Asymptotic homogenization of a mechanical equilibrium problem of a functionally-graded Euler-Bernoulli beam with non-periodic microstructure

Douglas M. da Silva[1], Leslie D. Pérez-Fernández[2], Julián Bravo-Castillero[3]

[1]*Graduate Program on Mathematical Modeling, Federal University of Pelotas*
Campus Universitário s/n, Prédio 16, 96160-000, Capão do Leão – RS, Brazil
doumach99@gmail.com
[2]*Dept. of Mathematics and Statistics, Federal University of Pelotas*
Campus Universitário s/n, Prédio 5, 96160-000, Capão do Leão – RS, Brazil
leslie.fernandez@ufpel.edu.br
[3]*IIMAS Academic Unit at Yucatan, National Autonomous University of Mexico*
Car. Mérida-Tetiz km 4.5, 97357, Ucú – YU, Mexico
julian@mym.iimas.unam.mx

Abstract. To the best of our knowledge, the few classical applications of Keller's two-space method of non-periodic asymptotic homogenization are related to the effective behavior of heterogeneous media in the context of poroelasticity considering fluid flow and saturation. We believe that this is due to the alternative common approach of approximating random or non-periodic microstructures via the periodic replication of a representative volume element, as periodic structures are, generally speaking, much more tractable mathematically and computationally. However, more than 40 years later, a number of preliminary results of applications of the two-space method has arisen on various areas, namely, effective behavior of composite or functionally-graded bars, approximate solution of the electroencephalogram forward problem for neural imaging activity, and modeling of atmospheric pollutant dispersion. These recent applications deal with second-order elliptic or parabolic equations. In this contribution, we present the application of the two-space method to a mechanical equilibrium problem of a functionally-graded Euler-Bernoulli beam with non-periodic microstructure, which relies on a fourth-order elliptic equation. To the best of our knowledge, homogenization of fourth-order equations has been considered only in the periodic case.

Keywords: Micro-heterogeneous Euler-Bernoulli beam, Non-periodic homogenization, Keller's two-space method

1 Introduction

Micro-heterogeneous materials exhibit both separation of structural scales – which is characterized by the geometric parameter ε, $0 < \varepsilon \ll 1$ – and continuity of matter at the microscale. Under such considerations, the equivalent homogeneity hypothesis guarantees that the effective physical properties of a micro-heterogeneous material are the locally-constant physical properties of its equivalent ideal homogeneous material (Panasenko [1]). The process for obtaining such an equivalent homogeneous material is called *homogenization*.

From the mathematical point of view, the physical behavior of the micro-heterogeneous material is modeled by differential equations whose coefficients show microstructure-induced rapid oscillations. On the other hand, the coefficients of the differential equations that model the behavior of the equivalent homogeneous material do not depend on the microscale and are called *effective coefficients* of the micro-heterogeneous material.

Methods of mathematical homogenization – such as two-scale convergence (Allaire [2], Nguetseng [3]); asymptotic homogenization (Bakhvalov and Panasenko [4], Bensoussan et al. [5], Pobedrya [6]); Σ-, G-, Γ- and H-convergences (de Giorgi [7], Murat and Tartar [8], Nguetseng [9], Spagnolo [10]); second-order tangent homogenization (Ponte Castañeda [11], Ponte Castañeda and Tiberio [12]); and oscillating test functions (Tartar [13]) – suppose that the heterogeneous materials whose models they aim to solve exhibit periodic microstructures. Moreover, a typical approach to address randomly-microstructured materials consists of approximating such non-periodic microstructures by the periodic replication of appropriate representative volume elements (Lipton and Talbot [14], Talbot [15, 16], Talbot and Willis [17]) – as periodicity is, in general, much more tractable mathematically and computationally – so periodic homogenization can be used. Here, the alternative approach to homogenize non-periodic micro-heterogeneous materials via the so-called *two-space method* (TSM – Keller [18]) is adopted.

Initially developed in 1973, Keller's TSM is a non-periodic homogenization method based on the construction of a formal asymptotic solution (FAS) as a two-scale power series of ε. The unknown two-scale coefficients of the powers of ε are sought as the null-average bounded solutions of the recurrence of problems obtained by substituting the proposed FAS into the problem modeling the behavior of the micro-heterogeneous material. Moreover, such a recurrence produces the so-called *homogenized problem* for the first term of the FAS – which model the behavior of the equivalent homogeneous material – and the *local problems* for the local functions carrying the microstructural information into the other terms of the FAS. The homogenization process is mathematically justified whenever the difference between the solutions of the original and homogenized problems is of the order of a positive power of ε with respect to the norm of the space of functions in which they are sought for.

To the best of our knowledge, the few classical applications of the TSM were developed in the contexts of neutron transport and diffusion in nuclear reactors (Larsen [19, 20]) and poroelasticity considering fluid flow and saturation (Keller [21], Burridge and Keller [22]). However, more than 40 years later, a number of preliminary results of applications of the TSM has arisen on various areas, namely, effective behavior of composite (Campos-Suárez [23]) or functionally-graded bars (Pérez-Fernández et al. [24]), approximate solution of the electroencephalogram forward problem for neural imaging activity (Décio Jr. et al. [25]), modeling of atmospheric pollutant dispersion (Pérez-Campos [26]) and derivation of Biot's consolidation equations (Murley [27]). These recent applications deal with second-order elliptic or parabolic equations.

In the present contribution, the TSM is applied to a mechanical equilibrium problem of a functionally-graded Euler-Bernoulli beam with non-periodic microstructure, which relies on a fourth-order elliptic equation. To the best of our knowledge, homogenization of fourth-order problems has been considered only in the periodic case (Huang et al. [28], Kukushkin and Suslina [29], Pastukhova [30, 31], Sloushch and Suslina [32], Suslina [33, 34], Veniaminov [35]).

This work is organized as follows: in section 2, the boundary-value problem under study is stated and its exact solution is given; in section 3, the application of the TSM is developed; in section 4, an illustrative example is presented; finally, some concluding remarks are given in section 5.

2 Problem statement and its exact solution

Let $\varepsilon \in (0, \varepsilon_0)$, $0 < \varepsilon_0 \ll 1$, be a parameter. Consider the non-dimensional version of the problem of the mechanical equilibrium of a non-periodic functionally-graded micro-heterogeneous Euler-Bernoulli beam of unit length, with strictly-positive and bounded locally-oscillating flexural rigidity $EI^\varepsilon \in \mathcal{C}^2(0,1)$ – which means that $EI^\varepsilon(x) = EI(x, x/\varepsilon)$, $EI \in \mathcal{C}^2((0,1)\times(0,n))$, $n \in \mathbb{N}$, $n = 1/\varepsilon$ – clamped at both ends and subjected to the locally-distributed load $q^\varepsilon \in \mathcal{C}(0,1)$ – that is, $q^\varepsilon(x) = q(x, x/\varepsilon)$, $q \in \mathcal{C}((0,1)\times(0,n))$. This problem is stated as follows: for each ε, find the deflection $w^\varepsilon \in \mathcal{C}^4(0,1) \cap \mathcal{C}^1[0,1]$ that solves the boundary-value problem

$$\frac{d^2}{dx^2}\left(EI^\varepsilon(x)\frac{d^2 w^\varepsilon}{dx^2}\right) = q^\varepsilon(x),\ x \in (0,1), \quad w^\varepsilon(x) = \frac{dw^\varepsilon}{dx} = 0,\ x \in \{0,1\}. \tag{1}$$

The exact solution of problem in eq. (1) can be obtained formally by direct integration as

$$w^\varepsilon(x) = \int_0^x \left(C_1^\varepsilon I_1^\varepsilon(\chi) + C_2^\varepsilon I_2^\varepsilon(\chi) + I_3^\varepsilon(\chi)\right) d\chi, \tag{2}$$

where

$$\begin{aligned}
C_1^\varepsilon &= \frac{1}{\Delta^\varepsilon}\int_0^1 \left(I_3^\varepsilon(1)I_2^\varepsilon(\chi) - I_2^\varepsilon(1)I_3^\varepsilon(\chi)\right) d\chi, & I_1^\varepsilon(\chi) &= \int_0^\chi \frac{d\alpha}{EI^\varepsilon(\alpha)},\\
C_2^\varepsilon &= \frac{1}{\Delta^\varepsilon}\int_0^1 \left(I_1^\varepsilon(1)I_3^\varepsilon(\chi) - I_3^\varepsilon(1)I_1^\varepsilon(\chi)\right) d\chi, & I_2^\varepsilon(\chi) &= \int_0^\chi \frac{\alpha d\alpha}{EI^\varepsilon(\alpha)},\\
\Delta^\varepsilon &= \int_0^1 \left(I_2^\varepsilon(1)I_1^\varepsilon(\chi) - I_1^\varepsilon(1)I_2^\varepsilon(\chi)\right) d\chi, & I_3^\varepsilon(\chi) &= \int_0^\chi \frac{1}{EI^\varepsilon(\alpha)}\int_0^\alpha\int_0^\beta q^\varepsilon(\gamma) d\gamma d\beta d\alpha.
\end{aligned} \tag{3}$$

3 Non-periodic asymptotic homogenization via Keller's TSM

The TSM applied to the problem in eq. (1) aims to construct a FAS as

$$w^{\varepsilon}(x) \sim w^{(4)}(x,\varepsilon) = \sum_{k=0}^{4} \varepsilon^{k} w_{k}(x,y),\ (x,y) \in (0,1) \times (0,n), \tag{4}$$

where the local variable $y = x/\varepsilon$ represents the microscale and $w_k \in \mathcal{C}^4((x,y) \in (0,1) \times (0,n))$, $k = \overline{0,4}$, are bounded unknown functions whose derivatives up to the second order are also bounded. Substitution of the FAS in eq. (4) into eq. $(1)_1$ taking the chain rule $d^2/dx^2 = \partial^2/\partial x^2 + 2\varepsilon^{-1}\partial^2/\partial x \partial y + \varepsilon^{-2}\partial^2/\partial y^2$ into account and grouping by powers of ε yields

$$\begin{aligned}
0 &\sim \frac{d^2}{dx^2}\left(EI(x,y)\frac{d^2 w^{(4)}}{dx^2}\right) - q(x,y) \\
&= \varepsilon^{-4}\mathcal{L}^{yy}_{yy} w_0 + \\
&+ \varepsilon^{-3}\left\{\mathcal{L}^{yy}_{yy} w_1 + 2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_0\right\} + \\
&+ \varepsilon^{-2}\left\{\mathcal{L}^{yy}_{yy} w_2 + 2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_1 + \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_0\right\} + \\
&+ \varepsilon^{-1}\left\{\mathcal{L}^{yy}_{yy} w_3 + 2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_2 + \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_1 + 2\left(\mathcal{L}^{xx}_{xy} + \mathcal{L}^{xy}_{xx}\right) w_0\right\} + \\
&+ \varepsilon^{0}\left\{\mathcal{L}^{yy}_{yy} w_4 + 2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_3 + \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_2 + 2\left(\mathcal{L}^{xx}_{xy} + \mathcal{L}^{xy}_{xx}\right) w_1 + \mathcal{L}^{xx}_{xx} w_0 - q(x,y)\right\} + \\
&+ \mathcal{O}(\varepsilon)
\end{aligned}, \tag{5}$$

where $\mathcal{O}(\varepsilon)$ represents the collection of the terms corresponding to the positive powers of ε, and the fourth-order differential operators $\mathcal{L}^{ab}_{cd}$, $a,b,c,d \in \{x,y\}$, are defined as

$$\mathcal{L}^{ab}_{cd}(\cdot) = \frac{\partial^2}{\partial a \partial b}\left(EI(x,y)\frac{\partial^2(\cdot)}{\partial c \partial d}\right). \tag{6}$$

In order for the asymptotic equality in eq. (5) to be satisfied for the homogenization limit $\varepsilon \to 0^+$, the coefficients of the powers of ε must be null, which produces the following recurrence of differential equations for $w_k(x,y)$, $k = \overline{0,4}$:

$$\begin{aligned}
\varepsilon^{-4} &: \mathcal{L}^{yy}_{yy} w_0 = 0, \\
\varepsilon^{-3} &: \mathcal{L}^{yy}_{yy} w_1 = -2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_0, \\
\varepsilon^{-2} &: \mathcal{L}^{yy}_{yy} w_2 = -2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_1 - \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_0, \\
\varepsilon^{-1} &: \mathcal{L}^{yy}_{yy} w_3 = -2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_2 - \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_1 - 2\left(\mathcal{L}^{xx}_{xy} + \mathcal{L}^{xy}_{xx}\right) w_0, \\
\varepsilon^{0} &: \mathcal{L}^{yy}_{yy} w_4 = -2\left(\mathcal{L}^{yy}_{xy} + \mathcal{L}^{xy}_{yy}\right) w_3 - \left(\mathcal{L}^{yy}_{xx} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{xx}_{yy}\right) w_2 - 2\left(\mathcal{L}^{xx}_{xy} + \mathcal{L}^{xy}_{xx}\right) w_1 - \mathcal{L}^{xx}_{xx} w_0 + q(x,y).
\end{aligned} \tag{7}$$

In order to solve the recurrence in eq. (7), the global variable x is treated as a parameter and the mean value operator $\langle\cdot\rangle$ over the microscale, defined for some $y_0 \in (0,n)$ as

$$\langle(\cdot)(\eta)\rangle = \lim_{y \to +\infty} \frac{1}{y - y_0}\int_{y_0}^{y} (\cdot)(\eta) d\eta, \tag{8}$$

is applied conveniently after integrating over (y_0, y) taking boundedness of $w_k(x,y)$, $k = \overline{0,4}$, and their derivatives into account. With such considerations, from eq. $(7)_1$, it follows that $w_0(x,y) = w_0(x)$, that is, the first term of the FAS in eq. (4) does not depend on the microscale, so it represents the mean deflection – which does not depend on the microstructure – for macroscopic/effective behavior. This implies that eq. $(7)_2$ becomes identical to eq. $(7)_1$, so it follows that also $w_1(x,y) = w_1(x)$, so it represents an ε-perturbed contribution to the mean deflection – which contradicts the fact that the mean deflection does not depend on the microstructure – so the only admissible realization is $w_1(x) \equiv 0$. This implies that eq. $(7)_3$ greatly simplifies to $\mathcal{L}^{yy}_{yy} w_2 = -w_0''(x)(\partial^2 EI/\partial y^2)$, which suggests a generalization of *Bakhvalov's ansatz* (Panasenko [1]) to seek its solution as $w_2(x,y) = w_0''(x) N_2(x,y)$, where *local function* $N_2(x,y)$ is the solution of the *first local problem*

$$\mathcal{L}^{yy}_{yy} N_2 = -\frac{\partial^2 EI}{\partial y^2},\ y \in (0,n), \quad \langle N_2(x,\eta)\rangle = 0, \tag{9}$$

that is,

$$N_2(x,y) = \lim_{\xi\to+\infty} \frac{1}{\xi - y} \int_y^\xi \int_y^\eta \lim_{\zeta\to+\infty} \frac{1}{\zeta-\alpha} \int_\alpha^\zeta \int_\alpha^\beta \left(\frac{\widehat{EI}(x)}{EI(x,\gamma)} - 1 \right) d\gamma d\beta d\alpha d\eta, \tag{10}$$

where $\widehat{EI}(x)$ is the *effective flexural rigidity* given by

$$\widehat{EI}(x) = \left\langle \frac{1}{EI(x,\eta)} \right\rangle^{-1} = EI(x,y) \left(\frac{\partial^2 N_2}{\partial y^2} + 1 \right). \tag{11}$$

With such considerations, eq. $(7)_4$ becomes $\mathcal{L}^{yy}_{yy} w_3 = -2\left(w_0'''(x)(\partial^2/\partial y^2)(EI(x,y)\partial N_2/\partial y) + w_0''(x)\mathcal{L}^{yy}_{xy} N_2\right)$, which suggests seeking its solution as $w_3(x,y) = w_0'''(x) N_{31}(x,y) + w_0''(x) N_{32}(x,y)$, where local functions $N_{3j}(x,y)$, $j = 1,2$, are the solutions of the *second local problems*

$$\mathcal{L}^{yy}_{yy} N_{31} = -2\frac{\partial^2}{\partial y^2}\left(EI(x,y)\frac{\partial N_2}{\partial y}\right), \quad \mathcal{L}^{yy}_{yy} N_{32} = -2\mathcal{L}^{yy}_{xy} N_2,\ y \in (0,n), \quad \langle N_{3j}(x,\eta)\rangle = 0,\ j = 1,2, \tag{12}$$

respectively, that is,

$$N_{31}(x,y) = 2\lim_{\xi\to+\infty} \frac{1}{\xi-y}\int_y^\xi \int_y^\eta N_2(x,\alpha) d\alpha d\eta \equiv N_3(x,y), \qquad N_{32}(x,y) = \frac{\partial N_3}{\partial x}, \tag{13}$$

so $w_3(x,y) = w_0'''(x) N_3(x,y) + w_0''(x)(\partial N_3/\partial x) = (\partial/\partial x)(w_0''(x) N_3(x,y))$. Finally, application of $\langle\cdot\rangle$ to the updated eq. $(7)_5$ the so-called *homogenized equation* is obtained which, complemented with the boundary conditions resulting from substituting the SAF in eq. (4) into eq. $(1)_2$, defines the so-called *homogenized problem*

$$\frac{d^2}{dx^2}\left(\widehat{EI}(x)\frac{d^2 w_0}{dx^2}\right) = \widehat{q}(x),\ x \in (0,1), \quad w_0(x) = \frac{dw_0}{dx} = 0,\ x \in \{0,1\}, \tag{14}$$

where $\widehat{q}(x) = \langle q(x,\eta)\rangle$ is the mean load. As the homogenized problem in eq. (14) has the same structure as the original problem in eq. (1), its solution $w_0(x)$ can be obtained from eq. (2) with eq. (3) by changing $EI^\varepsilon(x)$ and $q^\varepsilon(x)$ by $\widehat{EI}(x)$ and $\widehat{q}(x)$, respectively. Observe that the updated eq. $(7)_5$ suggests seeking its solution as $w_4(x,y) = w_0^{\mathrm{iv}}(x) N_{41}(x,y) + w_0'''(x) N_{42}(x,y) + w_0''(x) N_{43}(x,y)$, where local functions $N_{4j}(x,y)$, $j = \overline{1,3}$, are the bounded null-mean solutions of the *third local problems*, which are not presented here for sake of space. However, as the main usefulness of eq. $(7)_5$ is to produce the homogenized equation, eq. $(14)_1$, the corresponding term in the updated eq. (4) can be neglected to produce the SAF $w^{(3)}(x,\varepsilon)$ given by

$$w^\varepsilon(x) \sim w^{(3)}(x,\varepsilon) = w_0(x) + \varepsilon^2 N_2^\varepsilon(x)\frac{d^2 w_0}{dx^2} + \varepsilon^3 \frac{d}{dx}\left(N_3^\varepsilon(x)\frac{d^2 w_0}{dx^2}\right), \tag{15}$$

where $N_j^\varepsilon(x) = N_j(x, x/\varepsilon)$, $j = 2, 3$, are given by eq. (10) and eq. $(13)_1$, respectively, with $y = x/\varepsilon$. In fact, for sufficiently small ε, it suffices to consider the FAS $w^{(2)}(x, \varepsilon)$ obtained by taking the first two terms on the right-hand side of eq. (15), as the contribution of the third term becomes negligible, that is

$$w^\varepsilon(x) \sim w^{(2)}(x, \varepsilon) = w_0(x) + \varepsilon^2 N_2^\varepsilon(x) \frac{d^2 w_0}{dx^2}. \tag{16}$$

4 Example

In order to illustrate the fact that the exact solution w^ε of the original problem in eq. (1) converges to the solution w_0 of the homogenized problem in eq. (14), consider the unit load $q^\varepsilon(x) \equiv 1$ (so the mean load is also $\widehat{q}(x) \equiv 1$) and the flexural rigidity

$$EI^\varepsilon(x) = \frac{\sqrt{1 + \left(\dfrac{x}{\varepsilon}\right)^2}}{1 + \dfrac{x}{\varepsilon}}, \tag{17}$$

whose behavior is presented in Fig. 1 for $\varepsilon \to 0^+$. The corresponding effective flexural rigidity, as calculated via eq. (11), is $\widehat{EI}(x) \equiv 1$.

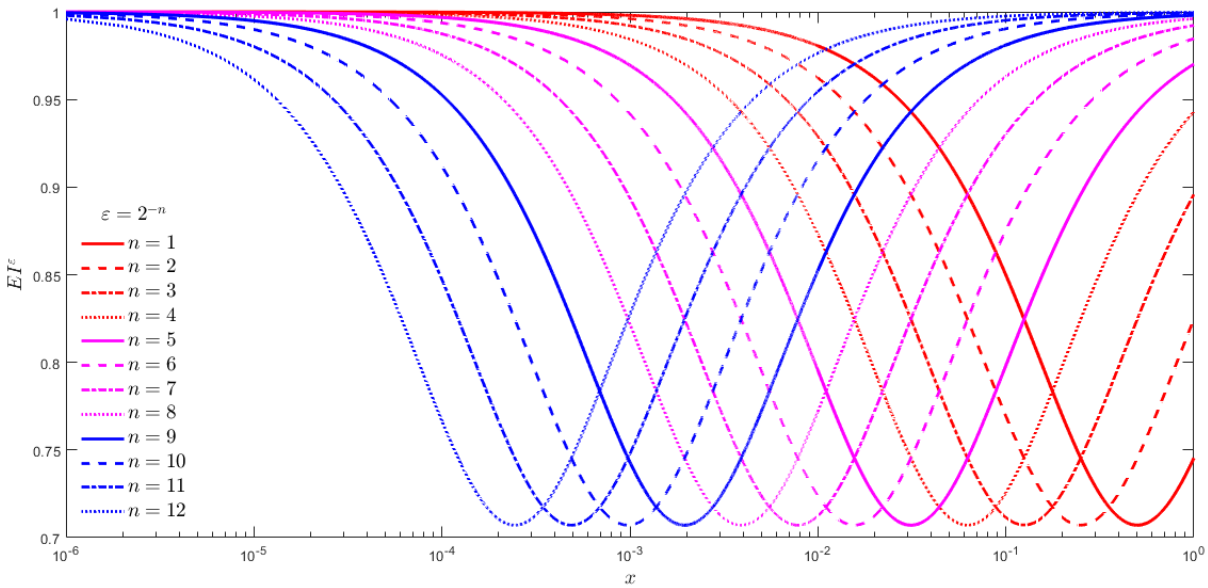

Figure 1. Behavior of flexural rigidity $EI^\varepsilon(x)$ as $\varepsilon \to 0^+$.

In this case, the exact solution w^ε of the original problem in eq. (1) is given by eq. (2) with eq. $(3)_{2,4,6}$ specialized as

$$I_k^\varepsilon(\chi) = \frac{1}{(k-1)!} \int_0^\chi \frac{\alpha^{k-1} d\alpha}{EI^\varepsilon(\alpha)}, \quad k = \overline{1, 3}, \tag{18}$$

with eq. (17), and its behavior is shown in Fig. 2 together with the solution w_0 of the homogenized problem in eq. (14), which in this case is $w_0(x) = x^2(x-1)^2/24$. Observe that, as expected, $w^\varepsilon \to w_0$ as $\varepsilon \to 0^+$.

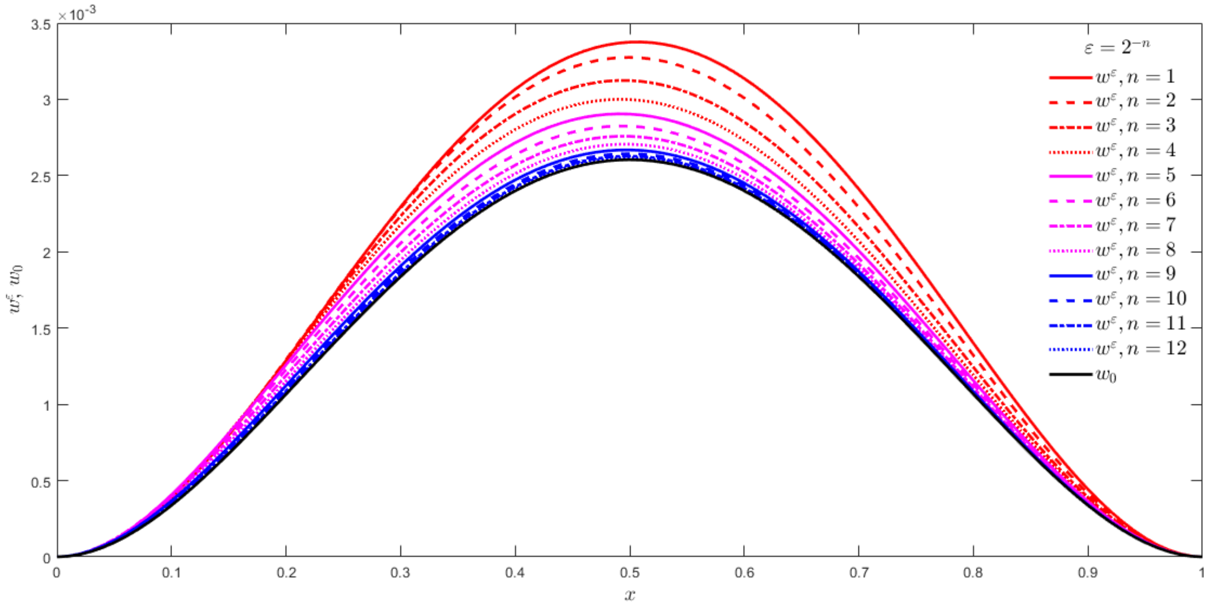

Figure 2. Convergence of the exact solution w^ε to the homogenized solution w_0 as $\varepsilon \to 0^+$.

5 Conclusions

This work presented non-periodic asymptotic homogenization via Keller's TSM applied to a boundary-value problem with a fourth-order differential equation. Such an application seems to be original as, to the best of our knowledge, homogenization of higher-order problems has been addressed only in periodic settings.

Acknowledgements. Support provided by Brazil's National Council for Scientific and Technological Development (CNPq) via Universal Project Nº 402857/2021-6 and Coordination of Superior Level Staff Improvement (CAPES) via PRAPG Program Nº 23038.003836/2023-39, and State of Sergipe's Foundation for Research and Technological Innovation (FAPITEC/SE) via Universal Project Nº 019203.01702/2024-6 is gratefully acknowledged.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] G. P. Panasenko. Homogenization for periodic media: from microscale to macroscale. *Physics of the Atomic Nuclei*, vol. 71, n. 4, pp. 681–694, 2008.
[2] G. Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, vol. 23, n. 6, pp. 1482–1518, 1992.
[3] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, vol. 20, n. 3, pp. 608–623, 1989.
[4] N. S. Bakhvalov and G. P. Panasenko. *Homogenisation: averaging processes in periodic media*. Kluwer, 1989.
[5] A. Bensoussan, J.-L. Lions, and G. Papanicolau. *Asymptotic analysis for periodic structures*. North-Holland, 1978.
[6] B. E. Pobedrya. *Mechanics of Composite Materials*. Moscow University Press, 1984 (in Russian).
[7] E. de Giorgi. G-operators and Γ-convergence. *Proceedings of the International Congress of Mathematicians*, vol. 2, pp. 1175–1191, 1984.
[8] F. Murat and L. Tartar. H-convergence. *Topics in the Mathematical Modelling of Composite Materials. Progress in Nonlinear Differential Equations and Their Applications*, vol. 31, pp. 21–43, 1997.

[9] G. Nguetseng. Homogenization structures and applications i. *Zeitschrift für Analysis und ihre Anwendungen*, vol. 22, n. 1, pp. 73–108, 2003.
[10] S. Spagnolo. Convergence in energy for elliptic operators. *Proceedings of the Symposium on Numerical Solution of Partial Differential Equations*, vol. III, pp. 469–499, 1976.
[11] P. Ponte Castañeda. Second-order homogenization estimates for nonlinear composites incorporating field fluctuations: I-theory. *Journal of the Mechanics and Physics of Solids*, vol. 50, n. 4, pp. 737–757, 2002.
[12] P. Ponte Castañeda and E. Tiberio. A second-order homogenization method in finite elasticity and applications to black-filled elastomers. *Journal of the Mechanics and Physics of Solids*, vol. 48, n. 6-7, pp. 1389–1411, 2000.
[13] L. Tartar. *The General Theory of Homogenization: A Personalized Introduction*. Springer, 2009.
[14] R. Lipton and D. R. S. Talbot. Bounds for the effective conductivity of a composite with an imperfect interface. *Proceedings of the Royal Society of London A*, vol. 457, n. 2010, pp. 1501–1517, 2001.
[15] D. R. S. Talbot. Bounds which incorporate morphological information for a nonlinear composite dielectric. *Proceedings of the Royal Society of London A*, vol. 455, n. 1990, pp. 3617–3628, 1999.
[16] D. R. S. Talbot. Improved bounds for the effective properties of a nonlinear two-phase elastic composite. *Journal of the Mechanics and Physics of Solids*, vol. 48, n. 6-7, pp. 1285–1294, 2000.
[17] D. R. S. Talbot and J. R. Willis. Bounds for the effective constitutive relation of a nonlinear composite. *Proceedings of the Royal Society of London A*, vol. 460, n. 2049, pp. 2705–2723, 2004.
[18] J. B. Keller. Effective behavior of heterogeneous media. *Proceedings of the Symposium on Statistical Mechanics and Statistical Methods in Theory and Applications*, pp. 631–644, 1977.
[19] E. W. Larsen. Neutron transport and diffusion in inhomogeneous media i. *Journal of Mathematical Physics*, vol. 16, pp. 1421–1427, 1975.
[20] E. W. Larsen. Neutron transport and diffusion in inhomogeneous media ii. *Nuclear Science and Engineering*, vol. 60, pp. 357–368, 1976.
[21] J. B. Keller. Darcy's law for flow in porous media and the two-space method. *Nonlinear Partial Equations in Engineering and Applied Science: Lecture Notes in Pure and Applied Mathematics*, vol. 54, pp. 429–443, 1980.
[22] R. Burridge and J. B. Keller. Poroelasticity equations derived from microstructure. *Journal of the Acoustic Society of America*, vol. 70, n. 4, pp. 1140–1146, 1981.
[23] N. E. Campos-Suárez. *Non-periodic homogenization and the two-space method*. BSc in Mathematics Thesis, University of Havana, 2016 (in Spanish).
[24] L. D. Pérez-Fernández, J. Bravo-Castillero, F. C. da Rocha, and M. S. M. Sampaio. Homogenization of a non-periodic functionally-graded bar via the two-space method. *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics*, vol. 11, 2024 (in Portuguese - accepted).
[25] R. M. S. Décio Jr., A. de Cezaro, L. D. Pérez-Fernández, and J. Bravo-Castillero. A preliminary study on the application of the two-space nonperiodic asymptotic homogenization method to the EEG forward problem with continuously differentiable coefficient. *Ciência e Natura*, vol. 45, n. 3, pp. e75138–1–e75138–16, 2023.
[26] I. E. Pérez-Campos. *Mathematical modeling of atmospheric pollutant dispersion via non-periodic homogenization*. BSc in Mathematics Thesis, National Autonomous University of Mexico, 2018 (in Spanish).
[27] J. Murley. *The two-space homogenization method*. MSc in Mathematics Thesis, University of Waterloo, 2012.
[28] Z. W. Huang, Y. F. Xing, and Y. H. Gao. A two-scale asymptotic expansion method for periodic composite euler beams. *Composite Structures*, vol. 241, pp. 112033–1–112033–14, 2020.
[29] A. A. Kukushkin and T. A. Suslina. Homogenization of high order elliptic operators with periodic coefficients. *St. Petersburg Mathematical Journal*, vol. 28, n. 1, pp. 65–108, 2017.
[30] S. E. Pastukhova. Operator error estimates for homogenization of fourth order elliptic equations. *St. Petersburg Mathematical Journal*, vol. 28, n. 2, pp. 273–289, 2017.
[31] S. E. Pastukhova. Approximation of resolvents in homogenization of fourth-order elliptic operators. *Sbornik: Mathematics*, vol. 212, n. 1, pp. 111–134, 2021.
[32] V. A. Sloushch and T. A. Suslina. Homogenization of the fourth-order elliptic operator with periodic coefficients with correctors taken into account. *Functional Analysis and Its Applications*, vol. 54, n. 3, pp. 224–228, 2020.
[33] T. A. Suslina. Homogenization of the neumann problem for higher order elliptic equations with periodic coefficients. *Complex Variables and Elliptic Equations*, vol. 63, n. 7-8, pp. 1185–1215, 2018a.
[34] T. A. Suslina. Homogenization of the dirichlet problem for higher-order elliptic equations with periodic coefficients. *St. Petersburg Mathematical Journal*, vol. 29, n. 2, pp. 325–362, 2018b.
[35] N. Veniaminov. Homogenization of periodic differential operators of high order. *St. Petersburg Mathematical Journal*, vol. 22, n. 5, pp. 751–775, 2011.