Analytical formulae for the higher-order effective coefficients of periodic laminated elastic composites with imperfect contact

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Abstract. Analytical formulae are derived for the effective coefficients of linear elastic second-order laminate composite materials made of any finite number of linear elastic first-order layers. Imperfect contact conditions are considered at the interfaces, and small strains are assumed at the micro and macro scales. The "micro-macro" homogenization procedure used here is reported in Yang et al. [4] wherein only numerical studies are presented. Asymptotic homogenization method results at the micro-scale level are combined with the macro-scale parameters based on the equivalence of the stored energies on the periodic cell (i.e., the energy-averaging theorem known as the Hill-Mandel condition). Some comparisons are considered for validation. To the best of our knowledge, the fully analytical application of this methodology to the case of laminate media has not been reported previously, that is, with the analytical solution of the local problems. The formulas obtained could be useful to control numerical codes in more complex periodic cells.

Keywords: asymptotic homogenization method, higher-order effective coefficients, linear elastic second-order laminate composite, imperfect contact.

1 Introduction

The problem is to determine the effective coefficients of a 1D linear elastic second-order homogeneous material (Mindlin [1]) as a function of the effective coefficient of a 1D linear elastic first-order heterogeneous material at the "micro-scale". The solution combines asymptotic homogenization method results at the micro-scale level by considering the equivalence of the energy at macro- and micro-scales within a periodic cell (Hill [2]). The "micro-macro" method was first reported by Li [3]. We follow the variant developed by Yang et al. [4,5], where the methodology of Li [3] is combined with results from the asymptotic homogenization method (Pobedrya [6]).

2 Brief description of a methodology relating the "micro" and "macro" fields

We will base our calculations on the relations derived from the methodology of Yang et al. [4,5]. Such methodology assumes relationships between the "micro" fields of a linear elastic heterogeneous periodic medium of known elastic properties with mechanical displacement defined by

$$u^m(x,\varepsilon) \sim u^{(0)}(x) + \varepsilon N^{(1)}(y)\frac{du^{(0)}}{dx} + \varepsilon^2 N^{(2)}(y)\frac{d^2u^{(0)}}{dx^2}, \quad y = \varepsilon^{-1}(x - x^c), \tag{1}$$

where x^c is the geometric center of the periodic cell $Y=(0,l)$, ε is the reciprocal of the number of times the periodic cell is replicated in the elastic composite bar. The local functions $N^{(1)}(y)$ and $N^{(2)}(y)$ are the null-average l-periodic solutions of the following equations on $Y-\Gamma$ being $\Gamma = \left\{y^1,\ldots,y^p\right\} \subset Y$ the set of point of discontinuity of $c(y)$ and p is the number of laminas:

$$\frac{d}{dy}\left(c(y)\frac{dN^{(1)}}{dy} + c(y)\right) = 0, \tag{2}$$

$$\frac{d}{dy}\left(c(y)\frac{dN^{(2)}}{dy} + c(y)N^{(1)}(y)\right) = 0. \tag{3}$$

We consider the following imperfect conditions on each point $y^q \in \Gamma \ (q = 1,\ldots,p)$:

$$c(y)\frac{dN^{(1)}}{dy} + c(y) = K^{(q)}\left[\!\left[N^{(1)}(y) \right]\!\right], \quad \left[\!\left[c(y)\frac{dN^{(1)}}{dy} + c(y) \right]\!\right] = 0, \tag{4}$$

$$c(y)\frac{dN^{(2)}}{dy} + c(y) = K^{(q)}\left[\!\left[N^{(2)}(y) \right]\!\right], \quad \left[\!\left[c(y)\frac{dN^{(2)}}{dy} + c(y)N^{(1)}(y) \right]\!\right] = 0, \tag{5}$$

where $K^{(q)}$ is the imperfection constant at $y^q \in \Gamma$ and $[\![\cdot]\!]$ denotes the jump of the enclosed function across the points of Γ.

On the other hand, the energy at the "macro" level (the homogenized medium) is giving by

$$\omega^M(x) = \frac{1}{2}C^M\left(\frac{du^M}{dx}\right)^2 + G^M\frac{du^M}{dx}\frac{d^2u^M}{dx^2} + \frac{1}{2}D^M\left(\frac{d^2u^M}{dx^2}\right)^2, \tag{6}$$

where C^M, G^M, D^M are the second-order effective coefficients to be determined. The energy at the "micro" level is defined by

$$\omega^m(x) = \frac{1}{2}c(y)\left(\frac{du^m}{dx}\right)^2. \tag{7}$$

The "micro-macro" methodology that we will use was reported for the first time in Li [3]. It is based on a series of hypotheses, some of which we summarize below.

(i) The fundamental hypothesis consists of assuming that the deformation energy densities coincide, on a representative volume element, of the "micro" and "macro" media, that is:

$$\left\langle \omega^m(x,\varepsilon)\right\rangle = \left\langle \omega^M(x)\right\rangle \Rightarrow \frac{1}{2}\left\langle c(y)\left(\frac{du^m(x,\varepsilon)}{dx}\right)^2\right\rangle = \left\langle \frac{1}{2}C^M\left(\frac{du^M}{dx}\right)^2 + G^M\frac{du^M}{dx}\frac{d^2u^M}{dx^2} + \frac{1}{2}D^M\left(\frac{d^2u^M}{dx^2}\right)^2\right\rangle, \quad (8)$$

where $\left\langle \varphi(x,y)\right\rangle = \frac{1}{l}\int_0^l \varphi(x,y)dy.$

(ii) It is considered that the "macro" displacements and their derivatives coincide with their corresponding ones at the "micro" level. Then, as stated by Yang et al. [5], as $u^{(0)}$ depends only on the macroscopic coordinates, from eq. (8), by comparing coefficients, we obtain that it may be chosen as the macroscopic displacement $u^{(0)}(x) = u^M(x)$. In particular,

$$u^{(0)}(x) = u^M(x),\ \frac{du^{(0)}}{dx} = \frac{du^M}{dx},\ \frac{d^2u^{(0)}}{dx^2} = \frac{d^2u^M}{dx^2}. \quad (9)$$

(iii) From (9) we obtain that

$$\left\langle u^{(0)}(x)\right\rangle = \left\langle u^M(x)\right\rangle,\ \left\langle \frac{du^{(0)}}{dx}\right\rangle = \left\langle \frac{du^M}{dx}\right\rangle,\ \left\langle \frac{d^2u^{(0)}}{dx^2}\right\rangle = \left\langle \frac{d^2u^M}{dx^2}\right\rangle. \quad (10)$$

(iv) In addition, taking the Taylor series of u^M around the geometric center of the periodic cell, the following results of interest in the connection are concluded:

$$\frac{du^M}{dx} = \left\langle \frac{du^M}{dx}\right\rangle + \left\langle \frac{d^2u^M}{dx^2}\right\rangle(x-x^c), \left\langle \frac{du^M}{dx}\right\rangle = \frac{du^M}{dx}(x^c), \left\langle \frac{d^2u^M}{dx^2}\right\rangle = \frac{d^2u^M}{dx^2}(x^c) = \frac{d^2u^M}{dx^2}. \quad (11)$$

As a consequence, the "macro" effective energy density is of the form

$$\left\langle \omega^M(x)\right\rangle = \frac{1}{2}\left[C^M\left\langle \frac{du^M}{dx}\right\rangle^2 + G^M\left\langle \frac{du^M}{dx}\right\rangle\left\langle \frac{d^2u^M}{dx^2}\right\rangle + \left(C^M\bar{I} + D^M\right)\left\langle \frac{d^2u^M}{dx^2}\right\rangle^2\right], \quad (12)$$

where

$$\bar{I} = \left\langle \left(x-x^c\right)^2\right\rangle. \quad (13)$$

Now, we will explain the dependence of the effective coefficients C^M, G^M, D^M of the "macro" medium (homogenized material) with respect to the local functions $N^{(1)}(y)$ and $N^{(2)}(y)$ appearing in eq. (1). Differentiating in eq. (1) and using the chain rule, we obtain

$$\frac{du^m}{dx} \sim \left(1+\frac{dN^{(1)}}{dy}\right)\frac{du^{(0)}}{dx} + \varepsilon\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right)\frac{d^2u^{(0)}}{dx^2}. \quad (14)$$

Substituting eq. (9) into eq. (14), we have

$$\frac{du^m}{dx} \sim \left(1+\frac{dN^{(1)}}{dy}\right)\frac{du^M}{dx} + \varepsilon\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right)\frac{d^2u^M}{dx^2}. \quad (15)$$

Substituting eq. (10) into eq. (15), results

$$\frac{du^m}{dx} \sim \left(1+\frac{dN^{(1)}}{dy}\right)\left(\left\langle\frac{du^M}{dx}\right\rangle+\left\langle\frac{d^2u^M}{dx^2}\right\rangle(x-x^c)\right)+\varepsilon\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right)\left\langle\frac{d^2u^M}{dx^2}\right\rangle. \tag{16}$$

Substituting $y=\varepsilon^{-1}(x-x^c)$ into eq. (16), we obtain

$$\frac{du^m}{dx} \sim \left(1+\frac{dN^{(1)}}{dy}\right)\left\langle\frac{du^M}{dx}\right\rangle+\varepsilon\left[y\left(1+\frac{dN^{(1)}}{dy}\right)+\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right)\right]\left\langle\frac{d^2u^M}{dx^2}\right\rangle. \tag{17}$$

Introducing the following notations

$$L(y)=1+\frac{dN^{(1)}}{dy},\quad M(y)=yL(y)+\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right)=y\left(1+\frac{dN^{(1)}}{dy}\right)+\left(N^{(1)}(y)+\frac{dN^{(2)}}{dy}\right), \tag{18}$$

so eq. (17) takes the form

$$\frac{du^m}{dx} \sim L(y)\left\langle\frac{du^M}{dx}\right\rangle+\varepsilon M(y)\left\langle\frac{d^2u^M}{dx^2}\right\rangle. \tag{19}$$

Now, we can derive the deformation energy at the “micro” level, as follows:

$$\begin{aligned}\omega^m(x,\varepsilon)&=\frac{1}{2}c(y)\left[\frac{du^m(x,\varepsilon)}{dx}\right]^2\\ &\sim\frac{1}{2}c(y)\left[L(y)\left\langle\frac{du^M}{dx}\right\rangle+\varepsilon M(y)\left\langle\frac{d^2u^M}{dx^2}\right\rangle\right]\left[L(y)\left\langle\frac{du^M}{dx}\right\rangle+\varepsilon M(y)\left\langle\frac{d^2u^M}{dx^2}\right\rangle\right]\\ &=\frac{1}{2}c(y)L^2(y)\left\langle\frac{du^M}{dx}\right\rangle^2+\varepsilon c(y)L(y)M(y)\left\langle\frac{du^M}{dx}\right\rangle\left\langle\frac{d^2u^M}{dx^2}\right\rangle+\frac{\varepsilon^2}{2}c(y)M^2(y)\left\langle\frac{d^2u^M}{dx^2}\right\rangle^2,\end{aligned} \tag{20}$$

where

$$\overline{C}=\left\langle c(y)L^2(y)\right\rangle,\ \overline{G}=2\varepsilon\left\langle c(y)L(y)M(y)\right\rangle,\ \overline{D}=\varepsilon^2\left\langle c(y)M^2(y)\right\rangle. \tag{21}$$

Taking into account eq. (18), we can see that the effective coefficients $\overline{C},\overline{G},\overline{D}$ depend on the small geometrical parameter ε and the solution of the local problems previously defined via the asymptotic homogenization method. With such considerations, from eqs. (12)-(13) and eqs. (20)-(21), the “macro” effective coefficients C^M, G^M, D^M are given by

$$C^M=\overline{C},\quad G^M=\overline{G},\quad D^M=\overline{D}-\overline{C}\,\overline{I},\quad \overline{I}=\left\langle\left(x-x^c\right)^2\right\rangle=\varepsilon^2\left\langle y^2\right\rangle. \tag{22}$$

Remark 1: In order to obtain the final formulas for the effective coefficients it is necessary to solve the two recurrent local problems stated by eqs. (2)-(5). The solutions to such problems can be found, as a particular case, in Appendix A of Chaki and Bravo-Castillero [7], and are given by

$$N^{(1)}(y)=\overline{c}_K\sum_{q=1}^{p}\left(K^{(q)}\right)^{-1}+\int_0^y\left(c^{-1}(\eta)\overline{c}_K-1\right)d\eta-\left\langle\overline{c}_K\sum_{q=1}^{p}\left(K^{(q)}\right)^{-1}+\int_0^y\left(c^{-1}(\eta)\overline{c}_K-1\right)d\eta\right\rangle=-\frac{dN^{(2)}}{dy}, \tag{23}$$

and the effective coefficient of the first order elastic medium with imperfect contact is obtained as

$$\bar{c}_K = \sum_{q=1}^{p} \left[\left(K^{(q)} \right)^{-1} + \left\langle c^{-1}(y) \right\rangle \right]^{-1}. \tag{24}$$

Then, the functions $L(y)$ and $M(y)$ introduced in eq. (18) take the form

$$L(y) = c^{-1}(y)\bar{c}_K, \quad M(y) = yc^{-1}(y)\bar{c}_K. \tag{25}$$

Finally, substituting eq. (25) into eq. (21) and using eq. (24), we obtain:

$$C^M = \bar{c}_K, G^M = 2\varepsilon \left\langle yc^{-1}(y) \right\rangle \left(\bar{c}_K \right)^2, \ D^M = \varepsilon^2 \bar{c}_K \left\langle y^2 \left(\bar{c}_K c^{-1}(y) - 1 \right) \right\rangle. \tag{26}$$

3 Example

In this section we will be compare eq. (26) with eqs. (42) and (43) of Li [3] for two different periodic cells for the particular case of perfect contact condition, that is when $K^{(q)} \to \infty, \forall q$, and $\bar{c}_K \to \left\langle c^{-1}(y) \right\rangle^{-1} \equiv \bar{c}_\infty$.

First, we consider the matrix-inclusion-matrix periodic cell defined by

$$Y = (-l/2, l/2) = (-l/2, -l(1-\theta)/2) \cup (-l(1-\theta)/2, l(1-\theta)/2) \cup (l(1-\theta)/2, l/2) \tag{27}$$

which is illustrated in Figure 1.

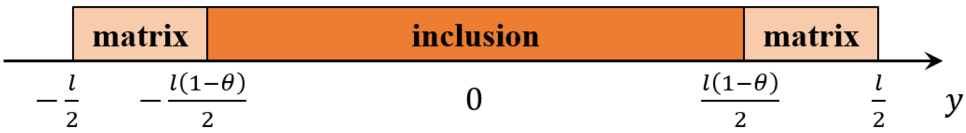

Figure 1. Periodic cell for matrix-inclusion-matrix case.

The matrix and inclusion constituents and the effective coefficient at the micro-level are, respectively,

$$c(y) = \begin{cases} E_m, & y \in (-l/2, -l(1-\theta)/2) \cup (l(1-\theta)/2, l/2) \\ E_i, & y \in (-l(1-\theta)/2, l(1-\theta)/2), \end{cases} \quad C^M \equiv \bar{c}_\infty = \bar{E} = \left[\frac{1-\theta}{E_i} + \frac{\theta}{E_m} \right]^{-1}. \tag{28}$$

Then,

$$D^M = \varepsilon^2 \bar{c}_\infty \left\langle y^2 \left(\bar{c}_\infty c^{-1}(y) - 1 \right) \right\rangle = \frac{\varepsilon^2 \bar{E}^2 \theta_m \theta_i}{12} \left[\frac{1}{E_m} - \frac{1}{E_i} \right] (2-\theta), \tag{29}$$

being $\theta_m = \theta l$, and $\theta_i = (1-\theta)l$, the matrix and inclusion volume fractions, respectively. For the particular case when $\varepsilon^2 = (1+\theta)/(2-\theta)$ from the above formula we obtain

$$D^M = \frac{\theta_m \theta_f \bar{E}^2 (1+\theta)}{12} \left[\frac{1}{E_m} - \frac{1}{E_f} \right], \tag{30}$$

which coincides with eq. (42) of Li [3].

Following similar calculations, we obtain eq. (43) of Li [3] for the matrix-inclusion-matrix-inclusion-matrix-inclusion-matrix periodic cell case as is illustrated in Figure 2.

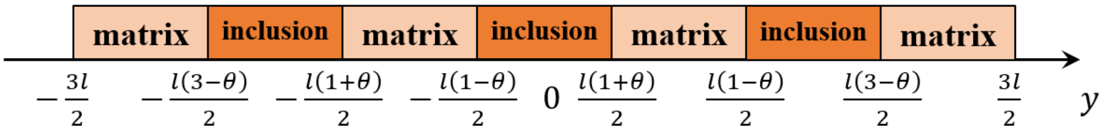

Figure 2. Periodic cell for matrix-inclusion-matrix-inclusion-matrix-inclusion-matrix case.

4 Conclusions

We study a well-established methodology to predict a linear infinitesimal elastic second-order homogenized material from periodic heterogeneous media whose constituents are linear infinitesimal elastic first-order materials. Imperfect contact condition of spring type as interfaces were considered. To the best of our knowledge, the fully analytical application of this methodology to the case of laminated media has not been reported previously, that is, with the analytical solution of the local problems. This methodology can be extended to three-dimensional laminated media with any finite number of layers and imperfect contact at the interfaces.

Acknowledgements. Projects PAPIIT DGAPA UNAM IN101822, Mexico, and CNPq Universal Nº 402857/2021-6, Brazil, are gratefully acknowledged. CAPES Brazil Program (PRAPG) – Edital No 14/2023 is also recognized.

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