A proposal for a unified high-order quasi-3D kinematic model coupled with a zigzag function to model laminated composite beams

Jéssica S. Rodrigues[1], Leonardo F. P. Leite[1], Fabio C. da Rocha[1], Maria do S. M. Sampaio[2], Leslie D. Pérez-Fernández[3], Julián Bravo-Castillero[4]

[1] *Postgraduate Program in Civil Engineering, Federal University of Sergipe*
Av. Marcelo Deda Chagas, 49107-230, São Cristóvão - SE, Brazil
jessica_simão@academico.ufs.br, leolvp@academico.ufs.br, fabiocrocha@academico.ufs.br
[2]*Dept. of Civil Engineering, University of Amazonas State*
Av. Darcy Vargas, 69050-020, Manaus - AM, Brazil
msampaio@uea.edu.br
[3]*Postgraduate Program in Mathematical Modeling, Federal University of Pelotas*
Campus Universitário, Capão do Leão, 96160-000, Pelotas - RS, Brazil
leslie.fernandez@ufpel.edu.br
[4]*Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Alcaldía Álvaro Obregón, Apartado Postal 20-126, 01000, Mérida - CDMX, México*
julian@mym.iimas.unam.mx

Abstract. Laminated composite materials have become increasingly prevalent in various engineering applications due to their remarkable stiffness/mass ratio and ease of manufacture. However, the classical theory fails to address certain limitations in modeling laminated composite beams, such as the absence of shear stress at the top and bottom edges, the non-uniform distribution of the shear stress field, and the zigzag effect in the field axial displacement. This study presents a newly proposed unified high-order quasi-3D kinematic model, which is coupled with a novel zigzag function to overcome the limitations encountered when it uses classical beam theory. This approach eliminates the need for correction factors required in Timoshenko beam theory and accurately captures the cross-section warping and transverse normal deformation resulting from the incorporated quasi-3D effect. Finally, the results were validated by comparing the displacement fields, normal stresses, transverse normal stress, and shear stress with reference values in the literature.

Keywords: unified high-order quasi-3D kinematic, zigzag function; laminated composite beams.

1 Introduction

Composite materials are widely used due to their high stiffness-to-weight ratio, ease of manufacturing and structural efficiency (Chen and Huang [1]). However, the increasing use of laminated composites presents challenges in the analysis of structures, especially due to the zigzag phenomenon in longitudinal displacements caused by the difference in stiffness between layers, requiring a more complex analysis (Gherlone and Marco [2]).

The Euler-Bernoulli beam theory does not consider transverse shear, while the Tymoshenko beam theory includes it, but needs correction factors for the shear stress (Gherlone and Marco [2]). Given this, high-order theories were developed to correctly describe the shear stress distribution without these factors (Gherlone and Marco [2]).

Although these theories are efficient, they can be inaccurate for laminated beams, as they do not consider the zigzag phenomenon in longitudinal displacement, applying only to an Equivalent Cross Section (ELS) (Prado Leite and da Rocha [3]). An alternative to this limitation is the Layerwise Theory, as it considers each layer separately, providing precision, but with high computational cost due to the dependence of the quantity of unknowns on the number of layers (Prado Leite and da Rocha [3], Pathan and Singh [4]).

As an alternative to the limitations presented, the Zigzag Theory emerges, which incorporates the zigzag phenomenon into the ESL theory without losing precision or increasing computational cost. This paper proposes a new zigzag function coupled to a unified formulation for beams with a quasi-3D effect, comparing it with existing formulations and referring to the elasticity theory results by Pagano [5].

2 Mathematical model

2.1 Definitions

Let be a beam of length L, height h and thickness b under the variable distributed loading Q_z, as shown in Figure 1. The superscript T denotes the position of the load on the upper surface of the beam and $T_x(0)$, $T_x(L)$, $T_z(0)$ and $T_z(L)$ are surface forces. The beam is composed of N orthotropic layers perfectly glued together, denoted by the superscript (k).

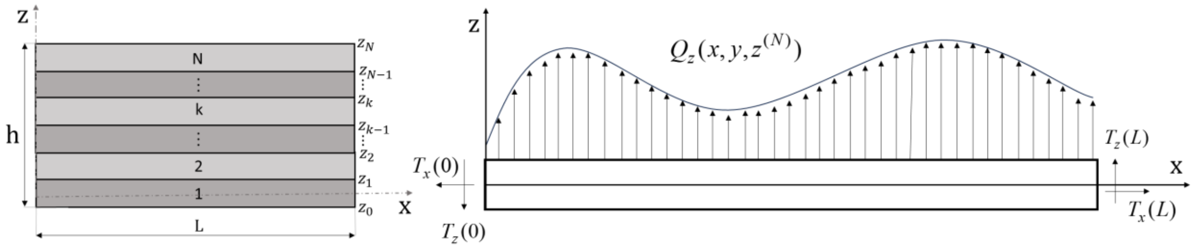

Figure 1. Geometric properties and applied forces of the laminated beam.

2.2 Kinematics

The displacement field proposed in eqs. (1) and (2) incorporate the quasi-3D effect, the zigzag effect, and the cross-section warping:

$$u_x^{(k)}(x,z) = [T_1^{(k)}(z)]\{d_1(x)\} + [T_2(z)]\{d_{2,x}(x)\}, \tag{1}$$

$$u_z(x,z) = [T_g(z)]\{d_2(x)\}, \tag{2}$$

where the subscripted comma notation indicates differentiation with respect to the variable(s) to its right, and

$$\begin{gathered} [T_1^{(k)}(z)] = \begin{bmatrix} 1 & f(z) & \phi_{zz}^{(k)}(z) \end{bmatrix}, \ [T_2(z)] = \begin{bmatrix} -z & 0 \end{bmatrix}, \ [T_g(z)] = \begin{bmatrix} 1 & g(z) \end{bmatrix}, \\ \{d_1(x)\} = \begin{bmatrix} u_0(x) & \phi_s(x) & \psi(x) \end{bmatrix}^T, \ \{d_2(x)\} = \begin{bmatrix} w_0(x) & \varphi_z(x) \end{bmatrix}^T. \end{gathered} \tag{3}$$

The functions $u_0(x)$ and $w_0(x)$ represent, respectively, the axial and the transverse displacements along the beam's centroidal axis. The function $f(z)$ incorporates the high-order theories for beams. The variable $w_{0,x}(x)$ corresponds to the derivative of the elastic line and $\phi_S(x)$ represents the rotation of the normal section to the midplane due to shear. The function $g(z)$ is the derivative of $f(z)$, satisfying the boundaries conditions of upper and lower surfaces free of transverse shear stress. The $\phi_{zz}^{(k)}(z)$ is the zigzag function and $\psi(x)$ is related to the amplitude of the zigzag effect along the length of the beam. The parameter $\varphi_z(x)$ is associated with the quasi-3D effect.

Considering small rotations and displacements in the kinematic model, we have the deformation fields shown in eqs. (4)-(6):

$$\varepsilon_x^{(k)} = u_{x,x} = [T_1^{(k)}(z)]\{d_{1,x}(x)\} + [T_2(z)]\{d_{2,xx}(x)\}, \tag{4}$$

$$\varepsilon_z = u_{z,z} = [T_{g,z}(z)]\{d_2(x)\}, \tag{5}$$

$$\gamma_{xz}^{(k)} = u_{x,z}^{(k)} + u_{z.x} = [T_{1,z}^{(k)}(z)]\{d_1(x)\} + [\bar{T}_2(z)]\{d_{2,x}(x)\}, \tag{6}$$

where $[\bar{T}_2(z)] = [0 \quad g(z)]$.

Considering orthotropic linear elastic material with fiber insertion, the constitutive model for the k-th layer is given by:

$$\sigma_x^{(k)} = \overline{\overline{C}}_{11}^{(k)}[T_1^{(k)}(z)]\{d_{1,x}(x)\} + \overline{\overline{C}}_{11}^{(k)}[T_2(z)]\{d_{2,xx}(x)\} + \overline{\overline{C}}_{13}^{(k)}[T_{g,z}(z)]\{d_2(x)\}, \tag{7}$$

$$\sigma_z^{(k)} = \overline{\overline{C}}_{13}^{(k)}[T_1^{(k)}(z)]\{d_{1,x}(x)\} + \overline{\overline{C}}_{13}^{(k)}[T_2(z)]\{d_{2,xx}(x)\} + \overline{\overline{C}}_{33}^{(k)}[T_{g,z}(z)]\{d_2(x)\}, \tag{8}$$

$$\tau_{xz}^{(k)} = \overline{\overline{C}}_{55}^{(k)}[T_{1,z}^{(k)}(z)]\{d_1(x)\} + \overline{\overline{C}}_{55}^{(k)}[\bar{T}_2(z)]\{d_{2,x}(x)\}, \tag{9}$$

where, $\sigma_x^{(k)}(x,z)$, $\sigma_z^{(k)}(x,z)$ and $\tau_{xz}^{(k)}(x,z)$ represent the normal, the transverse and the shear components of the stress vector; the $\overline{\overline{C}}_{ij}^{(k)}$ are the stiffness constants of the material (Nguyen et al.[6]) and $\varepsilon_x^{(k)}(x,z)$, $\varepsilon_z(x,z)$ and $\gamma_{xz}^{(k)}(x,z)$ represent the normal, the transverse and the shear components of the strain vector.

Table 1 shows the zigzag functions present in the literature and the proposed model applied to the high-order beam model developed in this paper.

Table 1. Zigzag functions.

Models	$\phi_{zz}^{(k)}(z)$
Murakami [7]	$\phi_{MUR}^{(k)}(z) = \dfrac{z - \left(z^{(k-1)} + h^{(k)}\right)}{h^{(k)}}(-1)^k$
Zhen [8]	$\phi_{MUR}^{(k)}(z) - \left[\dfrac{z^2}{2z_0} + \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}\right]\dfrac{d\phi_{MUR}^{(0)}}{dz} - \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}\dfrac{d\phi_{MUR}^{(N)}}{dz}$
Sine wave shape — Leite and da Rocha [3]	$\sin\left(\phi_{MUR}^{(k)}(z)\right) - \left[\dfrac{z^2}{2z_0} + \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}\right]\dfrac{d\phi_{MUR}^{(0)}}{dz} - \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}\dfrac{d\phi_{MUR}^{(N)}}{dz}$
Hyperbolic-exponential shape — Proposed Model	$\sinh\left(\phi_{MUR}^{(k)}(z)\right) - \left[\dfrac{z^2}{2z_0} + \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}\right]e^{\frac{4}{3}h}\dfrac{d\phi_{MUR}^{(0)}}{dz} - \dfrac{2z^3 - 3z_0 z^2}{12z_N^2}e^{\frac{4}{3}h}\dfrac{d\phi_{MUR}^{(N)}}{dz}$

2.3 Governing equations

To obtain the equilibrium equations and their boundary conditions, the principle of minimum energy was used by equating to zero the first variation of the total energy functional. This functional is given by the internal deformation energy, U, and the potential energy due to external forces, V, providing the eq. (10):

$$\delta\Pi = \delta U + \delta V = 0. \tag{10}$$

The first variation of the internal energy and the potential energy are given by eqs. (11) and (12), respectively, with $\Omega = [0, L] \times \left[-\dfrac{b}{2}, \dfrac{b}{2}\right] \times [0, h]$.

$$\delta U = \int_{\Omega} \left(\delta\varepsilon_x^{(k)}\sigma_x^{(k)} + \delta\varepsilon_z^{(k)}\sigma_z^{(k)} + \delta\gamma_{xz}^{(k)}\tau_{xz}^{(k)}\right) dV, \tag{11}$$

$$\delta V = -\left(\int_0^L \{\delta u_z\}^T q_z(x)dx + \int_A \{\delta u_x(L,z)\}^T T_x(L)dA + \int_A \{\delta u_z(L,z)\}^T T_z(L) + \right. \\ \left. -\int_A \{\delta u_x(0,z)\}^T T_x(0)dA - \int_A \{\delta u_z(0,z)\}^T T_z(0)dA \right). \tag{12}$$

Using the integration by parts technique, and the fundamental lemma of variational calculus, it is possible to obtain the Euler Equation, which provides the restriction required for the domain equation (eq. (13)) and its boundary conditions (eq. (14)):

$$\{M_{1,x}\} - \{M_3\} = 0, \quad -\{V_{1,xx}\} - \{M_2\} + \{V_{2,x}\} + [T_g(z)]^T q_z(x) = 0, \tag{13}$$

$$\{d_1\} = \{\bar{d}_1\} ou \{M_1\} = \{\bar{V}_1\}, \quad \{d_{2,x}\} = \{\bar{d}_{2,x}\} ou \{V_1\} = \{\bar{V}_2\}, \quad \{d_2\} = \{\bar{d}_2\} ou \{V_2\} - \{V_{1,x}\} = \{\bar{V}_g\}, \tag{14}$$

where

$$\{M_1\} = \int_A [T_1^{(k)}(z)]^T \left[\sigma_x^{(k)}\right] dA, \quad \{M_2\} = \int_A [T_{g,z}(z)]^T \left[\sigma_z^{(k)}\right] dA, \\ \{M_3\} = \int_A [T_{1,z}^{(k)}(z)]^T \left[\tau_{xz}^{(k)}\right] dA, \quad \{V_1\} = \int_A [T_2(z)]^T \left[\sigma_x^{(k)}\right] dA = \begin{bmatrix} V_z & 0 \end{bmatrix}^T, \\ \{V_2\} = \int_A [\bar{T}_2(z)]^T \left[\tau_{xz}^{(k)}\right] dA, \quad \{\bar{V}_1\} = \int_A \left([T_1(z)]^T T_x(x)\right) dA, \quad \{\bar{V}_2\} = \int_A \left(T_2(z)]^T T_x(x)\right) dA. \tag{15}$$

Replacing eqs. (7)-(9) in eq. (15) and then substituting in eq. (13), we obtain the domain equation given by eq. (16):

$$\begin{cases} [H_1]\{d_{1,xx}(x)\} + [H_2]\{d_{2,xxx}(x)\} + [H_3]\{d_{2,x}(x)\} - [H_4]\{d_1(x)\} - [H_5]\{d_{2,x}(x)\} = 0, \\ [H_9]\{d_{1,xxx}(x)\} + [H_{10}]\{d_{2,xxxx}(x)\} + [H_{13}]\{d_{2,xx}(x)\} + [H_6]\{d_{1,x}(x)\} + [H_7]\{d_{2,xx}(x)\} \\ \quad + [H_8]\{d_2(x)\} - [H_{11}]\{d_{1,x}(x)\} - [H_{12}]\{d_{2,xx}(x)\} = [T_g(z)]^T q_z(x), \end{cases} \tag{16}$$

where

$$[H_1] = \int_A \left([T_1^{(k)}]^T \bar{\bar{C}}_{11}^{(k)} [T_1] \right) dA, \quad [H_2] = \int_A \left([T_1^{(k)}]^T \bar{\bar{C}}_{11}^{(k)} [T_2] \right) dA, \quad [H_3] = \int_A \left([T_1^{(k)}]^T \bar{\bar{C}}_{13}^{(k)} [T_{g,z}] \right) dA, \\ [H_4] = \int_A \left([T_{1,z}^{(k)}]^T \bar{\bar{C}}_{55}^{(k)} [T_{1,z}] \right) dA, \quad [H_5] = \int_A \left([T_{1,z}^{(k)}]^T \bar{\bar{C}}_{55}^{(k)} [\bar{T}_2] \right) dA, \quad [H_6] = \int_A \left([T_{g,z}]^T \bar{\bar{C}}_{13}^{(k)} [T_1] \right) dA, \\ [H_7] = \int_A \left([T_{g,z}]^T \bar{\bar{C}}_{13}^{(k)} [T_2] \right) dA, \quad [H_8] = \int_A \left([T_{g,z}]^T \bar{\bar{C}}_{33}^{(k)} [T_{g,z}] \right) dA, \quad [H_9] = \int_A \left([T_2]^T \bar{\bar{C}}_{11}^{(k)} [T_1] \right) dA, \\ [H_{10}] = \int_A \left([T_2]^T \bar{\bar{C}}_{11}^{(k)} [T_2] \right) dA, \quad [H_{11}] = \int_A \left([\bar{T}_2]^T \bar{\bar{C}}_{55}^{(k)} [T_{1,z}] \right) dA, \quad [H_{12}] = \int_A \left([\bar{T}_2]^T \bar{\bar{C}}_{55}^{(k)} [\bar{T}_2] \right) dA, \\ [H_{13}] = \int_A \left([T_2]^T \bar{\bar{C}}_{13}^{(k)} [T_{g,z}] \right) dA. \tag{17}$$

2.4 Analytical solution

The Navier procedure is used to validate the proposed formulation for a simply supported beam, such that the unknown functions are represented by the Fourier Series given by eq. (18):

$$u_0 = \sum_{n=1}^{\infty} u_n \cos\alpha_n x, \quad \phi_s = \sum_{n=1}^{\infty} \phi_n \cos\alpha_n x, \quad \psi = \sum_{n=1}^{\infty} \psi_n \cos\alpha_n x, \\ w_0 = \sum_{n=1}^{\infty} w_n \sin\alpha_n x, \quad \varphi_z = \sum_{n=1}^{\infty} \varphi_n \sin\alpha_n x, \quad q_z = \sum_{n=1}^{\infty} Q_n \sin\alpha_n x, \tag{18}$$

with $\alpha_n = n\pi / L$.

Replacing eq. (18) in eq. (3), we have:

$$\{d_1\} = \sum_{n=1}^{\infty}\{\overline{d}_1\}\cos\alpha_n x, \{d_2\} = \sum_{n=1}^{\infty}\{\overline{d}_2\}\sin\alpha_n x, \tag{19}$$

where

$$\{\overline{d}_1\} = \begin{bmatrix} u_n & \phi_n & \psi_n \end{bmatrix}^T, \{\overline{d}_2\} = \begin{bmatrix} w_n & \varphi_n \end{bmatrix}^T. \tag{20}$$

Then, substituting eq. (19) in eq. (16), we obtain:

$$\begin{cases} \left[\overline{H}_1\right]_{3x3}\{\overline{d}_1\} + \left[\overline{H}_2\right]_{3x2}\{\overline{d}_2\} = 0, \\ \left[\overline{H}_3\right]_{2x3}\{\overline{d}_1\} + \left[\overline{H}_4\right]_{2x2}\{\overline{d}_2\} = [T_g(z)]^T Q_n(x), \end{cases} \tag{21}$$

where

$$\begin{gathered} \left[\overline{H}_1\right] = [H_1]\alpha_n^2 + [H_4],\ \left[\overline{H}_2\right] = [H_5]\alpha_n + [H_2]\alpha_n^3 - [H_3]\alpha_n, \left[\overline{H}_3\right] = [H_9]\alpha_n^3 - [H_6]\alpha_n + [H_{11}]\alpha_n, \\ \left[\overline{H}_4\right] = [H_{10}]\alpha_n^4 - [H_{13}]\alpha_n^2 - [H_7]\alpha_n^2 + [H_{12}]\alpha_n^2 + [H_8]. \end{gathered} \tag{22}$$

3 Results and discussion

A comparative analysis was carried out using Pagano [5] as the benchmark solution for comparison. In this analysis, the accuracy of the results for longitudinal displacement and shear stress for both, regular and non-regular layers (2-1-1) was calculated. Additionally, the effect on the results when the shear shape function and the zigzag function in high-order kinematics are varied was analyzed. After analyzing the accuracy when the shear shape function and the zigzag function are varied, the behavior of the results was observed when considering or not the quasi-3D effect. The non-dimensionalized equations were used:

$$U_a^{(k)}(x,z) = \frac{u_x^{(k)}(x,z)E_y}{bhq_0},\quad \tau_{xza}^{(k)}(x,z) = \frac{\tau_{xz}^{(k)}(x,z)}{q_0}. \tag{23}$$

The relative errors for this formulation are presented in Tabs. 2 and 3. The abbreviations Red and Sol, in those Tables, refer to the functions, adapted here, to represent the proposal in Reddy [9] and Soldatos [10], respectively. The abbreviations Mur, Zhen, Sine and Hiper refer, respectively, to the zigzag functions of Murakami [7], Zhen [8], Leite and Da Rocha [3] and the present proposed model. The errors were calculated based on the WAPE (Weighted Absolute Percentage Error) metric, given by eq. (24):

$$WAPE(\%) = 100\frac{\sum_{j=1}^{n}\left|x_j - X_j\right|}{\sum_{j=1}^{n}\left|X_j\right|}. \tag{24}$$

Table 2. WAPE for axial displacements, $U_a(L,z)$.

Analysis type	**Settings**	**Red +Mur**	**Red +Zhen**	**Red +Sine**	**Red +Hiper**	**Sol +Mur**	**Sol +Zhen**	**Sol +Sine**	**Sol +Hiper**
With quasi-3D effect	1-1-1	11,84%	7,41%	7,15%	4,09%	11,78%	7,27%	7,02%	3,91%
	2-1-1	26,80%	16,25%	14,40%	10,62%	26,52%	15,81%	13,97%	10,15%
No quasi-3D effect	1-1-1	11,79%	7,14%	7,12%	4,28%	11,73%	7,00%	7,01%	4,19%
	2-1-1	26,24%	15,34%	13,53%	10,05%	25,97%	14,92%	13,13%	9,64%

Table 3. WAPE for shear stress by equilibrium equation, $\tau_{xza}(0,z)$.

Analysis type	**Settings**	**Red +Mur**	**Red +Zhen**	**Red +Sine**	**Red +Hiper**	**Sol +Mur**	**Sol +Zhen**	**Sol +Sine**	**Sol +Hiper**
With quasi-3D effect	1-1-1	1,44%	2,11%	1,66%	0,93%	1,43%	2,07%	1,61%	0,85%
	2-1-1	5,31%	3,06%	2,86%	2,38%	5,28%	3,01%	2,82%	2,32%
No quasi-3D effect	1-1-1	1,28%	1,91%	1,46%	0,73%	1,26%	1,87%	1,42%	0,66%
	2-1-1	5,40%	2,87%	2,68%	2,20%	5,37%	2,83%	2,64%	2,15%

From Tabs. 2 and 3, it is observed that the Soldatos shear function [10], together with the hyperbolic-exponential zigzag function produced the best results in all response fields, guaranteeing a lower error. This conclusion is corroborated by the qualitative analysis presented in Figs. 2 and 3. In these figures, a stacking in the 1-1-1 configuration is considered, where the image on the left side displays the result of the proposed model and the one on the right side displays the model by Pagano [5].

The model with non-regular height layers maintains a behavior similar to that of regular layers, although it presents larger errors when compared to regular layers. This discrepancy is more pronounced for axial displacement.

Comparing the errors between the analyses with and without the quasi-3D effect, it is clear that the difference among the two types of analyzes was small. For regular layers, the model that considers the quasi-3D effect presented lower values for axial displacement in the WAPE metric. However, for the shear stress fields, the model without the quasi-3D effect presented slightly lower values in the WAPE metric. It appears that the consideration or not of the quasi-3D effect did not cause significant changes in the example analyzed by the equilibrium equation represented in Figs. 2 and 3. Additional results, not presented here, were obtained using the constitutive model. These results indicate that accounting for the quasi-3D effect enhances the accuracy of the stress field predictions.

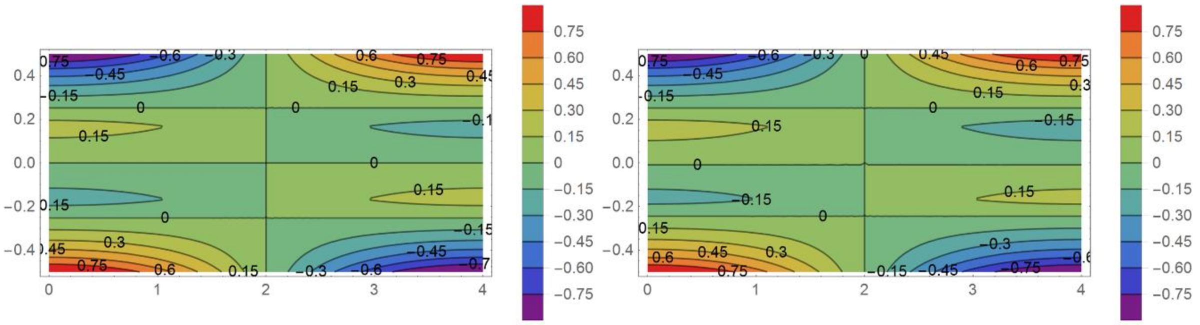

Figure 2. Variation of axial displacement for the 1-1-1 configuration: Sol+Hiper model on the left and model by Pagano [5] on the right.

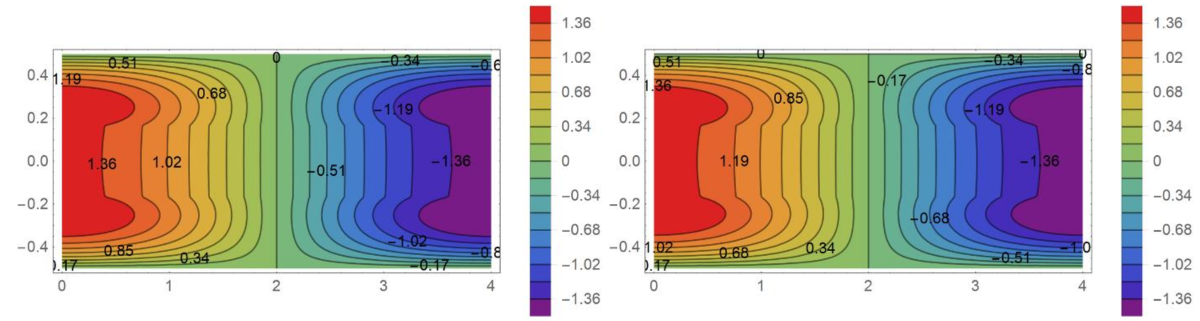

Figure 3. Variation of shear stress by equilibrium equation for the 1-1-1 configuration: Sol+Hiper model on the left and model by Pagano [5] on the right.

4 Conclusions

Analysis of the results reveals that the proposed model, combining the high-order beam theory of Soldatos [10] and the high-order zigzag function developed in this paper, called Sol+Hiper, provides more accurate results in all fields. These results were compared with those obtained by the elasticity theory proposed in Pagano [5] and with other approximate models from the literature, suggesting that the hyperbolic-exponential zigzag function, when combined with the Soldatos shape function, offers a more accurate representation beam behavior, resulting in displacements and stresses fields closer to the reference values.

For displacement fields with regular layer heights, the formulation that considers the quasi-3D effect showed smaller errors when using the hyperbolic-exponential zigzag function, while in other cases, the formulation without the quasi-3D effect had smaller errors. However, the difference among the two types of analysis was not significant, indicating that the quasi-3D effect is not a determining factor for the accuracy of the model in this context.

The model with non-regular height layers, although maintaining a behavior similar to that of regular layers, presented larger errors, especially in the axial displacement. This points to a loss of precision when there is variation in layer heights. The qualitative results indicate good accuracy of the Sol+Hiper model along the beam for regular layer configuration.

Acknowledgments. The authors, J.S.R., and L.F.P.L., would like to thank CAPES for the master's scholarships granted. The authors F.C.R., M.S.M.S., L.D.P.F., and J.B.C., thank CNPq for the financial support through Universal Project No 402857/2021-6. Additionally, the authors F.C.R., L.D.P.F., and J.B.C., would like to thank FAPITEC for the financial support through Universal Project No. 019203.01702/2024-6 and the author F.C.R., extends its thanks to FAPITEC/SE/FUNTEC (Notice No. 02/2024).

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] C. D. Chen and B. F. Huang. “A novel higher-order refined zigzag theory for static bending analysis in sandwich composite beam”. *Applied Mathematical Modelling*, 119, pp. 586-604, Jul 2023.
[2] M. Gherlone, “On the use of zigzag functions in equivalent single layer theories for laminated composite and sandwich beams: A comparative study and some observations on external weak layers”. *Journal of Applied Mechanics*, vol. 80, n. 1, pp. 011004, 2013.
[3] L. F. Prado Leite and F. C. da Rocha, "A novel exponential zigzag function coupled high-order beam theory for advanced laminated composite analysis". *Journal of Applied and Computational Mechanics*, vol. 10, n. 2, pp. 330-341, 2024.
[4] F. Pathan, S. Singh, S. Natarajan, and G. Watts, “An analytical solution for the static bending of smart laminated composite and functionally graded plates with and without porosity”. *Archive of Applied Mechanics*, vol. 92, n. 3, pp. 903-931, 2022.
[5] N. J. Pagano, “Exact solutions for composite laminates in cylindrical bending”. *Journal of Composite Materials*, vol. 3, n. 3, pp. 398-411, 1969.
[6] N. D. Nguyen, T. K. Nguyen, T. P. Vo and H. T. Thai. “Ritz-Based Analytical Solutions for Bending, Buckling and Vibration Behavior of Laminated Composite Beams. *International Journal of Structural Stability and Dynamics*, vol. 18, n. 11, 2018.
[7] H. Murakami, A. Maewal and G. A. Hegemier. “A mixture theory with a director for linear elastodynamics of periodically laminated media”. *International Journal of Solids and Structures*, vol. 17, n. 2, pp. 155-173, 1981.
[8] W. Zhen, C. Yang, H. G. Zhang and X. T. Zheng. “Stability of laminated composite and sandwich beams by a Reddy-type higher-order zig-zag theory”. *Mechanics of Advanced Materials and Structures*, vol. 26, n. 19, p. 1622-1635, 2018.
[9] J. N. Reddy. “A general non-linear third-order theory of plates with moderate thickness”. *International Journal of Non-linear Mechanics*, vol. 25, p. 677-686, 1990.
[10] K. P. Soldatos. “A transverse-shear deformation-theory for homogeneous monoclinic plates”. *Acta Mechanica*, vol. 94, n. 3-4, pp. 195-220, 1992.