities, and a single pressure, thus having better spectral properties and being easier to solve compared to the semi-hybrid weak formulation.

4 Examples

Two different $\mathrm{H}(\mathrm{div}, \Omega)$ spaces are used in this project, the so-called Hdiv-S (for Standard) and Hdiv-C (from Constant). The Hdiv-S is the default way to create $\mathrm{H}(\mathrm{div}, \Omega)$ spaces using NeoPZ. The Hdiv-C space is created using the De Rham exact sequence theory, replacing face functions with RT0 (Raviart Thomas zero order functions) with element-wise constant divergence. The result is that Hdiv-C space has a reduced number of shape functions when compared to Hdiv-S, yielding, nevertheless, the same result for the velocity field and an element-wise constant pressure.

4.1 Annular Couette flow

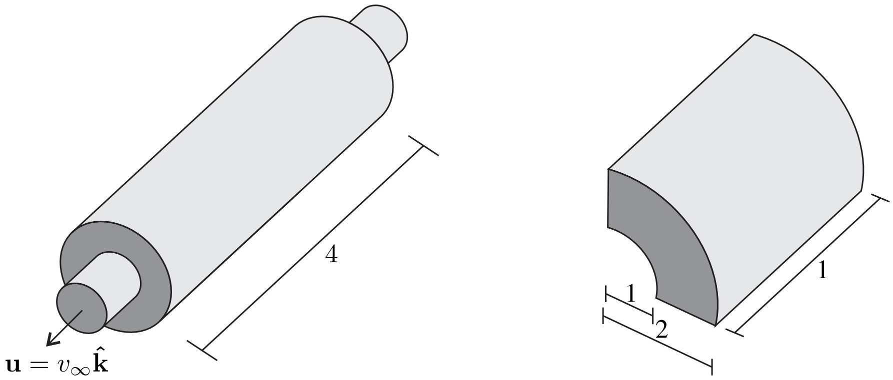
The Annular Couette flow problem is used to verify the three-dimensional formulation's rates of convergence. The problem involves two cylinders positioned concentrically. The inner cylinder is pushed in the z-direction with a constant velocity $v_{zinner} = v_\infty$, while the outer cylinder remains stationary with $v_{zouter} = 0$. As a result, the velocity profile in the radial direction follows a logarithmic pattern. The analytical solutions for the velocity and pressure can be described by equations (16) and (17), respectively

$$\boldsymbol{u} = v_\infty \left(\frac{\log\left(\frac{R_o}{r}\right)}{\log\left(\frac{R_o}{R_i}\right)} \right) \hat{\boldsymbol{k}}, \tag{16}$$

$$p = 1, \tag{17}$$

in which, r is the radial distance from the center of the annulus, R_o is the radius of the outer cylinder and R_i is the radius of the inner cylinder. The polynomial order for the facet velocity k is set to 1, 2, 3, and 4.

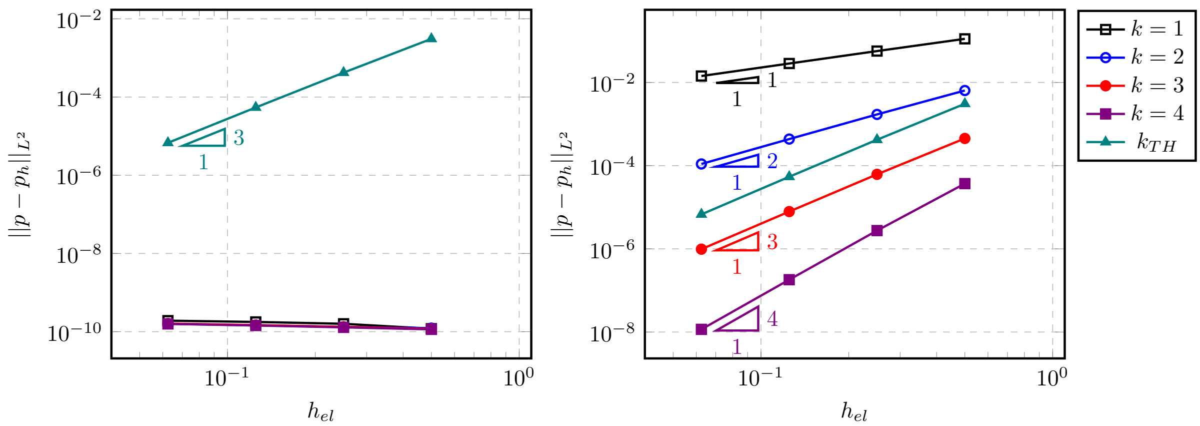
The domain is given by the annulus comprehended between the inner and outer cylinders, with $R_i = 1$ and $R_o = 2$. Only a quarter of the annulus is simulated (see Figure 1) due to its symmetry. The boundary conditions are normal stress on the front and back surfaces ($\boldsymbol{\sigma n} \cdot \boldsymbol{n} = 1$), on every surface of the annulus, tangential velocity ($\boldsymbol{u} \cdot \boldsymbol{t}$) as a function of r is applied. Finally, on the inner, outer, and surfaces of symmetry the no penetration condition ($\boldsymbol{u} \cdot \boldsymbol{n} = 0$) is enforced.



(a) Annular Couette Domain (b) Annular Couette Domain

Figure 1. Domain for the three-dimensional verification test.

The rates of convergence are displayed in Figures 2 - 5, respectively.



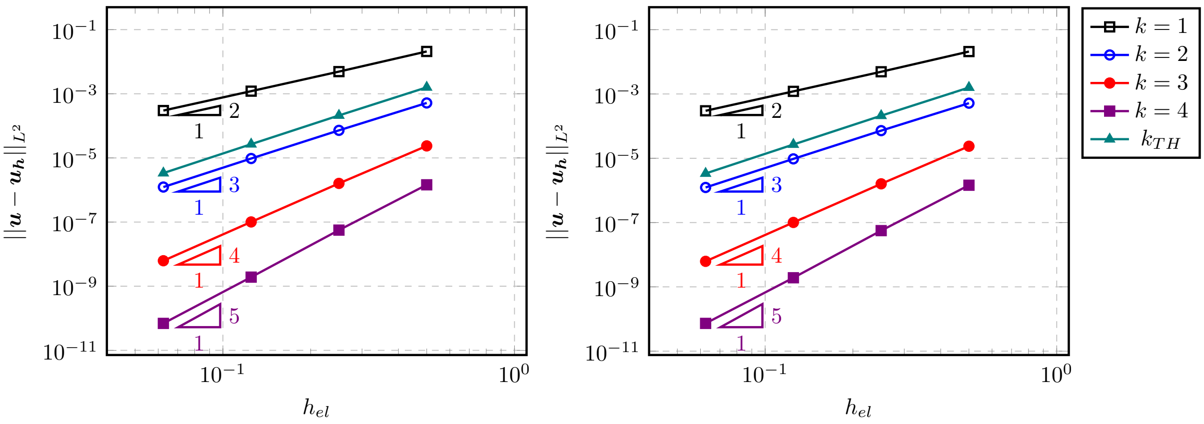
(a) Hdiv-C (b) Hidv-S

Figure 2. Three-dimensional verification test: Convergence analysis for pressure. k indicates the polynomial order of normal velocities.

4.2 Flow through an obstruction device

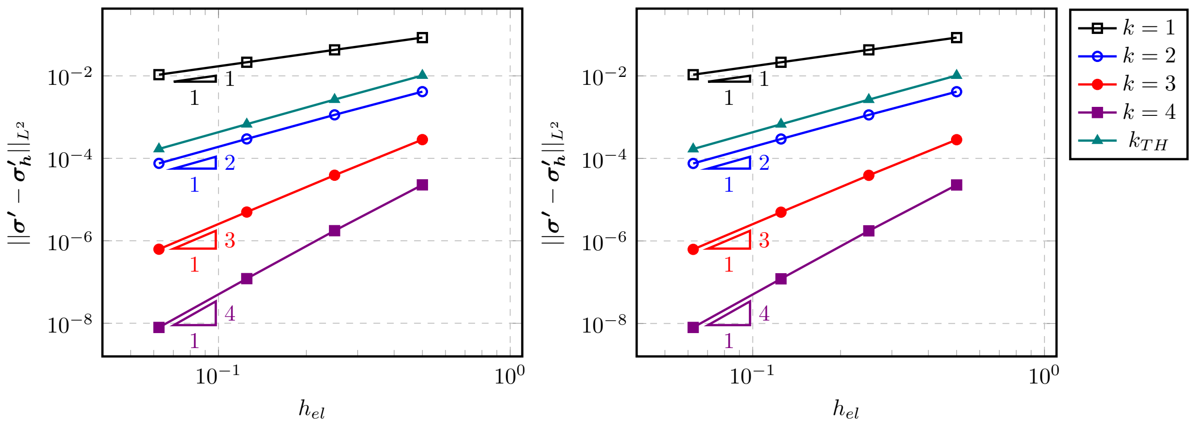
Obstructed flows can be easily solved by the combination of mesh generator algorithms and the double hybrid formulation. An example of an obstructed domain is shown next. The following boundary conditions are considered: no-slip on every wall ($\boldsymbol{u} \cdot \boldsymbol{t} = 0$), inlet and outlet normal stress ($(\boldsymbol{\sigma n}) \cdot \boldsymbol{n} = 10$ and $(\boldsymbol{\sigma n}) \cdot \boldsymbol{n} = 0$, respectively), and no-penetration along the channel's axis ($\boldsymbol{u} \cdot \boldsymbol{n} = 0$). The obstruction is placed at the center of the channel, and the flux is not allowed to pass through them ($\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{t} = 0$). Each channel has a unit length and radius and viscosity is also set to $\mu = 1$ (see Figure 6).

The pressure field and the streamlines are both depicted in Figure 7. The most noticeable pressure drop occurs at the circular obstruction. The streamlines show that the flow is accelerated as it passes through the circular region.



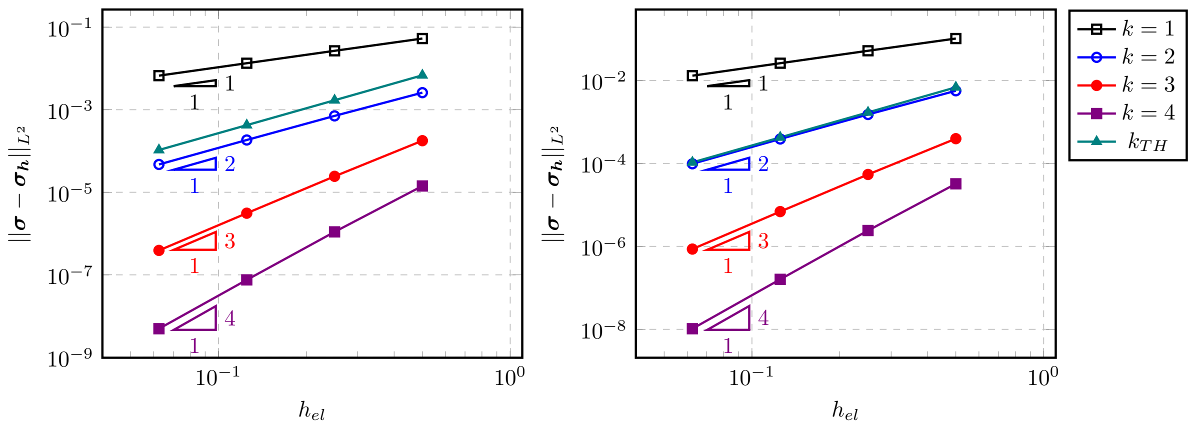
(a) Hdiv-C (b) Hidv-S

Figure 3. Three-dimensional verification test: Convergence analysis for velocity. k indicates the polynomial order of normal velocities.



(a) Hdiv-C (b) Hidv-S

Figure 4. Three-dimensional verification test: Convergence analysis for deviatoric stress. k indicates the polynomial order of normal velocities.



(a) Hdiv-C (b) Hidv-S

Figure 5. Three-dimensional verification test: Convergence analysis for stress. k indicates the polynomial order of normal velocities.

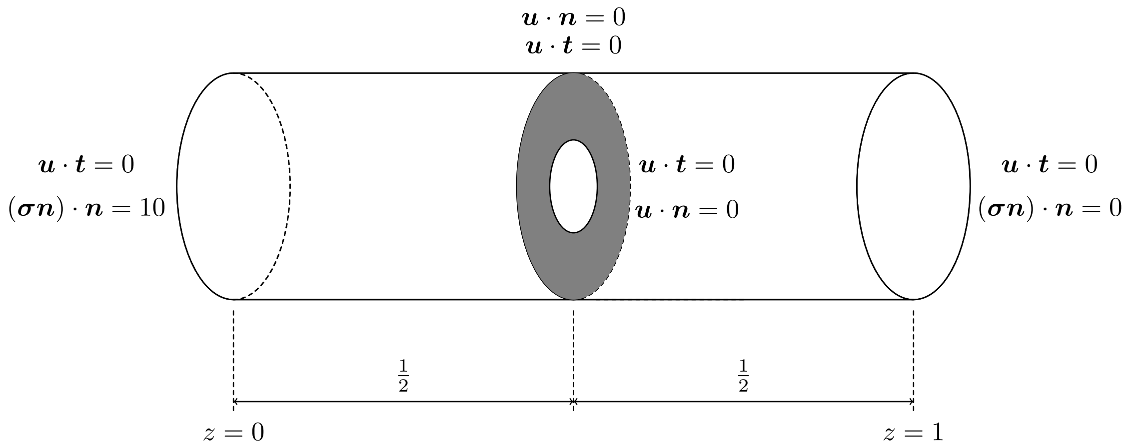
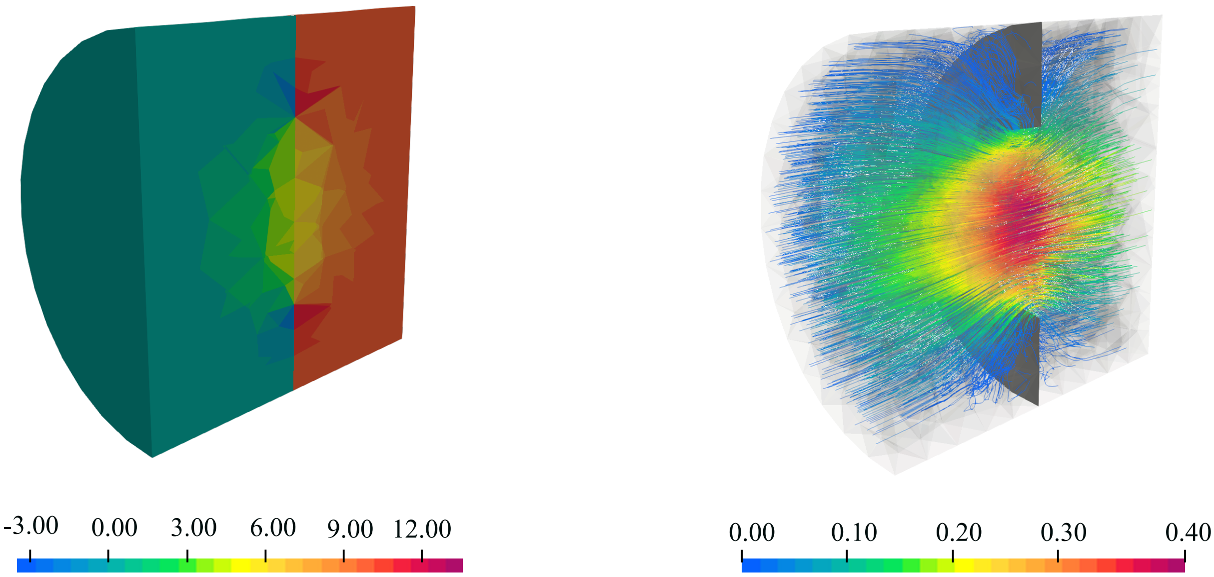


Figure 6. Obstructed domain: geometry and boundary conditions.



(a) Pressure field.

(b) Streamlines.

Figure 7. Results for circular obstruction.

5 Conclusion

The proposed method is optimal for cases in which local mass conservation is paramount. The new double hybrid formulation is a suitable alternative for the semi-hybrid formulation, without compromising computational performance. The $\mathrm{H}(\mathrm{div},\Omega)$ spaces presented have demonstrated to be competitive in terms of error, after applying static condensation, when compared to the Mixed Taylor-Hood. Regarding the matrix structure, the proposed method only has one pressure per element, while the Taylor-Hood presents one pressure per node, resulting in a more complex system.

The condensing process when employing Hdiv-C is straightforward, while for the Hdiv-S additional procedures are required to avoid pivoting during decomposition. As expected, the Hdiv-C results for velocity and deviatoric stress are identical to the results obtained with the Hdiv-S. The approximation error for velocity and deviatoric stress, for both Hdiv-S and Hdiv-C, are better than the Taylor-Hood element scheme. Not to mention the fact that, by its point-wise conservation of the mass property, double hybrid formulation combined with the proper choice of the $\mathrm{H}(\mathrm{div},\Omega)$- $\mathrm{L}^2(\Omega)$ spaces leads to a more accurate solution for the divergence of the velocity field than the Taylor-Hood element.

Many examples of applications of the double hybrid formulation are presented in this work to show the range of possibilities that the method can be used. All the examples are solved with the same Object-Oriented code, which is now incorporated into the NeoPZ library. The double hybrid results show that the formulation is verified. For the Annular Couette flow problem, the convergence rates for pressure using the Hdiv-C space are super convergent (i.e. more precise than expected). This is possible because the analytical solution is included in the approximation space in the Hdiv-C space. The results verify the accuracy of the formulation.

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