Asymptotic homogenization of energy-limited nonlinear microperiodic composites with imperfect interfaces to model failure of masonry structures under uniaxial compression

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Abstract. A masonry prismatic structure made of alternating layers of mortar and bricks is modeled here as a periodic two-phase elastic composite with one-dimensional heterogeneity along the layering direction. Predicting its mechanical failure is important to estimate the admissible stress and stiffness used in its design. At the mortar-brick interfaces, both ideally perfect and spring-type imperfect contacts are considered. Also, both classical and failure behaviors are modeled simultaneously, for which a nonlinear constitutive relation resulting from the substitution of the classical Hookean energy into the so-called softening hyperelasticity energy is adopted. The model is a two-point boundary value problem stated by subjecting the corresponding mechanical equilibrium nonlinear differential equation to the contact conditions at the interfaces and the mixed boundary conditions corresponding to uniaxial compression in the layering direction. The masonry structure exhibits separations of scales, so its mechanical properties are rapidly oscillating, and so it satisfies the equivalent homogeneity hypothesis. Here, the effective law, that is, the constitutive relation of the equivalent homogeneous structure, is obtained via the asymptotic homogenization method. Finally, comparisons with experimental results from the literature are provided, which show qualitative agreement and that the model with imperfect contact is more accurate.

Keywords: Failure of masonry structures, Periodic two-phase elastic composites, Softening hyperelasticity model, Spring-type imperfect contact, Asymptotic homogenization method

1 Introduction

In this work, the mechanical failure under uniaxial compression of masonry prismatic structures made of alternating layers of mortar and bricks is studied (Peralta et al. [1], Yang et al. [2]). Such an heterogeneous structure is considered as a periodic two-phase (mortar-brick) composite with one-dimensional heterogeneity in the layering direction, which is orthogonal to the loading direction. It is assumed that there is a sufficiently large number of mortar-brick bilayers in order to guarantee separation of scales, that is, the height of the bilayer functioning as a periodicity cell, is much smaller than the height of the composite. This assumption, together with the continuity of both material phases at the microscale, ensures that the equivalent homogeneity hypothesis is satisfied. This implies that the solution of the corresponding two-point boundary-value problem (whose differential equation has rapidly oscillating coefficients) can be approximated via some mathematical homogenization method that provides the effective mechanical behavior of the composite via the behavior of its equivalent ideally-homogeneous material whose problem has effective (constant) coefficients (Peralta et al. [1], Drougkas et al. [3]).

Here, both classical and failure mechanical behaviors are modeled simultaneously by considering the constitutive law resulting form the substitution of the classical Hookean energy into the energy of the so-called softening hyperelasticity model by Volokh [4], which makes use of energy limiters in order to account for failure phenomena. In fact, Volokh [5] introduced the approach of energy limiters into the constitutive modeling of materials in the context of fracture of isotropic brittle solids, which is related to the object of this work. Later on, Volokh [6] formalized such and approach in the softening hyperelasticity for modeling materials failure. Since then, several applications of this energy-limiter based model have arisen, for instance, on the subjects of: cavitation under hydrostatic pressure in hyperelastic materials (Lev and Volokh [7], Volokh [8]), multiscale modeling of failure (Volokh [9]), arterial failure and aneurysm rupture (Volokh [10], Volokh and Vorp [11]), dynamic propagation of cracks and failure is elastic and soft materials (Abu-Qbeitah et al. [12], Trapper and Volokh [13]), failure of natural and synthetic rubbers in elastic and thermoelastic settings (Lev et al. [14], Volokh [15, 16]), and failure in elastic and viscoelastic composites under the finite strains (Aboudi and Volokh [17, 18]). To the best of our knowledge, there are no other applications of this model to composites besides our own (Décio Jr. [19]) which applies the asymptotic homogenization method (AHM – Bakhvalov and Panasenko [20]) to nonlinear microperiodic functionally graded materials (Décio Jr. et al. [21]) and composites considering imperfect contact at the interfaces (Décio Jr. et al. [22]) or failure with perfect contact (Décio Jr. et al. [23]). Here, the contribution of the imperfect contact at the brick/mortar interfaces to the mechanical failure of the composite is also considered via the spring-type model, in which tractions orthogonal to the interface are continuous and proportional to the jump in the mechanical displacement across the interface (Hashin [24]). It should be noticed that the applications of the AHM are two-fold: providing accurate approximations of the exact solution, and obtaining the effective coefficients or, at least, the effective stress-strain law, which is an intermediary step of the former and is the mathematical goal of this work.

This work is organized as follows: section 2 presents the formulation of the problem of uniaxial compression of masonry prismatic structures with spring -type imperfect contact at the brick/mortar interfaces, and the formal application of the AHM to that problem; section 3 presents and discusses the results of the AHM application for real brick and mortar constituents in comparison with experimental results from the literature; and section 4 presents some concluding remarks.

2 Methodology

2.1 Problem formulation

Let a masonry prismatic structure of non-dimensional unit height be made by stacking several bricks with mortar layers between them for structural integrity. Assuming that all bricks and mortar layers are respectively equal and homogeneous, such a structure is a periodic two-phase composite with periodicity cell of non-dimensional height $\varepsilon \ll 1$ made of a single brick (phase $r = 1$) and the mortar layer immediately below it (phase $r = 2$), so $\varepsilon = \ell_1 + \ell_2$, where ℓ_r is the non-dimensional height of phase $r = 1, 2$. In order to constitutively account for both mechanical equilibrium and failure, a formal nonlinear stress-strain relation $\sigma^\varepsilon(x, \epsilon^\varepsilon(x))$ is adopted, which reproduces both classical Hookean behavior for small strains and yield for sufficiently large strains. Also, it is assumed that the contact at the brick/mortar interfaces is of linear spring-type, that is, tractions in the stacking direction are continuous at the brick/mortar interfaces $x \in \Gamma^\varepsilon$, whereas the displacement u^ε jumps proportionally to the traction with spring constant $\beta^\varepsilon = \beta/\varepsilon$. The composite is under uniaxial compression in the stacking direction with a unit uniform load at the top ($x = 0$) while fixed at the bottom ($x = 1$) and having no body forces (see Fig. 1).

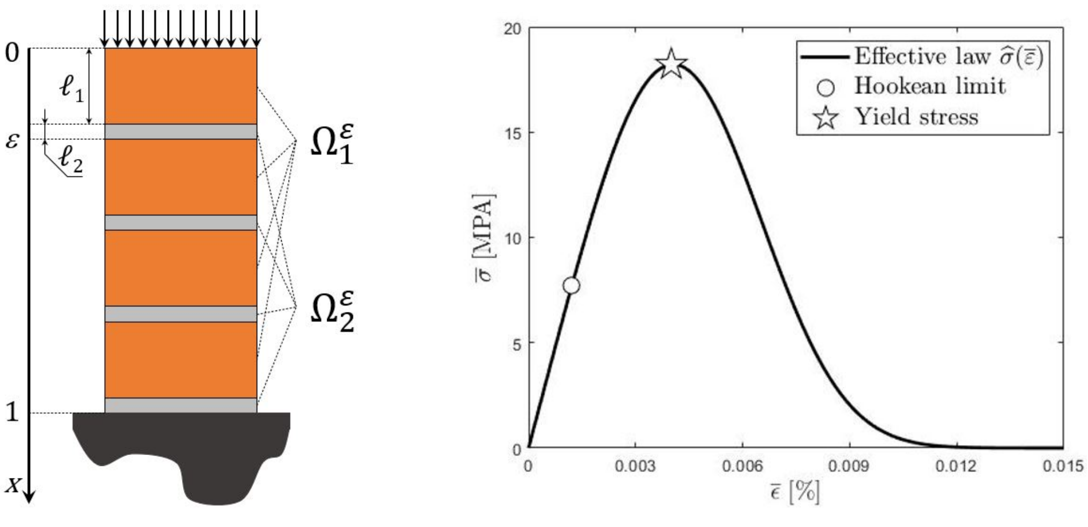

Figure 1. Depiction of the masonry prismatic structure under uniaxial compression for $\varepsilon = 1/4$ (left) and effective law via Volokh's model fitted to an experiment in Peralta et al. [1] (right).

With such considerations, this is a one-dimensional homogenization problem for the effective mechanical equilibrium of the composite with vertical dimension $x \in [0,1] = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$, where Ω_r^ε is the region occupied by phase $r = 1, 2$ and Γ^ε contains the locations of the spring-type imperfect brick/mortar interfaces. Such a problem is stated as follows: find the piecewise twice continuously differentiable displacement u^ε such that

$$\frac{d}{dx}\left[\sigma^\varepsilon\left(x, \frac{du^\varepsilon}{dx}\right)\right] = 0, \quad x \in (0,1) \setminus \Gamma^\varepsilon \tag{1}$$

$$\sigma^\varepsilon\left(x, \frac{du^\varepsilon}{dx}\right) = \sigma_r\left(\frac{du^\varepsilon}{dx}\right), \quad x \in \Omega_r^\varepsilon,\ r = 1, 2, \tag{2}$$

$$\left[\!\!\left[\sigma^\varepsilon\left(x, \frac{du^\varepsilon}{dx}\right)\right]\!\!\right] = 0, \quad x \in \Gamma^\varepsilon, \tag{3}$$

$$[\![u^\varepsilon(x)]\!] = \frac{1}{\beta^\varepsilon}\sigma_1^\varepsilon\left(\frac{du^\varepsilon}{dx}\right) \quad x \in \Gamma^\varepsilon, \tag{4}$$

$$-\sigma_1^\varepsilon\left(\left.\frac{du^\varepsilon}{dx}\right|_{x=0}\right) = 1, \quad u^\varepsilon(1) = 0, \tag{5}$$

where $\epsilon^\varepsilon = du^\varepsilon/dx$ is the strain, $\sigma^\varepsilon(x, \epsilon^\varepsilon) = \sigma(x/\varepsilon, \epsilon^\varepsilon)$ is the nonlinear constitutive stress-strain relation being $\sigma_r(\epsilon^\varepsilon)$ its realization in phase $r = 1, 2$, and $[\![\cdot]\!] = (\cdot)^+ - (\cdot)^-$ is the jump operator at the interfaces $x \in \Gamma^\varepsilon$. Note that $\beta^\varepsilon \to +\infty$ implies ideally perfect contact between mortar and bricks, as the jump of u^ε in eq. (4) becomes zero. Conversely, $\beta^\varepsilon \to 0^+$ implies perfect debonding, as the jump of u^ε becomes unbounded. In particular, it is assumed that both phases follow Volokh's generalized model with energy limiters for Hookean materials, that is,

$$\sigma_r(\epsilon^\varepsilon) = E_r \epsilon^\varepsilon \exp\left\{-\Phi_r^{-m_r} W_r^{m_r}(\epsilon^\varepsilon)\right\}, \quad W_r(\epsilon^\varepsilon) = \frac{1}{2}E_r(\epsilon^\varepsilon)^2, \quad r = 1, 2, \tag{6}$$

where E_r, Φ_r and m_r are the Young modulus, the critical energy and the adjustment parameter of phase $r = 1, 2$. For instance, the plot on the right of figure 1 was obtained with the effective Young modulus $\widehat{E} = 6.43$ GPa calculated via AHM for the configuration on the left of figure 1 for perfect contact ($\beta \to +\infty$) and with $E_1 = 11$ GPa, $E_2 = 2.2$ GPa, $\varepsilon = \ell_1 + \ell_2 = 1/4$, $\ell_1 = 0.211$, $\ell_2 = 0.039$, and adjusted effective critical energy $\widehat{\Phi} = 0.11$ MPa and effective parameter $\widehat{m} = 1.4$ for the uniaxial compression test for specimen 3 of Peralta et al. [1].

2.2 AHM application

The AHM seeks for a formal asymptotic solution (FAS) of problem in eqs. (1)-(5) as a two-scale power series of ε with unknown functional coefficients. Here, it suffices to take the FAS

$$u^\varepsilon(x) \sim u^{(2)}(x, \varepsilon) = v_0(x) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y), \quad y = \frac{x}{\varepsilon} \tag{7}$$

where $v_0, u_k,\ k = 1, 2$, are twice continuously differentiable in x and u_k are 1-periodic and piecewise twice continuously differentiable in y. Substitution of FAS in eq. (7) into eq. (1) considering the chain rule $d/dx = \partial/\partial x + \varepsilon^{-1}\partial/\partial y$ and Taylor linearization of $\sigma(y, du^{(2)}/dx)$ around $\zeta = dv_0/dx + \partial u_1/\partial y$ produces

$$\frac{d}{dx}\left[\sigma\left(\frac{x}{\varepsilon}, \frac{du^{(2)}}{dx}\right)\right] = \varepsilon^{-1}\left\{\frac{\partial\sigma}{\partial y}(y, \zeta)\right\} + \varepsilon^0\left\{\frac{\partial\sigma}{\partial x}(y, \zeta) + \frac{\partial}{\partial y}\left[\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\frac{\partial\sigma}{\partial\epsilon}(y, \zeta)\right]\right\} + O(\varepsilon), \tag{8}$$

where $O(\varepsilon)$ gathers the terms corresponding to the positive powers of ε. Thus, for the asymptotic equality in eq. (8) to be satisfied as $\varepsilon \to 0^+$, the coefficients of the non-positive powers of ε must be zero, which, for x and v_0 fixed, produces the following recurrence over the periodic cell $(0, 1) \ni y$ for obtaining u_k, $k = 1, 2$:

$$\frac{\partial\sigma}{\partial y}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right) = 0, \qquad \frac{\partial\sigma}{\partial x}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\frac{\partial\sigma}{\partial\epsilon}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right)\right] = 0, \tag{9}$$

which in turn must be complemented with the corresponding conditions obtained by substituting the FAS (7) into conditions in eqs. (2)-(4) and conditions for periodicity and uniqueness. In particular, the problem for eq. $(9)_1$ is: for x and v_0 fixed, find u_1 as the null-average 1-periodic solution of

$$\frac{\partial \sigma}{\partial y}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right) = 0, \quad y \in (0,n) \setminus \Gamma, \tag{10}$$

$$\left[\!\left[\sigma\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right) \right]\!\right] = 0, \quad y \in \Gamma, \tag{11}$$

$$[\![u_1(x,y)]\!] = \frac{1}{\beta}\sigma_1\left(\frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right), \quad y \in \Gamma, \tag{12}$$

where $n = \varepsilon^{-1}$, $n \in \mathbb{N}$, and $\Gamma = \varepsilon^{-1}\Gamma^{\varepsilon}$. Existence and uniqueness of the null-average 1-periodic solution of the problem in eqs. (10)-(12), are guaranteed by the following Lemma (Décio Jr. et al. [22]):

Lemma: Let $\overline{\epsilon}$ be a parameter. Let $\sigma(y, \epsilon)$ be a piecewise continuously differentiable function in $(0,1) \ni y$. Then, there exists a unique 1-periodic function $N_1(y, \overline{\epsilon})$ that solves the so-called local problem stated as

$$\frac{\partial \sigma}{\partial y}\left(y, \overline{\epsilon} + \frac{\partial N_1}{\partial y}\right) = 0, \quad y \in (0,1) \setminus \{c_1\}, \tag{13}$$

$$\left[\!\left[\sigma\left(y, \overline{\epsilon} + \frac{\partial N_1}{\partial y}\right) \right]\!\right] = 0, \quad y = c_1, \tag{14}$$

$$[\![N_1(y,\overline{\epsilon})]\!] = \frac{1}{\beta}\sigma_1\left(\overline{\epsilon} + \frac{\partial N_1}{\partial y}\right), \quad y = c_1 \tag{15}$$

$$\langle N_1(y,\overline{\epsilon}) \rangle = 0, \tag{16}$$

where $c_1 = \ell_1/\varepsilon$, and $\langle\cdot\rangle$ is the mean value operator on the periodic cell, so eq. (16) is the null-average condition for uniqueness of solution N_1, whose periodic extension to $(1,n)$ provides u_1 for each $\overline{\epsilon} = dv_0/dx$ fixed.

In order to obtain the effective stress-strain law of the composite, that is, the relation between the mean stress $\overline{\sigma}$ and the mean strain $\overline{\epsilon}$ that models the behavior of the equivalent homogeneous medium, which is the mathematical goal of this work, proceed as follows: first, observe from eqs. (13) and (14) that $\sigma(y, \overline{\epsilon} + \partial N_1/\partial y) = \overline{\sigma}$; then, use the implicit function theorem in eq. (13) to isolate the second argument and obtain $\partial N_1/\partial y = \epsilon(y, \overline{\sigma}) - \overline{\epsilon}$, where $\epsilon(y,\sigma)$ is the inverse of $\sigma(y,\epsilon)$ with respect to the second argument; now, apply the mean value operator taking eq. (15) and the 1-periodicity in into account to obtain the inverse effective law $\overline{\epsilon} = \langle \epsilon(y, \overline{\sigma}) \rangle + \overline{\sigma}/\beta \equiv \widehat{\epsilon}(\overline{\sigma})$, that is, it defines the effective law implicitly; and, finally, the effective law $\overline{\sigma} = \widehat{\sigma}(\overline{\epsilon})$ follows by taking the inverse of $\overline{\epsilon} = \widehat{\epsilon}(\overline{\sigma})$, which, from the practical point of view, involves the solution o algebraic equations. Note that the effective law does not require the full solution of the local problem, but its derivative.

Finally, for completeness, the so-called homogenized problem for v_0, which models the effective behavior of the composite, that is, the physical behavior of the equivalent homogeneous medium, is presented and stated as

$$\frac{d}{dx}\left[\widehat{\sigma}\left(\frac{dv_0}{dx}\right)\right] = 0, \quad x \in (0,1), \qquad -\widehat{\sigma}\left(\left.\frac{dv_0}{dx}\right|_{x=0}\right) = 1, \quad v_0(1) = 0, \tag{17}$$

where the so-called homogenized equation $(17)_1$ is obtained by applying Lemma 1, p. 1124, of Álvarez Borges et al. [25], to eq. $(9)_2$ for u_2, written as

$$\frac{d}{dy}\left[E(y)\frac{dN_2}{dy}\right] = F_0(y) + \frac{dF_1}{dy}, \tag{18}$$

where N_2 is u_2 for x fixed and

$$E(y) = \frac{\partial \sigma}{\partial \epsilon}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right), \quad F_0(y) = -\frac{\partial \sigma}{\partial x}\left(y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right), \quad F_1(y) = -E(y)\frac{\partial u_1}{\partial x}, \tag{19}$$

for x and v_0 fixed, and the 1-periodic solution u_1 of problem in eqs. (10)-(12) obtained via the local problem in eqs. (13)-(16). The Lemma of Álvarez Borges et al. [25] provides the necessary and sufficient condition $\langle F_0(y)\rangle = 0$ for the existence of 1-periodic solutions for equations with the same structure as eq. (18), so application to eq. $(19)_2$ produces eq. $(17)_1$. Therefore, it is possible to construct the FAS $u^{(1)}$ from the first two terms of eq. (7). However, the construction of FAS $u^{(1)}$ is not the goal of this work, but it will be addressed in future ones.

3 Results and discussion

When values for the physical properties of real brick and mortar are unavailable (here, parameters E_r, Φ_r and m_r, $r = 1, 2$, in eq. (6)), it is possible obtain them from their uniaxial compression tests stress-strain curves. Yang et al. [2] present the individual curves for one type of brick and three types of mortar, besides the curves for the masonry prismatic structures produced with these components. The data from these curves were extracted with the online software WebPlotDigitizer (Rohatgi [26]), and the parameters E_r, Φ_r and m_r of brick ($r = 1$) and mortar ($r = 2$) were calculated by a fitting algorithm, where the experimental curves were fitted to Volokh's constitutive model in eq. (6), with respect to E_r, Φ_r and m_r, $r = 1, 2$, as shown in Fig. 2. For brick, the fitted parameter values are $E_1 = 6.471$ GPa, $\Phi_1 = 0.03691$ MPa and $m_1 = 0.5767$, whereas for the type of mortar arbitrarily chosen, the fitted values are $E_2 = 17.42$ GPa, $\Phi_2 = 0.08621$ MPa and $m_2 = 0.8353$.

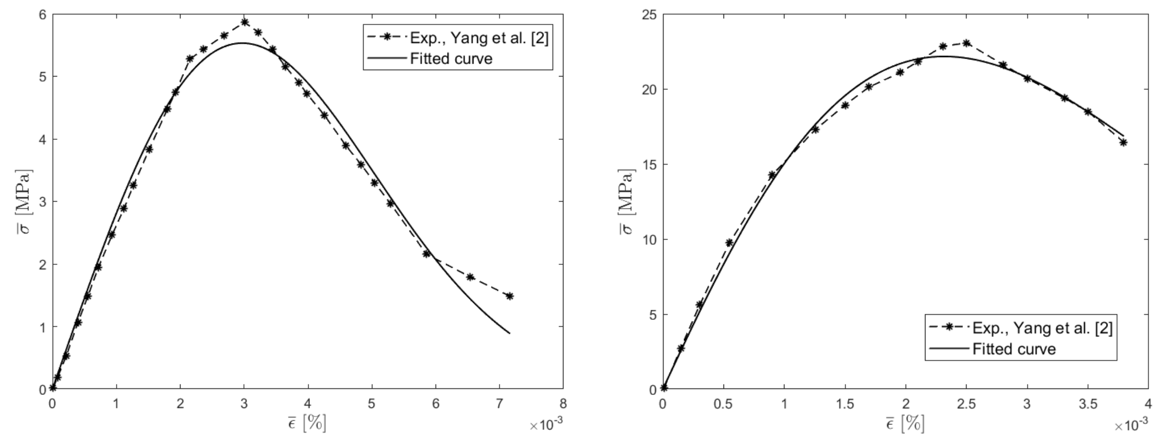

Figure 2. Experimental stress-strain curves of constituents fitted to Volokh's model: brick (left) and mortar (right).

The constituents proportions of the masonry prismatic structure used in the uniaxial compression tests of Yang et al. [2] are $c_1 = 57/67 \approx 0.851$ of brick and $c_2 = 10/67 \approx 0.149$ of mortar, respectively, which, together with their physical properties E_r, Φ_r and m_r, $r = 1, 2$, allows obtaining the effective law $\overline{\sigma} = \widehat{\sigma}(\overline{\epsilon})$ via AHM, as described in the previous section. The effective law curve obtained computationally considering perfect contact ($\beta \to +\infty$) also follows a Volokh's model with fitted effective properties $\widehat{E} = 7.029$ GPA, $\widehat{\Phi} = 0.03339$ MPA and $\widehat{m} = 0.6065$ (imperfect contact effective laws did not fit well to Volokh's models), whereas the corresponding experimental stress-strain follows a different Volokh's model with fitted effective properties $\widehat{E} = 2.927$ GPA, $\widehat{\Phi} = 0.02646$ MPA and $\widehat{m} = 1.105$, as shown in Fig. 3.

Note that linear (Hookean) regimes occur in roughly the same mean strain interval, whereas yield (maximum) mean stresses occur for almost the same value of the mean strain: $\overline{\epsilon} = 0.0026$ in the effective law by AHM and $\overline{\epsilon} = 0.0030$ in the experiment, while the AHM curves overestimate the experimental ones in $2 - 2.5$ MPa, in the yield mean stress point. In other words, whereas the AHM approach is capable of reproducing the qualitative behavior of the real situation, it overestimates the compression strength of the real masonry prismatic structure in this case. Such a quantitative inaccuracy of the AHM approach can have various single or combined causes, for instance: classical Hookean behavior on the small strains regime (it seems suitable for the real situation at hand, but power-law or polynomial nonlinearities can also be considered); the scale of the imperfect contact (the typical size of the imperfections is much smaller than the size of the periodic cell, which suggests using more than two-scale models and reiterated homogenization); the imperfect contact was considered periodic (real periodic composites can exhibit non-periodic interfacial properties); the imperfect contact is uniform (real contact regions can contain local defects such as cracks and pores, and transition zones created by chemical reactions, which suggests using

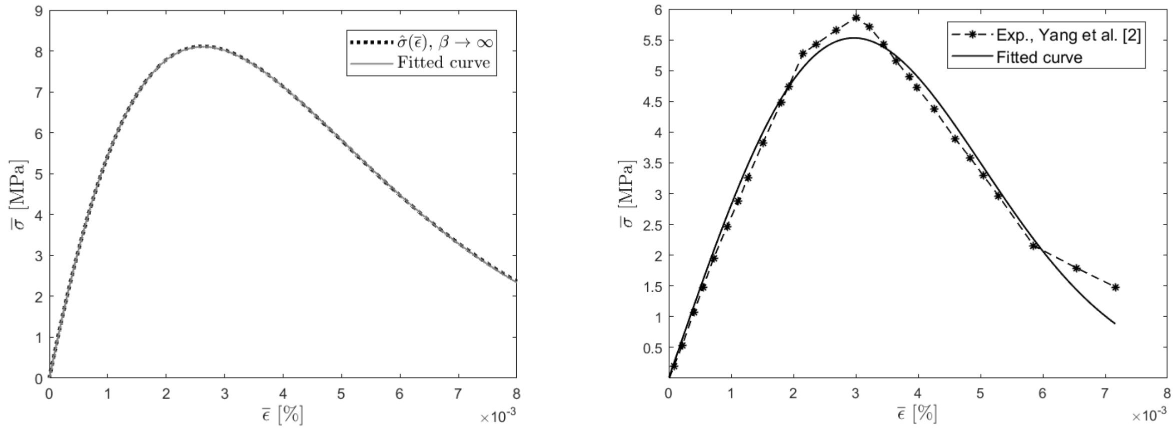

Figure 3. Theoretical (left) and experimental (right) stress-strain curves, follow its fitted Volokh's models.

non-one-dimensional models); the type of imperfect contact considered (there are several imperfect contact models that can be considered); and the number of periodic cells of the real masonry prismatic structure is insufficient to guarantee separation of scales (size effects can occur).

4 Conclusions

Volokh's model proved to be suitable to represent local and effective behaviors with failure at least qualitatively. Even though the AHM approach addressed here was insufficient to reproduce accurately the real behavior in the experiment, it shows great potential to study this type of situations. Therefore, whichever the cause or causes of the inaccuracy of the AHM approach, the boundary-value problem considered here must be enriched in order to overcome such a deficiency.

Acknowledgements. Support provided by Brazil's National Council for Scientific and Technological Development (CNPq) via Universal Project Nº 402857/2021-6 and Coordination of Superior Level Staff Improvement (CAPES) via PRAPG Program Nº 23038.003836/2023-39, and State of Sergipe's Foundation for Research and Technological Innovation (FAPITEC/SE) via Universal Project Nº 019203.01702/2024-6 is gratefully acknowledged. Also, the reviewers' useful and relevant comments are welcomed.

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