A micromechanical approach for homogenization of multiphase elastic periodic composites

Eduardo N. Lages, Severino P. C. Marques

Laboratory of Scientific Computing and Visualization, Center of Technology, Federal University of Alagoas
Campus A. C. Simoes, Maceió, 57072-900, Alagoas, Brazil.
enl@ctec.ufal.br, smarques@ctec.ufal.br

Abstract. This paper presents a novel micromechanical procedure for the linear elastic homogenization of composites with periodic microstructures. The procedure is developed for composites with an arbitrary number of phases and geometric shapes of the inhomogeneities, in contrast with most existing homogenization approaches. Also, no restriction is made in relation to the mismatch between the properties of the phases and volume fractions of the inhomogeneities. The proposed procedure is based on the Eshelby equivalent inclusion approach and extends a model originally derived for evaluating the effective elastic moduli of periodic two-phase composites. The procedure represents the fluctuating elastic fields within each multiphase repeating unit cell (RUC) using Fourier series, resulting in Lippmann-Schwinger integral equations governing the unknown eigenstrain fields of the inclusions. Unlike traditional iterative algorithms used in Fast Fourier Transform (FFT)-based approaches, the procedure solves the integral equations straightforwardly from a scheme of partition of the domain of each inclusion. The efficiency of the proposal procedure is demonstrated through applications to composites with different arrays of coated fibers and constituent materials.

Keywords: elastic homogenization, periodic multiphase composite, eigenstrain, equivalent inclusion.

1 Introduction

The great interest in materials with special properties manifested by the more diverse industrial sectors has motivated a considerable increase in the investigations on composites. This is justified because the composites can be designed and built to exhibit advanced and multifunctional performances. In the last decades, a large number of experimental and theoretical studies have been developed aiming at understanding the behavior of new and existing composites [1,2]. Among the theoretical approaches with this objective, the analytical models based on the mean-field micromechanics deserve to be highlighted [2]. These micromechanical models are constructed from the equivalent inclusion method proposed by Eshelby [3] assuming statistical homogeneity at the macroscopic scale. Meantime, studies have shown that the traditional mean-field approaches usually provide different results among themselves for composites with high volume fractions of inhomogeneities or large mismatches between the properties of the constituent phases, even for two-phase composites with relatively simple microstructures. Then, the evaluation of the effective elastic behavior of composites exhibiting such conditions requires more elaborated micromechanical approaches [4]. A common category of composites includes those constituted by inhomogeneities periodically or regularly distributed inside the matrix. Many approaches have been proposed to evaluate the effective properties of periodic composites, most based on the behavior of a repeating unit cell subjected to periodic boundary conditions. Many of these homogenization approaches use numerical tools, such as finite-element method (FEM) [5], finite-volume theory (FVT) [6] and Fast Fourier Transform (FFT)-based algorithms [7]. Considering the periodicity of the fluctuating fields in a RVE of periodic composite under homogeneous boundary conditions, Fourier series have been conveniently employed in the construction of analytical micromechanical models for homogenization of such composite materials [2]. It is worth noting that most of the analytical models mentioned above are developed for two-phase composites reinforced with inhomogeneities of particular geometric shapes. Nonetheless, homogenization procedures for composites with three or more phases are very important and often needed. For periodic multiphase composites with more sophisticated microstructures, numerical homogenization models based on the finite-element method or finite-volume method, for instance, can be employed. However, such procedures require the use of discretization meshes over all the phases of the RUC domain that, depending on the composite microstructures, can demand high level of refinement.

The current work presents a theoretical approach for predicting the effective elastic moduli of multiphase/multilayer composites with periodic microstructures. Such composites may have an arbitrary number of constituent phases with no restriction relative to the geometric form of the inhomogeneities. The model formulation is based on the concept of eigenstrain field [3] and uses Fourier series to represent the elastic fluctuating fields in the repeating unit cell that characterizes the material microstructure. The problem resulting is governed by a set of Lippmann-Schwinger integral equations whose solution is obtained by an approximate straightforward procedure in the current work [8]. The ability of the proposed model is shown through applications in numerical examples of three-phase composites with distinct microstructural characteristics.

2 Homogenization model for periodic elastic multiphase composites

2.1 Generalization of the elastic equivalent inclusion method

Consider a representative volume element (RVE) of a periodic multiphase composite, with volume V and surface S, subjected to a homogeneous boundary condition defined by the displacement

$$\boldsymbol{u}^0(\boldsymbol{x}) = \boldsymbol{E}^{\boldsymbol{0}}\boldsymbol{x} \qquad \text{for } \boldsymbol{x} \in S \tag{1}$$

where $\boldsymbol{E}^{\boldsymbol{0}}$ and $\boldsymbol{x}$ indicate a constant strain matrix and the coordinate vector of points in V, respectively. The composite microstructure is characterized by a repeating unit cell (RUC) composed of a matrix embedding inhomogeneities Ω_r $(r = 1,2,\cdots N)$ with arbitrary shapes and spatial distribution (Fig. 1(a)). The matrix and inhomogeneities are homogeneous and linear elastic with stiffness matrices $\boldsymbol{C}$ and $\boldsymbol{C}_r$ $(r = 1,2,\cdots N)$, respectively.

The displacement field inside the RUC can be expressed in two-scale representation as

$$\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{E}^{\boldsymbol{0}}\boldsymbol{x} + \tilde{\boldsymbol{u}}(\boldsymbol{y}) \tag{2}$$

where $\boldsymbol{y}$ represents the RUC local coordinates, the first term on the right side corresponds to the macroscopic contribution and $\tilde{\boldsymbol{u}}$ is the periodic fluctuating displacement vector generated by the presence of the inhomogeneities. Here, $\tilde{\boldsymbol{u}}$ is expanded in a Fourier series in the form [2]

$$\tilde{\boldsymbol{u}}(\boldsymbol{y}) = \sum_{\xi}^{\pm\infty} \hat{\boldsymbol{u}}(\boldsymbol{\xi})\exp\,(i\boldsymbol{\xi}\cdot\boldsymbol{y}) \tag{3}$$

where $\hat{\boldsymbol{u}}(\boldsymbol{\xi}) = \frac{1}{U}\int_U \tilde{\boldsymbol{u}}(\boldsymbol{y})\exp\,(-i\,\boldsymbol{\xi}\cdot\boldsymbol{y})dU$ and the components of the vector $\boldsymbol{\xi}$ are defined by $\xi_k = \pi n_k/a_k$, $(k = 1,2,3)$, $2a_k$ indicating the RUC side dimensions (Fig. 1) and $n_k = 0, \pm 1, \pm 2, \ldots \pm\infty$. It is worth remarking that $\boldsymbol{\xi} = \boldsymbol{0}$ must be excluded in the summation, because the constant displacement field is considered in the macroscopic contribution of Eq. (2). Here, the RUC volume is denoted by U. The RUC strain field can be derived from Eq. (2) in the form $(\boldsymbol{y}) = \boldsymbol{\varepsilon}^0 + \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{y})$, where $\boldsymbol{\varepsilon}^0$ is the macroscopic strain vector related to the matrix $\boldsymbol{E}^{\boldsymbol{0}}$ and $\tilde{\boldsymbol{\varepsilon}}$ is the fluctuating strain field corresponding to the displacement field $\tilde{\boldsymbol{u}}$. The total strain field $\tilde{\boldsymbol{\varepsilon}}$ is related to the fluctuating strain fields $\tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{r}}$ $(r = 1,2,\cdots N)$ generated by the inhomogeneities Ω_r, i.e.

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{y}) = \sum_{r=1}^{N} \tilde{\boldsymbol{\varepsilon}}_r(\boldsymbol{y}) \tag{4}$$

The generalization of the equivalent inclusion method consists in replacing each inhomogeneity Ω_r by its corresponding eigenstrain field $\boldsymbol{\varepsilon}_r^*(\boldsymbol{x})$ imposed on the homogenized unit cell. Then, the actual repeating unit cell is substituted by an equivalent unit cell constituted by the matrix material subjected to the same boundary conditions and eigenstrain fields $\boldsymbol{\varepsilon}_r^*(\boldsymbol{x})$ corresponding to the inhomogeneities Ω_r, as shown in Fig. 1(b). The primary problem is to determine the volume-averaged eigenstrains $\bar{\boldsymbol{\varepsilon}}_r^*$ $(r = 1,2,\cdots N)$ over the inclusion domains imposing the elastic equivalency between the original RUC and the homogenized RUC shown in Fig. 1. This equivalency is enforced by the following consistency conditions involving the equality of the local stress fields in the two systems:

$$\boldsymbol{C}_s\left[\boldsymbol{\varepsilon}^0+\sum_{r=1}^{N}\tilde{\boldsymbol{\varepsilon}}_r(\boldsymbol{y})\right]=\boldsymbol{C}\left[\boldsymbol{\varepsilon}^0+\sum_{r=1}^{N}\tilde{\boldsymbol{\varepsilon}}_r(\boldsymbol{y})-\boldsymbol{\varepsilon}_s^*(\boldsymbol{y})\right]\quad \text{for } \boldsymbol{y}\in\Omega_s\ (s=1,2,\cdots N) \tag{5}$$

Similarly, the fluctuating strain vector can be expressed in Fourier series as

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{y})=\sum_{\boldsymbol{\xi}}^{\pm\infty}\hat{\boldsymbol{\varepsilon}}(\boldsymbol{\xi})\exp\,(i\boldsymbol{\xi}\cdot\boldsymbol{y}) \tag{6}$$

where $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{\xi})=\frac{1}{U}\int_U \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{y})\exp\,(-i\,\boldsymbol{\xi}\cdot\boldsymbol{y})dU$. Applying the relation strain-displacement in Eq. (2) and using Eq. (6), results $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{\xi})=i\boldsymbol{L}^{\varepsilon}(\boldsymbol{\xi})\hat{\boldsymbol{u}}(\boldsymbol{\xi})$ with $\boldsymbol{L}^{\varepsilon}(\boldsymbol{\xi})$ representing a (6×3) matrix [8].

The equilibrium equation of the stress field $\boldsymbol{\sigma}(\boldsymbol{y})$ in the RUC, neglecting the body forces, can be expressed in the form $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{y})=\boldsymbol{0}$, where $\boldsymbol{\nabla}$ represents the well-known differential del operator. Considering this equilibrium condition and the equivalent inclusion strategy shown in Fig. 1(b), results from Eq. (5)

$$\boldsymbol{\varepsilon}^0=-(\boldsymbol{C}_s-\boldsymbol{C})^{-1}\boldsymbol{C}\boldsymbol{\varepsilon}_s^*(\boldsymbol{y})-\sum_{\boldsymbol{\xi}}^{\pm\infty}\boldsymbol{S}(\boldsymbol{\xi})\boldsymbol{C}\left[\sum_{r=1}^{N}\frac{1}{U}\int_{\Omega_r}\boldsymbol{\varepsilon}_r^*(\boldsymbol{y}')\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{y}')\,d\Omega_r\right]\exp(i\boldsymbol{\xi}\cdot\boldsymbol{y}) \tag{7}$$

for $s=1,2,\cdots N$. Here, $\boldsymbol{S}(\boldsymbol{\xi})=\boldsymbol{L}^{\varepsilon}(\boldsymbol{\xi})[\boldsymbol{L}^{\sigma}(\boldsymbol{\xi})\boldsymbol{C}\boldsymbol{L}^{\varepsilon}(\boldsymbol{\xi})]^{-1}\boldsymbol{L}^{\sigma}(\boldsymbol{\xi})$ with $\boldsymbol{L}^{\sigma}(\boldsymbol{\xi})=\boldsymbol{L}^{\varepsilon}(\boldsymbol{\xi})^t$.

The solution to Eq. (7) is carried out through an approximate scheme that directly evaluates the average values of the eigenstrain fields, $\bar{\boldsymbol{\varepsilon}}_r^*$, over the inclusion domains Ω_r. This can be justified because the used homogenization approach only requires the average values of the strain fields. In the present work, the straightforward approximate solution of Eq. (7) is obtained using an efficient and general strategy that consists in partitioning the domain Ω_r of each inclusion into M_r partitions $\Omega_r^{(j)}$ $(j=1,2,\ldots M_r)$, inside which the variable eigenstrain field $\boldsymbol{\varepsilon}_r^*$ is substituted by its average value $\bar{\boldsymbol{\varepsilon}}_r^{*(j)}$ defined by

$$\bar{\boldsymbol{\varepsilon}}_r^{*(j)}=\frac{1}{\Omega_r^{(j)}}\int_{\Omega_r^{(j)}}\boldsymbol{\varepsilon}_r^*\,(\mathbf{y})d\Omega_r^{(j)} \tag{8}$$

As shown in the reference [8], this partition scheme is particularly important for the cases of composites with high volume fractions of inhomogeneities and strong contrasts between the properties of the constituent phases. Employing the mentioned partition scheme, Eq. (7) can be rewritten in the form

$$c_s^{(j)}\boldsymbol{\varepsilon}^0=-c_s^{(j)}(\boldsymbol{C}_s-\boldsymbol{C})^{-1}\boldsymbol{C}\bar{\boldsymbol{\varepsilon}}_s^{*(j)}-\sum_{r=1}^{N}\frac{1}{U\Omega_s}\sum_{m=1}^{M_r}\sum_{\xi}^{\pm\infty}\boldsymbol{S}(\boldsymbol{\xi})\boldsymbol{C}\,g_r^{(m)}(-\boldsymbol{\xi})g_s^{(j)}(\boldsymbol{\xi})\bar{\boldsymbol{\varepsilon}}_r^{*(m)} \tag{9}$$

where $c_s^{(j)}=\Omega_s^{(j)}/\Omega_s$ $(s=1,2,\ldots N)$ and

$$g_s^{(j)}(\boldsymbol{\xi})=\int_{\Omega_s^{(j)}}\exp(i\boldsymbol{\xi}\cdot\boldsymbol{y})\,d\Omega_s^{(j)}\qquad\qquad g_r^{(m)}(-\boldsymbol{\xi})=\int_{\Omega_r^{(m)}}\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{y})\,d\Omega_r^{(m)} \tag{10}$$

It is observed that Eq. (9) corresponds to a system of $6n$ linear equations with $6n$ unknowns, which are the components of $\bar{\boldsymbol{\varepsilon}}_r^{*(m)}$ of each partition $\Omega_r^{(m)}$ and $n=\sum_{r=1}^{N}M_r$. After the solution of these system of equations, the average eigenstrain vector for the r-th inclusion, $\bar{\boldsymbol{\varepsilon}}_r^*$, can be readily obtained by the summation

$$\bar{\boldsymbol{\varepsilon}}_r^*=\sum_{m=1}^{M_r}c_r^{(m)}\bar{\boldsymbol{\varepsilon}}_r^{*(m)} \tag{11}$$

with $c_r^{(m)}=\Omega_r^{(m)}/\Omega_r$. For computation convenience, the system of equations (9) can be expressed in the compact form

$$\bar{\boldsymbol{\varepsilon}}^*=\boldsymbol{L}^{-1}\boldsymbol{F}\boldsymbol{\varepsilon}^0 \tag{12}$$

where $\bar{\boldsymbol{\varepsilon}}^*=\left[\bar{\boldsymbol{\varepsilon}}_1^{*(1)}\ \bar{\boldsymbol{\varepsilon}}_1^{*(2)}\ldots\bar{\boldsymbol{\varepsilon}}_1^{*(M_1)},\ \bar{\boldsymbol{\varepsilon}}_2^{*(1)}\ \bar{\boldsymbol{\varepsilon}}_2^{*(2)}\ldots\bar{\boldsymbol{\varepsilon}}_2^{*(M_2)},\ldots\bar{\boldsymbol{\varepsilon}}_N^{*(1)}\ \bar{\boldsymbol{\varepsilon}}_N^{*(2)}\ldots\bar{\boldsymbol{\varepsilon}}_N^{*(M_N)}\right]_{(6\times 6n)}^t$, $\boldsymbol{F}=[\boldsymbol{F}_1^t\ \boldsymbol{F}_2^t\ldots\boldsymbol{F}_N^t]_{(6\times 6n)}^t$, $\boldsymbol{F}_r=\left[c_r^{(1)}\boldsymbol{I}\ \ c_r^{(2)}\boldsymbol{I}\ldots c_r^{(M_r)}\boldsymbol{I}\right]_{(6\times 6M_r)}^t$ and $\boldsymbol{L}$ is a $(6n\times 6n)$ matrix with components

$$\boldsymbol{L}_{ij}^{(r,s)} = -c_i^{(r)}(\boldsymbol{C}_r - \boldsymbol{C})^{-1}\boldsymbol{C}\delta_{ij}\delta_{rs} - \frac{1}{U\Omega_r}\sum_{\xi}^{\pm\infty} \boldsymbol{S}(\boldsymbol{\xi})\boldsymbol{C}g_j^{(s)}(-\boldsymbol{\xi})g_i^{(r)}(\boldsymbol{\xi}) \tag{13}$$

$(r, s = 1, 2, \cdots N;\ i = 1, 2 \cdots M_r;\ j = 1, 2, \cdots M_s)$. δ_{ij} and δ_{rs} mean the Kronecker delta, $\boldsymbol{I}$ indicates the (6×6) identity matrix and the superscript t stands for the transpose of a matrix. It is observed from Eq. (12) that the average eigenstrain vector of the r-th inclusion $\bar{\boldsymbol{\varepsilon}}_r^*$ can be directly related to the macroscopic strain vector $\boldsymbol{\varepsilon}^0$ in the form

$$\bar{\boldsymbol{\varepsilon}}_r^* = \boldsymbol{D}_r\boldsymbol{\varepsilon}^{\mathbf{0}} \tag{14}$$

where $\boldsymbol{D}_r$ is a submatrix readily obtained by the product $\boldsymbol{L}^{-1}\boldsymbol{F}$. See Ref. [8] for more details.

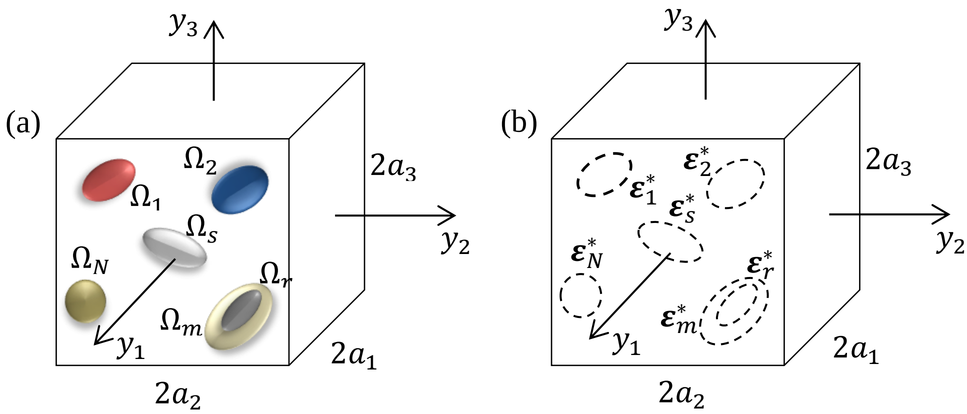

Figure 1. Generalized equivalent inclusion method; (a) Repeating unit cell with N inhomogeneities; (b) Homogenized unit cell embedding inclusions subjected to the eigenstrain fields.

2.2 Effective elastic stiffness tensor of multiphase composites

If $\bar{\boldsymbol{C}}$ represents the effective stiffness matrix of the material, the constitutive relation of the homogenized composite is defined in the form $\bar{\boldsymbol{\sigma}}_U = \bar{\boldsymbol{C}}\bar{\boldsymbol{\varepsilon}}_U$, where $\bar{\boldsymbol{\varepsilon}}_U$ and $\bar{\boldsymbol{\sigma}}_U$ are the volume-averaged strain and volume-averaged stress of the RUC. It is noticed that $\bar{\boldsymbol{\varepsilon}}_U = \boldsymbol{\varepsilon}^0$ by the average strain theorem [2]. Considering the equivalent inclusion method (Section 2.1)

$$\bar{\boldsymbol{\sigma}}_U = \frac{1}{U}\int_U \boldsymbol{C}\left(\boldsymbol{\varepsilon}^{\mathbf{0}} + \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{y}) - \sum_{r=1}^{N}\boldsymbol{\varepsilon}_r^*(\boldsymbol{y})\right)dU = \boldsymbol{C}\left(\boldsymbol{\varepsilon}^{\mathbf{0}} - \sum_{r=1}^{N} c_r\bar{\boldsymbol{\varepsilon}}_r^*\right) \tag{15}$$

where $c_r = \Omega_r/U$ indicates the volume fraction of the r-th inhomogeneity in the RUC. Now, introducing (14) into (15), the following relation is obtained for the effective elastic stiffness matrix of the composite:

$$\bar{\boldsymbol{C}} = \boldsymbol{C}\left(\boldsymbol{I} - \sum_{r=1}^{N} c_r\boldsymbol{D}_r\right) \tag{16}$$

It is worth mentioning that the effective stiffness matrix of the composite is readily generated when the matrices $\boldsymbol{D}_r$ are known and, for this, the assemblage of the matrix $\boldsymbol{L}$ plays a crucial role. The determination of the components of this matrix depends on the integrals of Eq. (10), which are evaluated over the subdomains of the inclusion partitions. In the present work, these integrals are computed by an efficient approach described in [8].

3 Numerical examples

3.1 Composites reinforced by unidirectional continuous coated fibers

Initially, the present micromechanical model is used to determine the macroscopic elastic properties of a composite constituted by an isotropic epoxy matrix reinforced by unidirectional long glass fibers distributed in a periodic hexagonal array along the y_1 direction. It is assumed the presence of an interphase layer (third phase) surrounding each fiber, as shown in Fig. 2. The Young's moduli and Poisson's ratios of the fibers and matrix are, respectively, $E_f = 84\,GPa$, $\nu_f = 0.22$ and $E_m = 4\,GPa$, $\nu_m = 0.34$. The fibers have radius $r_f = 8.5\,\mu m$ and volume fraction $c_f = 50\%$. The Young's modulus of the interphase material (E_i) is taken in a range between 4 GPa and 12 GPa, whereas its Poisson's ratio is $\nu_i = 0.34$. The interphase thickness is assumed as $t_i = 1\,\mu m$. Table 1 shows the results generated by the present approach for the effective in-plane Young's modulus $E_2^{eff} = E_3^{eff}$ and shear modulus G_{23}^{eff}, as well as for the effective out-of-plane shear modulus $G_{12}^{eff} = G_{13}^{eff}$, normalized by the corresponding matrix properties, considering five different interphase moduli. For comparison, Tab. 1 also presents the predictions obtained by the finite-element model (PMH) [9] and the elasticity-based homogenization theory LEHT (Locally Exact Homogenization Theory) [10]. As observed, the results generated by the present approach are in very good agreement with those obtained by both the finite-element calculations and the elasticity-based theory LEHT. This example shows that the present approach is capable of providing results with a quality of accuracy similar to that corresponding to the predictions of finite-element method, even using a scheme with few inclusion partitions.

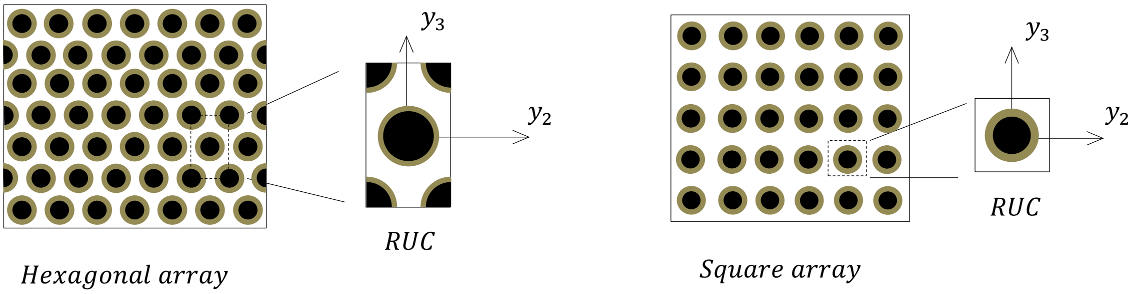

Fig. 2. Repeating unit cells for periodic composite reinforced by continuous fibers with interphase layers and different arrays.

Table 1. Effective elastic moduli for the composite with continuous coated fibers arranged in a hexagonal array.

E_i	E_2^{eff}/E_m			G_{23}^{eff}/G_m			G_{12}^{eff}/G_m		
(GPa)	PMH	LEHT	Present	PMH	LEHT	Present	PMH	LEHT	Present
4	2.6887	2.6636	2.6623	2.5495	2.5195	2.5182	2.7126	2.6362	2.7001
6	2.9112	2.9255	2.9199	2.7751	2.7840	2.7770	2.9367	2.9618	2.9630
8	3.0425	3.0841	3.0767	2.9108	2.9474	2.9379	3.0654	3.1181	3.1188
10	-	-	3.1827	-	-	3.0483	-	-	3.2221
12	3.1916	3.2677	3.2595	3.0667	3.1393	3.1289	3.2083	3.2480	3.2959

The second example aims to show the ability of the present approach for evaluating the effective elastic moduli of periodic composites reinforced by fibers with very thin interphase layer and wide range of fiber volume fraction. For this case, it is considered a composite material reinforced with unidirectional fibers distributed in a square array, as shown in Fig. 2. The elastic moduli of the fiber, matrix and interphase are, respectively: $E_f = 24\,GPa$, $\nu_f = 0.20$; $E_m = 2.7\,GPa$, $\nu_m = 0.35$; and $E_i = 3.03\,GPa$, $\nu_i = 0.50$, as found in [11]. The interphase thickness is equal to $t = 0.001$ and the square RUC has a unit side length, leading to a maximum geometrically possible fiber volume fraction $c_{f,max} \approx 78\%$. Table 2 shows the results of the normalized effective transverse shear modulus G_{23}^{eff}/G_m provided by the present approach, the LEHT [10] and an analytical three-phase model proposed in [11], for fiber volume fraction c_f ranging from 10% to 75%. Considering the wide range of the fiber volume fraction, as well as the very thin interphase layer, the homogenized elastic properties of that composite have been investigated by the present approach employing different partition schemes.

Table 2. Values of G_{23}^{eff}/G_m for the unidirectional composite with very thin interphase.

c_f(%)	10	20	30	40	50	60	70	75
LEHT [10]	1.139	1.287	1.455	1.662	1.939	2.353	3.109	3.852
Three-phase model [11]	1.142	1.293	1.461	1.671	1.948	2.360	3.129	3.817
Present approach	1.140	1.288	1.457	1.664	1.940	2.355	3.106	3.827

3.2 Hybrid composite reinforced by unidirectional continuous different fibers

The next example consists of a hybrid composite with a continuous isotropic matrix reinforced by a hexagonal array of unidirectional long fibers made of two different materials. The composite microstructure is characterized by the rectangular RUC shown in Fig. 3. The elastic moduli of the matrix and fibers are given in Tab. 3. All the fibers are assumed to have the same size. This composite has been analyzed by [12] using polarization approximations for the effective elastic moduli of multicomponent matrix composite, which are deduced from three-point correlation bounds based on minimum energy principles and Hashin–Shtrikman polarization trial fields. Finite-element predictions are also presented in [12] for comparison with the analytical approximations. The results provided by the present approach for the homogenized bulk (K^{eff}) and transverse shear (μ^{eff}) moduli are shown in Fig. 4 in function of the total fiber volume fraction $c_f = c_{f1} + c_{f2}$, with $2c_{f1} = c_{f2}$. Here, c_{fi} denotes the volume fraction of the fibers constituted by the material i. As can be seen in Fig. 4, the results generated by the present approach for those effective elastic moduli are in excellent agreement with the finite-element predictions in the entire wide range of reinforcement volume fractions. Then, considering the quality of those predictions and the computational efficiency, the proposal approach demonstrates be a competitive alternative to the finite-element technique for homogenization of hybrid periodic composites.

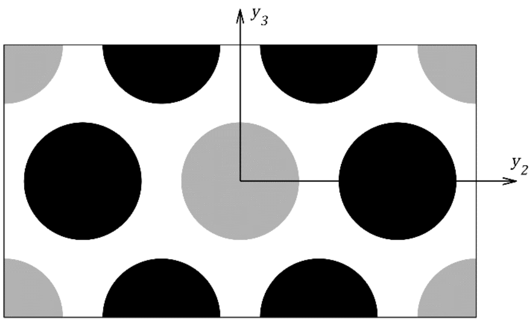

Figure 3. Repeating unit cell representative of the hybrid composite reinforced by two different types of continuous unidirectional fibers in a hexagonal array.

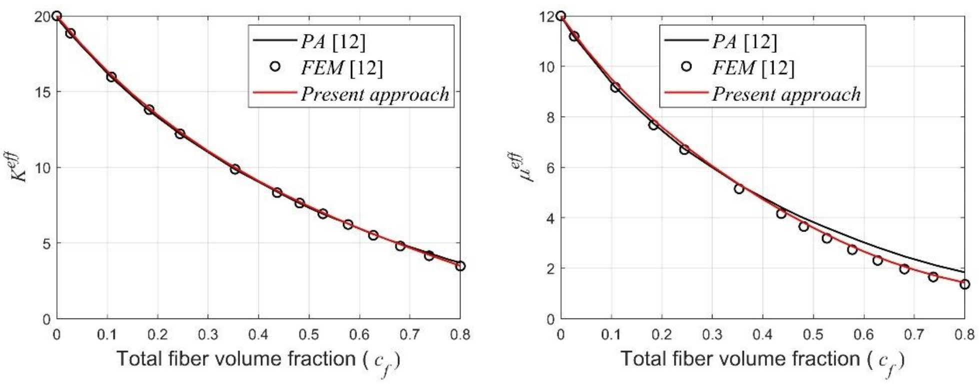

Figure 4. Effective elastic bulk and transverse shear moduli of the hybrid composite.

Table 3. Elastic moduli of the matrix and fibers of the hybrid composite [12].

Phase	Bulk modulus (K)	Shear modulus (μ)
Matrix	20	12
Fiber 1 (black)	4	2
Fiber 2 (gray)	1	0.4

4 Conclusions

An eigenstrain-based model was developed to predict the effective elastic properties of multiphase composites with periodic microstructures presenting an arbitrary number of phases and geometric shapes of the inhomogeneities. The model is capable of accurately evaluating the homogenized elastic properties irrespective of the volume fractions and contrast between the properties of the constituent phases. Comparisons with results obtained from different analytical models and finite element-based homogenization techniques showed that the proposed approach accurately evaluates the homogenized elastic moduli of periodic multiphase composites with various microstructural architectures.

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