Asymptotic homogenization method with fictitious multilayers applied to the mesoscale characterization of concrete beams

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Abstract. The escalating complexity of engineering challenges necessitates the utilization of increasingly efficient materials, often met through the adoption of composite materials as a viable solution. Concrete stands out as one of the most prevalent composite materials in civil engineering, evolving through alterations in its constituent components as researchers pursue enhanced durability, workability, and sustainability. However, traditional theories treating concrete as a homogenous isotropic material prove inadequate for predicting the mechanical properties of these innovative concretes, primarily due to the excessive costs associated with experimental analyses. To address this issue, the present contribution introduces the alternative approach of a high-order zigzag multilayer theory incorporating the asymptotic homogenization method. The variational formulation of a unified beam kinematics is used to carry out a multiscale analysis through the asymptotic expansion of the unknown variables. This methodology allows studying a beam composed of heterogeneous materials through its homogeneous equivalent with the same effective behavior. The findings of the methodology proposed here agree with well-established numerical and experimental formulations documented in the literature, even when employing a one-dimensional beam theory.

Keywords: asymptotic homogenization method, zigzag theory, high order beam theories, concrete beams.

1 Introduction

Composite material is the term given to the combination of two or more materials, at either macroscopic or microscopic levels, to form a new compound aimed at improving properties such as mechanical, thermal, acoustic, corrosion resistance, stiffness, color, weight, and fatigue life (Jones [1]). At the macro- and meso-structural levels, composite materials generally consist of a matrix and some sorts of reinforcement, with various classifications including particle-reinforced composites, fiber-reinforced composites, porous composites, laminates, and laminates with oriented fibers (Vison and Sierakoswski [2]). It is notable that most materials found in nature are heterogeneous at some scale level. For instance, concrete is often modeled as a homogeneous and isotropic material, yet at the mesoscale, it can be differentiated by aggregate type, mortar, void content (including pores, cracks and voids from constituents debonding) and reinforcing fibers. At the microscale, there exists a variety of chemical components in cement influencing the final behavior of the mixture. Therefore, concrete is considered as a particle-reinforced composite material.

In addition to laminates, the modeling of specific composite materials such as concrete is of utmost importance, given its status as the most widely used material in civil construction globally, including in Brazil (Monteiro et al. [3]). Therefore, optimizing the composition of this material is crucial for structural performance. Another critical aspect is that cement production is a significant source of carbon dioxide emissions, contributing to 8% of total pollution (Monteiro et al. [3]). Consequently, research into alternative components for concrete, such as recycled demolition aggregates, alternative types of cement, additives, and natural fibers, has been gaining prominence in laboratory investigations (Cantero et al. [4]). The study of Figueiredo [5] illustrates the effect of adding metallic fibers to concrete, reducing stress concentrations, and thereby mitigating potential structural issues.

This study proposes a methodology that combines the use of unified kinematics with higher-order beam and zigzag theories, along with the application of asymptotic homogenization methods to model heterogeneous problems exhibiting periodic or random characteristics.

2 Mathematical development

2.1 Problem statement

Consider a beam contained in a three-dimensional Cartesian space (x,y,z), with length L, width b and height h, which occupies the domain $\left[0,L\right]\times\left[-b/2,b/2\right]\times\left[-h/2,h/2\right] \ni \left(x,y,z\right)$. The beam in question is laminated with constituents considered orthotropic in the material reference axes, having a volume V; cross-sectional area S; longitudinal elasticity modules E_x, E_z and E_y; shear modules G_{xz}, G_{zy} and G_{xy}; Poisson coefficients v_{xz}, v_{zx}, v_{xy}, v_{yx}, v_{zy} and v_{yz}. In the cross-sectional area, the height of each layer is identified by $2h^{(k)}$, with $k = 1,2\ldots,N$ indicating the number of the layer. The interface coordinates between each layer are $z_{(i)}(i = 0,1,\ldots,N)$, with $z_{(0)} = -h$, $z_{(N+1)} = h$ e $z_{(i)} = z_{(i-1)} + 2h^{(i)}$, as presented in Fig. 1.

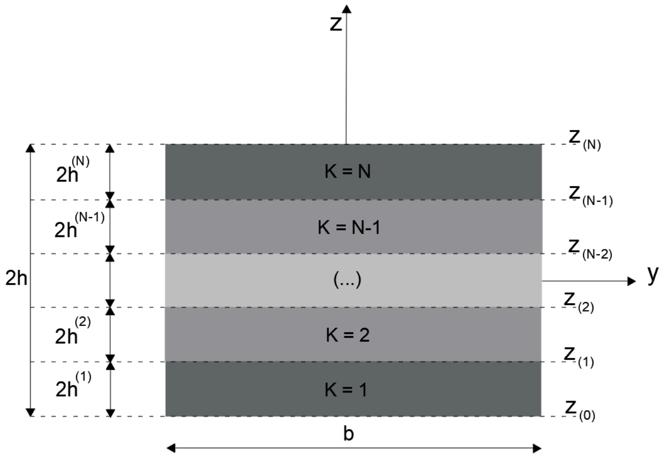

Figure 1. Section characteristics of a generic laminated beam.

2.2 Kinematics

The displacement field defined below is suitable for laminated composite beams and proposed in a unified manner, that is, it encompasses any type of beam theory $f(z)$, for shear distribution (Euler-Bernoulli theory in Sayyad [6], Timoshenko [7], Reddy [8], Kruszewski [9], Touratier [10], Soldatos [11], Karama et al. [12] and Akavci [13]), and any type of zigzag theory $\phi_{zz}^{(k)}(z)$, for layer effects (for example, Murakami et al. [14], Tessler et al. [15], Zhen et al. [16], Prado Leite and Rocha [17,18]):

$$u_x^{(k)}(x,z) = u(x) - z\frac{dw}{dx} + f(z)\phi(x) + \phi_{zz}^{(k)}(z)\psi(x),$$
$$u_z(x,z) = w(x). \tag{1}$$

In eq. (1), considering the Cartesian axes, u_x and u_z are the displacements of the beam along the x and z axes, respectively. The functions u and w are, respectively, the axial and transverse displacements with respect to the beam's centroidal axis. The terms $-\frac{dw}{dx}$ and ϕ represent the angles of rotation due to bending and shear effects, respectively. The function ψ represents the amplitude of the zigzag effect. The present work uses a linear elastic regime and considers small deformations and rotations; therefore, the deformation-displacement and stress-strain relationships are described by eq. (2) and eq. (3), respectively:

$$\varepsilon_x^{(k)}(x,z) = \frac{du}{dx} - z\frac{d^2w}{dx^2} + f(z)\frac{d\phi}{dx} + \phi_{zz}^{(k)}(z)\frac{d\psi}{dx},$$
$$\gamma_{xz}^{(k)}(x,z) = \frac{df}{dz}\phi(x) + \frac{d\phi_{zz}^{(k)}}{dz}\psi(x), \tag{2}$$

$$\sigma_x^{(k)}(x,z) = \overline{Q}_{11}^{(k)}\left[\frac{du}{dx} - z\frac{d^2w}{dx^2} + f(z)\frac{d\phi}{dx} + \phi_{zz}^{(k)}(z)\frac{d\psi}{dx}\right],$$
$$\tau_{xz}^{(k)}(x,z) = \overline{Q}_{55}^{(k)}\left[\frac{df}{dz}\phi(x) + \frac{d\phi_{zz}^{(k)}}{dz}\psi(x)\right], \tag{3}$$

the elasticity matrix $\overline{Q}_{11}^{(k)}$ and $\overline{Q}_{55}^{(k)}$ are calculated as shown in the work of Vison and Sierakowski [2] for orthotropic composite materials.

2.3 Energy functional

Following Reddy [19], the mechanical equilibrium of a structure occurs when the first variation of the total energy functional $\Pi = U_T + \Omega_T$ is zero. Thus, the internal energy U_T from eq. (2) and (3) is presented in eq. (4) and the external energy Ω_T, considering only longitudinally distributed transverse loads $q(x)$, is shown in eq. (6).

$$\begin{aligned} U_T = &\int_0^L A_1(x)\left(\frac{du}{dx}\right)^2 dx + \int_0^L A_2(x)\left(\frac{du}{dx}\frac{d^2w}{dx^2}\right)dx + \int_0^L A_3(x)\left(\frac{du}{dx}\frac{d\phi}{dx}\right)dx \\ &+ \int_0^L A_4(x)\left(\frac{du}{dx}\frac{d\psi}{dx}\right)dx + \int_0^L A_5(x)\left(\frac{d^2w}{dx^2}\right)^2 dx + \int_0^L A_6(x)\left(\frac{d\phi}{dx}\frac{d^2w}{dx^2}\right)dx \\ &+ \int_0^L A_7(x)\left(\frac{d\psi}{dx}\frac{d^2w}{dx^2}\right)dx + \int_0^L A_8(x)\left(\frac{d\phi}{dx}\right)^2 dx + \int_0^L A_9(x)\left(\frac{d\psi}{dx}\frac{d\phi}{dx}\right)dx \\ &+ \int_0^L A_{10}(x)\left(\frac{d\psi}{dx}\right)^2 dx + \int_0^L B_1(x)(\phi(x))^2 dx + \int_0^L B_2(x)(\phi(x)\psi(x))dx + \int_0^L B_3(x)(\psi(x))^2 dx, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &A_1(x) = \frac{1}{2}\int_S \overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_2(x) = -\int_S z\overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_3(x) = \int_S f(z)\overline{Q}_{11}^{(k)}(x,z)dS, \\ &A_4(x) = \int_S \phi_{zz}^{(k)}(z)\overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_5(x) = \frac{1}{2}\int_S z^2\overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_6(x) = -\int_S zf(z)\overline{Q}_{11}^{(k)}(x,z)dS, \\ &A_7(x) = -\int_S z\phi_{zz}^{(k)}(z)\overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_8(x) = \frac{1}{2}\int_S f(z)^2\overline{Q}_{11}^{(k)}(x,z)dS,\ \ A_9(x) = \int_S \phi_{zz}^{(k)}(z)f(z)\overline{Q}_{11}^{(k)}(x,z)dS, \\ &A_{10}(x) = \frac{1}{2}\int_S \phi_{zz}^{(k)}(z)^2\overline{Q}_{11}^{(k)}(x,z)dS,\ \ B_1(x) = \frac{1}{2}\int_S\left(\frac{df(z)}{dz}\right)^2\overline{Q}_{55}^{(k)}(x,z)dS, \\ &B_2(x) = \int_S \frac{df(z)}{dz}\frac{d\phi_{zz}^{(k)}(z)}{dz}\overline{Q}_{55}^{(k)}(x,z)dS,\ \ B_3(x) = \frac{1}{2}\int_S\left(\frac{d\phi_{zz}^{(k)}(z)}{dz}\right)^2\overline{Q}_{55}^{(k)}(x,z)dS. \end{aligned} \tag{5}$$

$$\Omega_T = \int_0^L q(x)w(x)dx. \tag{6}$$

To achieve the commented equilibrium situation, based on eq. (4) and (6), the application of the first energy theorem can be summarized as:

$$\begin{aligned}
&\delta\Pi = \delta\left[U_T + \Omega_T\right] = 0. \\
&\Pi^\xi\left[u^\xi, w^\xi, \phi^\xi, \psi^\xi\right] = \int_0^L A_1{}^\xi \left(\frac{du^\xi}{dx}\right)^2 dx + \int_0^L A_2{}^\xi \frac{du^\xi}{dx}\frac{d^2w^\xi}{dx^2} dx + \int_0^L A_3{}^\xi \frac{du^\xi}{dx}\frac{d\phi^\xi}{dx} dx \\
&\qquad + \int_0^L A_4{}^\xi \frac{du^\xi}{dx}\frac{d\psi^\xi}{dx} dx + \int_0^L A_5{}^\xi \left(\frac{d^2w^\xi}{dx^2}\right)^2 dx + \int_0^L A_6{}^\xi \frac{d\phi^\xi}{dx}\frac{d^2w^\xi}{dx^2} dx \\
&\qquad + \int_0^L A_7{}^\xi \frac{d\psi^\xi}{dx}\frac{d^2w^\xi}{dx^2} dx + \int_0^L A_8{}^\xi \left(\frac{d\phi^\xi}{dx}\right)^2 dx + \int_0^L A_9{}^\xi \frac{d\psi^\xi}{dx}\frac{d\phi^\xi}{dx} dx \\
&\qquad + \int_0^L A_{10}{}^\xi \left(\frac{d\psi^\xi}{dx}\right)^2 dx + \int_0^L B_1{}^\xi \left(\phi^\xi\right)^2 dx + \int_0^L B_2{}^\xi \phi^\xi \psi^\xi dx \\
&\qquad + \int_0^L B_3{}^\xi \left(\psi^\xi\right)^2 dx - \int_0^L q^\xi w^\xi dx.
\end{aligned} \tag{7}$$

The eq. (7) seeks to recognize the oscillating characteristics of the material's elastic properties and consequently its stiffness terms, therefore the unknown functions will also have oscillating responses. Consider ξ as a small parameter indicating a basic length and the superscript ξ indicates that the functions are rapidly oscillating in x, depending on a small parameter that relates a global or slow variable x and a local or fast variable $\overline{x}$. Knowing these oscillating characteristics of the functions and that X is the period, it is possible applies the asymptotic homogenization method to deal with this type of functional.

Asymptotic homogenization method

To deal with this type of functional, it is possible to approximate microscale heterogeneities through periodic behavior, solving local problems related to a repetitive cell and determining equivalent homogeneous properties for the macroscopic scale, through the asymptotic homogenization method (AHM - Bakhvalov and Panasenko [20]). The fundamental idea of AHM is to look for a formal asymptotic solution (FAS) of the original problem (Equation 7) a two-scale expansion in asymptotic series in powers of ξ for x and $\overline{x} = x / \xi$.

By substituting FAS into the original problem, a recurrence of problems is obtained, from which the homogenized problem is determined (it is proved to be independent of $\overline{x}$), the effective coefficients and the local problems on the periodicity cell (element basic microstructural structure whose periodic replication reproduces the structure of the micro-heterogeneous medium). Finally, when ξ tends to zero, the exact solution and the FAS of the original problem converge to the solution of the homogenized problem. From eq. (7), this process is done resulting in the homogenized functional Π^H shown in eq. (8) and its effective coefficients in eq. (9):

$$
\begin{aligned}
\Pi^H = & \int_0^L A_1{}^H \left(\frac{du^{(0)}}{dx}\right)^2 dx + \int_0^L A_2{}^H \frac{du^{(0)}}{dx}\frac{d^2w^{(0)}}{dx^2} dx + \int_0^L A_3{}^H \frac{du^{(0)}}{dx}\frac{d\phi^{(0)}}{dx} dx + \\
& \int_0^L A_4{}^H \frac{du^{(0)}}{dx}\frac{d\psi^{(0)}}{dx} dx + \int_0^L A_5{}^H \left(\frac{d^2w^{(0)}}{dx^2}\right)^2 dx + \int_0^L A_6{}^H \frac{d\phi^{(0)}}{dx}\frac{d^2w^{(0)}}{dx^2} dx + \\
& \int_0^L A_7{}^H \frac{d\psi^{(0)}}{dx}\frac{d^2w^{(0)}}{dx^2} dx + \int_0^L A_8{}^H \left(\frac{d\phi^{(0)}}{dx}\right)^2 dx + \int_0^L A_9{}^H \frac{d\phi^{(0)}}{dx}\frac{d\psi^{(0)}}{dx} dx + \\
& \int_0^L A_{10}{}^H \left(\frac{d\psi^{(0)}}{dx}\right)^2 dx + \int_0^L B_1{}^H \left(\phi^{(0)}\right)^2 dx \int_0^L B_2{}^H \left(\phi^{(0)}\psi^{(0)}\right) dx + \int_0^L B_3{}^H \left(\psi^{(0)}\right)^2 dx + \\
& -\int_0^L q^H w^{(0)} dx,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
& q^H = \frac{1}{X}\int_0^X q(\bar{x})d\bar{x}, \quad A_1{}^H = \frac{1}{X}\int_0^X \frac{\widehat{A_1}^2}{A_1(\bar{x})} d\bar{x} = \widehat{A_1}, \quad A_2{}^H = \frac{1}{X}\int_0^X A_2(\bar{x})\frac{\widehat{A_5}}{A_5(\bar{x})}\frac{\widehat{A_1}}{A_1(\bar{x})} d\bar{x}, \\
& A_3{}^H = \frac{1}{X}\int_0^X A_3(\bar{x})\frac{\widehat{A_1}}{A_1(\bar{x})}\frac{\widehat{A_8}}{A_8(\bar{x})} d\bar{x}, \quad A_4{}^H = \frac{1}{X}\int_0^X A_4(\bar{x})\frac{\widehat{A_1}}{A_1(\bar{x})}\frac{\widehat{A_{10}}}{A_{10}(\bar{x})} d\bar{x}, \\
& A_5{}^H = \frac{1}{X}\int_0^X \frac{\widehat{A_5}^2}{A_5(\bar{x})} d\bar{x} = \widehat{A_5}, \quad A_6{}^H = \frac{1}{X}\int_0^X A_6(\bar{x})\frac{\widehat{A_5}}{A_5(\bar{x})}\frac{\widehat{A_8}}{A_8(\bar{x})} d\bar{x}, \\
& A_7{}^H = \frac{1}{X}\int_0^X A_7(\bar{x})\frac{\widehat{A_5}}{A_5(\bar{x})}\frac{\widehat{A_{10}}}{A_{10}(\bar{x})} d\bar{x}, \quad A_8{}^H = \frac{1}{X}\int_0^X \frac{\widehat{A_8}^2}{A_8(\bar{x})} d\bar{x} = \widehat{A_8}, \\
& A_9{}^H = \frac{1}{X}\int_0^X A_9(\bar{x})\frac{\widehat{A_{10}}}{A_{10}(\bar{x})}\frac{\widehat{A_8}}{A_8(\bar{x})} d\bar{x}, \quad A_{10}{}^H = \frac{1}{X}\int_0^X \frac{\widehat{A_{10}}^2}{A_{10}(\bar{x})} d\bar{x} = \widehat{A_{10}}, \quad B_1{}^H = \frac{1}{X}\int_0^X B_1(\bar{x})d\bar{x}, \\
& B_2{}^H = \frac{1}{X}\int_0^X B_2(\bar{x})d\bar{x}, \quad B_3{}^H = \frac{1}{X}\int_0^X B_3(\bar{x})d\bar{x}.
\end{aligned}
\tag{9}
$$

where,

$$
\widehat{A_1} = \left\langle A_1(\bar{x},z)^{-1} \right\rangle^{-1}, \quad \widehat{A_5} = \left\langle A_5(\bar{x},z)^{-1} \right\rangle^{-1}, \quad \widehat{A_8} = \left\langle A_8(\bar{x},z)^{-1} \right\rangle^{-1}, \ \widehat{A_{10}} = \left\langle A_{10}(\bar{x},z)^{-1} \right\rangle^{-1}. \tag{10}
$$

Now this functional can be solved with numerical techniques more easily because its coefficients are constant with respect to x and no longer rapidly oscillating.

2.4 Results and discussion

To understand the behavior of the results generated by the AHM, a convergence analysis was carried out using a problem involving a beam with oscillating characteristics. A simply supported single-layer beam is considered, subjected to a load uniformly distributed along its length with magnitude $q(x) = 1000$ N/m and dimensions $L = 1\,\text{m}$, $b = 0.1\,\text{m}$ and $h = 0.3\,\text{m}$. Using the shear distribution of Reddy [8] and without considering layer effects ($\phi_{zz}^{(k)}(z) = 0$), the problem was solved for the homogenized solution and four inhomogeneous solutions varying the small parameter (Fig. 2). Also, the types of oscillation that occur in the elastic characteristics of the beam are the Young modulus $E^{\xi}(x) = 210 + 25 sin\left(2\pi x / \xi\right)$ and $G^{\xi}(x) = 80.77 + 10.42 sin\left(2\pi x / \xi\right)$.

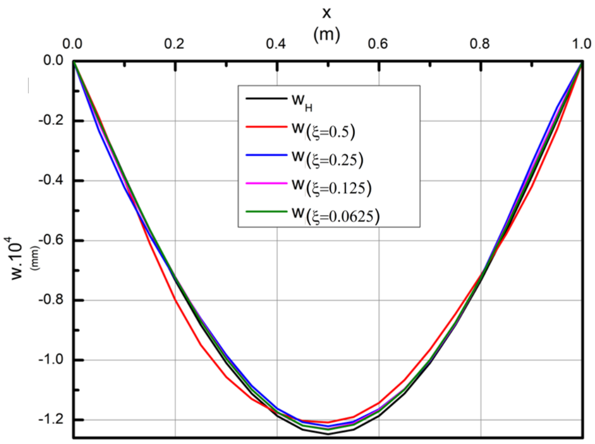

Figure 2. Convergence towards the homogenized solution w_H on maximum deflection of a beam with oscillating characteristics.

It is observed that the smaller ξ (that is, the faster the oscillations of the material characteristics), the better the approximation of the solution w_H the equivalent homogenized problem. In these cases, the homogenized formulation offers great advantages due to its simplicity and lower computational cost. Now, considering the layering effect, it is possible to approximate real rapidly oscillating behaviors. In Taj and Al-Zuhairi [21], a four-point bending experiment was carried out for a concrete test piece with dimensions $450mm \times 150mm \times 150mm$. Then, it is possible to use functions like $E^{\xi(k)}(x) = (E_{aggregate} - E_{mortar}) + E_{mortar} sin\left(T^{(k)}\pi\, x/\xi\right)$ and $G^{\xi(k)}(x) = (G_{aggregate} - G_{mortar}) + G_{mortar} sin\left(T^{(k)}\pi\, x/\xi\right)$ to simulate the mortar-aggregate variation and change the $T^{(k)}$ value, providing a simulation of the randomness material distribution. Table 1 shows the results of a 10-layer simulation, combining some high-order beam theories and zigzag functions. In this case, $T^{(k)} = 16$ for $k = 1,4,7,10$, $T^{(k)} = 24$ for $k = 2,5,8$, and $T^{(k)} = 12$ for $k = 3,6,9$.

Table 1. Results of maximum axial deformation $\varepsilon\left(L/2, h/2\right) \times 10^6$ for the four-point bending of a concrete beam.

Total force (N)	Experimental [21]	Numerical [21]	Reddy [8] + ZZsin [17]	Reddy [8] + ZZexp [18]	Soldatos [11] + ZZsin [17]	Soldatos [11] + ZZexp [18]
1000	3.34	4.82	3.02	3.032	3.017	3.034
5000	16.70	25.04	15.09	15.11	15.08	15.13
12500	41.75	58.64	36.22	36.86	36.21	36.92

It is noted that even though it is an approximation, the homogenization process combined with the multilayer theory allows two-dimensional problems to be modeled even with one-dimensional formulations, maintaining simplicity, and even so obtaining values comparable to the experimental ones in all combinations. All results using combinations of high-order theories were more accurate than the numerical result of Taj and Al-Zuhairi [21] using Euler-Bernoulli beam theory with finite elements. An alternative to approximate the randomness of the internal structure of concrete that can improve the results, consists of considering different representative volume elements with random microstructures as the periodic cell. Another important observation is that for layered problems with more different characteristics, the comparison of high-order theories with each other would be more relevant, as for problems like this the results are very similar.

3 Conclusions

Based on the analysis carried out, it is possible to conclude that the formulation of laminated beams using the asymptotic homogenization method has several applications in structural engineering, especially for materials with heterogeneities at different scales, through a simple approximation of a periodic behavior it is possible to obtain results comparable to experimental ones. Furthermore, the convergence study illustrated that the application is consistent with what is expected from the mathematical homogenization method, that is, problems with rapidly oscillating characteristics are well-posed, including those that model the behavior of concrete.

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