Asymptotic homogenization, domain decomposition and finite elements combined for calculating effective elastic properties of periodic fiber-reinforced composites with imperfect interfaces

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Abstract. A methodology is presented for calculating the effective elastic properties of periodic multi-phase composites made of an anisotropic linear elastic matrix reinforced with a periodical distribution of unidirectional fibers and exhibiting spring-type imperfect contacts at the interfaces. The periodicity cell contains any finite number of parallel fibers and exhibits arbitrary cross-section. Fibers also exhibit arbitrary cross-sections and are made of a different anisotropic linear elastic material each. The methodology uses asymptotic homogenization (AH) to obtain the mathematical expressions of the effective properties and to formulate the so-called local problems on the periodicity cell on whose solutions the effective properties depend on. In order to deal with the discontinuities arising from the spring-type interfaces, the local problems are then restated via domain decomposition (DD) in a way allowing for an iterative resolution scheme in which the solution of the problem to be solve in each iteration is obtained via finite elements (FE). Results in the examples are obtained via a computational implementation of the methodology based on the FreeFEM open-source software, which allows for the variational formulation of the iteration problem to be dealt with directly.

Keywords: Elastic composites with imperfect interfaces, Effective properties, AH-DD-FE methodology

1 Introduction

The design and production of composites with effective physical properties that improve over their constituents is an area of intensive development. The aim is to accurately predict the effective properties in terms of the physical properties of the constituents and on their interfaces, and their geometrical arrangement. In linear elasticity, the micro-heterogeneity of fiber-reinforced composites produces a rapid oscillation of the coefficients involved in the partial differential equations model. The analytical solution of this system of equations for general anisotropic constituents is impossible. Numerical methods, on the other hand, require a very fine discretization, of the domain, which considerably increases the computational cost and compromises the convergence of the methods. An alternative is to use mathematical homogenization, which allows obtaining approximations of the effective (homogenized) properties of heterogeneous media. Effective properties characterize an ideal homogeneous medium equivalent to the heterogeneous one under study. Here, the asymptotic homogenization method (AHM – Bakhvalov and Panasenko [1]), which is rigorous and relevant specially for periodic structures, is employed. The AHM presents a mathematical framework for the calculation of effective coefficients that rely on the solution of so-called local problems posed on the periodic cell. The local problems are analytically solvable only in exceptional cases, otherwise are numerically addressed mostly via the finite element method (FEM – Babuška [2]). However, most works are restricted to periodic cells with square or parallelogram shape and circular, elliptical or square fiber cross-section. It is our assumption that study of more complicated geometries have been limited by the meshing capabilities of the software being used. In this work, the effective elastic modulus of composites with microperiodic distribution of parallel unidirectional fibers imperfectly bonded to the matrix is addressed. The study of such composites is two-dimensional. The approach presented here allows arbitrary shapes for the periodic cell and the cross-sections of the fibers. The constituents are anisotropic and the linear-spring interface model (Duan et al. [3]) is adopted. The numerical solution of the local problems is obtained via a domain decomposition method (DDM – Dolean et al. [4], Mathew [5]) and the problems on the subdomains are solved via FEM using the free software FreeFEM (Hecht [6]), which is based on directly discretizing the variational formulation of the problem, which makes it very close to the mathematical formulation of FEM. The benefits of the methodology have been already shown in the computation of the effective thermal conductivity of composites (León-Mecías et al. [7]).

2 Problem statement

In what follows, arbitrary shapes for the cross-sections of the fibers are considered. The periodic cells can have arbitrary polygonal boundary. In order to simplify the presentation, only composites with two types of fibers are considered. The two fibers can be distinguish by cross-section shape or elastic properties and all fibers are aligned with the x_3-axis of a three dimensional Cartesian coordinate system.

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ be non-collinear, $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$, and $S \subset \mathbb{R}^2$ such that $\dim(S) = \max_{\mathbf{y}_1,\mathbf{y}_2 \in S} \|\mathbf{y}_1 - \mathbf{y}_2\|$. Define $\mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}}(S) = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = \mathbf{x} + z_1\mathbf{v} + z_2\mathbf{w}, \mathbf{x} \in S\}$. Let $\Omega_\# \subset \mathbb{R}^2$ be a Lipschitz domain with polygonal boundary $\Gamma_\# = \partial\Omega_\#$, which is a *periodic cell* if and only if $\mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}_1}(\Omega_\#) \cap \mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}_2}(\Omega_\#) = \emptyset$ for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{Z}^2$ such that $\mathbf{z}_1 \neq \mathbf{z}_2$ and $\cup_{\mathbf{z}\in\mathbb{Z}^2}\mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}}(\overline{\Omega_\#}) = \mathbb{R}^2$. The cross-section of the two fibers are the Lipschitz domains $\Omega_\#^{(1)}, \Omega_\#^{(2)} \subset \Omega_\#$ such that $\overline{\Omega_\#^{(1)}} \cap \overline{\Omega_\#^{(2)}} = \emptyset$. As $\Gamma_\#$ is arbitrarily shaped, it follows that $\partial\Omega_\#^{(1)} \cap \Gamma_\# = \partial\Omega_\#^{(2)} \cap \Gamma_\# = \emptyset$, so $\Omega_\#^{(0)} = \Omega_\# \setminus \{\overline{\Omega_\#^{(1)}} \cup \overline{\Omega_\#^{(2)}}\}$ is the matrix phase in the periodic cell. The cross-section orthogonal to the fibers of a 3D sample of the composite is the Lipschitz domain $\Omega \subset \mathbb{R}^2$ is such that $\Omega_\# \subsetneq \Omega$ with $\dim(\Omega_\#) \ll \dim(\Omega)$, so $\varepsilon = \dim(\Omega_\#)/\dim(\Omega) \ll 1$ is the relative size of the periodic cell, which is made explicit by denoting $\Omega^\varepsilon \equiv \Omega$. Note that $\varepsilon \to 0^+$ is the homogenization limit, which have no effect on the effective properties of the composite. Hereinafter, Einstein's summation convention over repeated indices is adopted for lowercase Latin and Greek indices taking values in the sets $\{1, 2, 3\}$ and $\{1, 2\}$, respectively, whereas index $\Phi \in \{0, 1, 2\}$ denotes the phase. Also, comma notation for spacial differentiation is adopted.

Linear elastic constitutive behavior is given by the generalized Hooke's law and Cauchy's law combined as

$$\sigma_{ij}^\varepsilon = C_{ijkl}^\varepsilon u_{k,l}^\varepsilon, \text{ in } \Omega^\varepsilon, \quad C_{ijkl}^\varepsilon \text{ is } \Omega_\#\text{-periodic in } \Omega^\varepsilon, \tag{1}$$

where C_{ijkl}^ε, σ_{ij}^ε and u_k are the components of the elastic modulus and the stress and displacement fields, respectively, the first two with the usual symmetries, and the first also having positive-definiteness and boundedness. Let $\Omega_\alpha^\varepsilon = \cup_{\mathbf{z}\in\mathbb{Z}^2}\{\mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}}(\Omega_\#^{(\alpha)}) \cap \Omega^\varepsilon\}$ be the phase of type-α fiber with boundary $\partial\Omega_\alpha^\varepsilon = \cup_{\mathbf{z}\in\mathbb{Z}^2}\{\mathcal{T}_{\mathbf{v},\mathbf{w},\mathbf{z}}(\partial\Omega_\#^{(\alpha)}) \cap \Omega^\varepsilon\}$, so the matrix phase is $\Omega_0^\varepsilon = \Omega^\varepsilon \setminus \{\overline{\Omega_1^\varepsilon} \cup \overline{\Omega_2^\varepsilon}\}$. Note that $\Omega_\alpha^\varepsilon$ is not connected, whereas Ω_0^ε is multiply connected. Assuming homogeneous constituents, the elastic properties are piecewice-constant, that is, $C_{ijkl}^\varepsilon(\mathbf{x}) = C_{ijkl}^{(\Phi)}$, $\mathbf{x} \in \Omega_\Phi^\varepsilon$, with $C_{ijkl}^{(\Phi)}$ constant. Then, the corresponding equilibrium problem in the absence of body forces is

$$\sigma_{i\alpha,\alpha}^\varepsilon = 0, \text{ in } \Omega^\varepsilon \setminus \{\partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon\}, \quad u_k^\varepsilon = 0, \text{ on } \partial_\mathbf{u}\Omega^\varepsilon, \quad \sigma_{i\alpha}^\varepsilon n_\alpha = 0, \text{ on } \partial_\mathbf{t}\Omega^\varepsilon, \tag{2}$$

where $\sigma_{i\alpha}^\varepsilon$ is given in eq. (1), n_α are the components of the unit outward normal vector to the boundary $\partial\Omega^\varepsilon$, and $\partial_\mathbf{u}\Omega^\varepsilon$ and $\partial_\mathbf{t}\Omega^\varepsilon$ are the portions of $\partial\Omega^\varepsilon$ in which surface displacement and traction fields are imposed, respectively, such that $\overline{\partial_\mathbf{u}\Omega^\varepsilon}\{\cup, \cap\}\overline{\partial_\mathbf{t}\Omega^\varepsilon} = \{\partial\Omega^\varepsilon, \emptyset\}$. Here, the aim is to obtain the effective elastic modulus, on which boundary conditions in eq. (2) have no effect. On the other hand, conditions at the interfaces $\partial\Omega_\alpha^\varepsilon$ are of great importance. In this work, *linear spring-type imperfect contact conditions* at $\partial\Omega_\alpha^\varepsilon$ are consider. Let $\mathcal{K}_{ij} = (C_{i\alpha j\beta}^\varepsilon) \in \mathcal{M}_2(\mathbb{R})$ be rank-2 matrices such that $\mathcal{K}_{ij} = \mathcal{K}_{ji}^{\mathrm{T}}$ for $i \neq j$ and $\mathcal{K}_{ij}$ is symmetric for $i = j$, so the system of differential equations in eq. (2) and the constitutive relations in eq. (1) can be rewritten, respectively, as

$$\nabla \cdot \sigma_i^\varepsilon = 0, \text{ in } \Omega^\varepsilon \setminus \{\partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon\}, \quad \sigma_i^\varepsilon = \mathcal{K}_{ij}\nabla u_j^\varepsilon. \tag{3}$$

Let $n_\alpha^{(\Phi)}$ be the components of the unit outward normal vector to phase Φ on $\partial\Omega_\Phi^\varepsilon$, so the traction field $\mathbf{t}_\Phi^\varepsilon$ directed by $\mathbf{n}^{(\Phi)}$ has components $t_{\Phi i}^\varepsilon = \sigma_i^\varepsilon|_{\Omega_\Phi^\varepsilon} \cdot \mathbf{n}^{(\Phi)}$, where $(\cdot)|_{\Omega_\Phi^\varepsilon}(\mathbf{x}) = \lim_{\mathbf{y}\in\Omega_\Phi^\varepsilon \to \mathbf{x}} (\cdot)(\mathbf{y})$ for $\mathbf{x} \in \partial\Omega_\Phi^\varepsilon$. The traction field $\mathbf{t}^{\varepsilon}_{\Phi}(\mathbf{x})$ is given in Cartesian components in $\mathbf{x} \in \cup_{\Phi\in\{0,1,2\}}\partial\Omega^{\varepsilon}_{\Phi}$ and can be decomposed into normal, tangential and axial components as $\mathbf{t}^{\varepsilon}_{\Phi\{n,t,a\}}(\mathbf{x}) = \mathcal{N}^{(\Phi)}\mathbf{t}^{\varepsilon}_{\Phi}(\mathbf{x})$, where $\mathcal{N}^{(\Phi)} \in \mathcal{M}_3$, $\mathcal{N}^{(\Phi)}_{11} = \mathcal{N}^{(\Phi)}_{22} = n^{(\Phi)}_1$, $\mathcal{N}^{(\Phi)}_{33} = 1$, $\mathcal{N}^{(\Phi)}_{12} = -\mathcal{N}^{(\Phi)}_{21} = n^{(\Phi)}_2$, and $\mathcal{N}^{(\Phi)}_{\alpha 3} = \mathcal{N}^{(\Phi)}_{3\alpha} = 0$. Similarly, the Cartesian displacement field $\mathbf{u}^{\varepsilon}_{\Phi}(\mathbf{x})$ is decomposed into normal, tangential and axial components as $\mathbf{u}^{\varepsilon}_{\Phi\{n,t,a\}}(\mathbf{x}) = \mathcal{N}^{(\Phi)}\mathbf{u}^{\varepsilon}_{\Phi}(\mathbf{x})$. With such considerations, the spring-type contact conditions are

$$\mathbf{t}^{\varepsilon}_{0\{n,t,a\}}(\mathbf{x}) = -\varepsilon^{-1}\mathcal{K}^{(\alpha)}[\![\mathbf{u}^{\varepsilon}_{\{n,t,a\}}]\!]_{\partial\Omega^{\varepsilon}_{\alpha}}(\mathbf{x}), \quad [\![\cdot]\!]_{\partial\Omega^{\varepsilon}_{\alpha}} = (\cdot)|_{\partial\Omega^{\varepsilon}_{0}}(\mathbf{x}) - (\cdot)|_{\partial\Omega^{\varepsilon}_{\alpha}}(\mathbf{x}) \text{ on } \partial\Omega^{\varepsilon}_{\alpha}, \tag{4}$$

where $\mathcal{K}^{(\alpha)} \in \mathcal{M}_3$ with $\mathcal{K}^{(\alpha)}_{11} = k^{(\alpha)}_n$, $\mathcal{K}^{(\alpha)}_{22} = k^{(\alpha)}_t$, $\mathcal{K}^{(\alpha)}_{33} = k^{(\alpha)}_a$ and $\mathcal{K}^{(\alpha)}_{ij} = 0$ for $i \neq j$, and $[\![\cdot]\!]_{\partial\Omega^{\varepsilon}_{\alpha}}$ is the jump or contrast operator across $\partial\Omega^{\varepsilon}_{\alpha}$. Note that ideal perfect contact between matrix and fibers corresponds to $k^{(\alpha)}_{\{n,t,a\}} \to +\infty$, which implies the continuity of tractions and displacements across the interfaces between the matrix and type-α fibers, whereas $\mathcal{K}^{(\alpha)} \equiv \mathbf{0}$ represents total debonding between them. The following problem summarizes these considerations:

Problem 1. In the domain Ω^{ε}, find the displacement field $\mathbf{u}^{\varepsilon}$ satisfying the system of partial differential equations in eq. (3), the boundary conditions in eq. $(2)_{2,3}$ and the imperfect contact conditions in eq. (4).

Problem 1 is not solvable analytically, whereas any numerical approach requires extremely dense meshes, which significantly increases the computational cost and compromises the convergence of such an approach. An alternative is to find approximate solutions for $\varepsilon \to 0^+$. In this regard, the AHM is an efficient choice.

3 Semianalytical approach to the calculation of the effective elastic modulus

The AHM provides a rigorous approach to obtain approximate solutions of problem 1 for any $\varepsilon > 0$. Moreover, as $\varepsilon \to 0^+$ those approximate solutions weakly converge to the solution of problem 1. The AHM approximates the solution $\mathbf{u}^{\varepsilon}$ as the following two-scale formal asymptotic solution:

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) \sim \mathbf{u}^{(\infty)}(\mathbf{x},\varepsilon) = \mathbf{u}_0(\mathbf{x}) + \sum_{k\geq 1}\varepsilon^k\mathbf{u}_k(\mathbf{x},\mathbf{y}), \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}, \tag{5}$$

where the unknown $\mathbf{u}_k(\mathbf{x},\mathbf{y})$ are $\varepsilon^{-1}\Omega_{\#}$-periodic in the local variable $\mathbf{y}$. Substituting eq. (5) into eq. (3), applying the chain rule and equating the coefficients of the powers of ε to zero, an infinite system of equations is obtained, to determine the $\mathbf{u}_k(\mathbf{x},\mathbf{y})$. From the equations for ε^{-1}, $\mathbf{u}_1(\mathbf{x},\mathbf{y})$ can be found in terms of the solutions of six local problems on the periodic cell indexed by related to indices $pq \in \{11, 22, 33, 23, 13, 12\}$, which only involve the local variable $\mathbf{y}$). The calculation of the components of the effective modulus depends on the solution of such problems, which are stated as follows:

Problem 2 (local problem pq). In the periodic cell $\varepsilon^{-1}\Omega_{\#}$, find $N^{pq}_j(\mathbf{y})$ satisfying

$$\begin{gathered}\nabla\cdot\sigma^{(\#)}_i = 0, \text{ in } \varepsilon^{-1}(\Omega_{\#}\setminus\{\partial\Omega^{(1)}_{\#}\cup\partial\Omega^{(2)}_{\#}\}), \quad \sigma^{(\#)}_i = \mathcal{K}_{ij}\nabla N^{pq}_j + \mathbf{k}^{pq}_i, \\ \sigma^{(\#)}_i|_{\varepsilon^{-1}\Omega^{(0)}_{\#}}\cdot\mathbf{n}_0 + \sigma^{(\#)}_i|_{\varepsilon^{-1}\Omega^{(\alpha)}_{\#}}\cdot\mathbf{n}_{\alpha} = 0, \text{ on } \varepsilon^{-1}\partial\Omega^{(\alpha)}_{\#}, \quad \mathbf{t}_{0\{n,t,a\}} = -k^{(\alpha)}_{\{n,t,a\}}[\![N^{pq}_{\{n,t,a\}}]\!]_{\varepsilon^{-1}\partial\Omega^{(\alpha)}_{\#}},\end{gathered} \tag{6}$$

where $\mathbf{k}^{pq}_i(\mathbf{y}) = (C_{i1pq}(\mathbf{y}), C_{i2pq}(\mathbf{y}))^{\mathrm{T}}$ with $C^{\varepsilon}_{ijkl}(\mathbf{x}) = C_{ijkl}(\mathbf{y})$, and $\mathbf{t}_{0\{n,t,a\}}$ are the normal, tangential and axial components of the traction field with Cartesian components $t_{0i}(\mathbf{y}) = \sigma^{(\#)}_i\cdot\mathbf{n}_0$ for $\mathbf{y} \in \varepsilon^{-1}\partial\Omega^{(\alpha)}_{\#}$.

Once problem 2 is solved, the components $\widehat{C}_{ijkl}$ of the effective elastic modulus are obtained as

$$\widehat{C}_{ijkl} = \left\langle C_{ijkl} + C_{ijpq}N^{pq}_{i,j}\right\rangle_{\varepsilon^{-1}\Omega_{\#}}, \tag{7}$$

where $\langle\cdot\rangle_S$ is the mean value operator over set S. Note that, for each $pq \in \{11, 22, 33, 23, 13, 12\}$, the corresponding solution of problem 2 allows calculating six components of the effective elastic modulus. Also, note that problem 2 has analytical solution only in exceptional cases, so it is usually solved numerically, being FEM one of the more employed for this (Babuška [2]). However, the linear spring-type interface conditions in eq. $(6)_{3,4}$ can not be easily imposed in standard FEM software implementations because these lacks appropriate jump operators on interfaces. Moreover, interface conditions eq. $(6)_4$ are not stated in the directions useful for the variational formulation of the problem needed by FEM. In what follows, the adequate form of such conditions is obtained.

Note that $\mathbf{t}_0 = (\mathcal{N}^{(\alpha)})^{-1}\mathcal{N}^{(\alpha)}\mathcal{K}^{(\alpha)}[\![\mathbf{N}^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}$, so

$$\begin{gathered} t_{0\beta} = -k_{\beta\gamma}^{(\alpha)}[\![N_\gamma^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}, \quad t_{03} = -k_a^{(\alpha)}[\![N_3^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}, \\ k_{11}^{(\alpha)} = k_n(n_1^{(\alpha)})^2 + k_t(n_2^{(\alpha)})^2, \quad k_{12}^{(\alpha)} = (k_n - k_t)n_1^{(\alpha)}n_2^{(\alpha)}, \quad k_{22}^{(\alpha)} = k_t(n_1^{(\alpha)})^2 + k_n(n_2^{(\alpha)})^2, \end{gathered} \tag{8}$$

from which, noting that $t_{0\beta}$ depends on $[\![N_\gamma^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}$ linearly, condition in eq. $(6)_3$ can be replaced by

$$t_{\alpha\beta} = -k_{\beta\gamma}^{(\alpha)}[\![N_\gamma^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}, \quad t_{\alpha 3} = -k_a^{(\alpha)}[\![N_3^{pq}]\!]_{\varepsilon^{-1}\partial\Omega_{\#}^{(\alpha)}}, \tag{9}$$

as eq. $(6)_2$ can be decomposed as

$$\sigma_i^{(\#)} = \mathcal{K}_{ij}^{(\Phi)}\nabla N_j^{pq} + \mathbf{k}_i^{(\Phi)pq}, \text{ in } \varepsilon^{-1}\Omega_{\#}^{(\Phi)}, \tag{10}$$

which is the motivation for employing DDM to solve problem 2, as eq. $(6)_1$ becomes a system of equations with constant coefficients $\mathcal{K}_{ij}^{(\Phi)}$ in each phase $\varepsilon^{-1}\Omega_{\#}^{(\Phi)}$.

The imperfect contact conditions at the interfaces in eq. (8) and eq. (9) are approached via a variant of DDM. The basic idea consists of decomposing a region into subdomains in which the solution is approximated independently. Then, the solution over the whole region can be calculated by iterative substructuring (Jelassi et al. [8]), or by linking the subdomains via interface conditions (Lions [9]). Here, the latter approach is adopted.

Considering eq. (10), decompose the solution of problem 2 as $\mathbf{N}^{pq}(\mathbf{y}) = \mathbf{N}^{(\Phi)pq}(\mathbf{y})$ for $\mathbf{y} \in \varepsilon^{-1}\Omega_{\#}^{(\Phi)}$, so $\sigma_i^{(\Phi)} = \mathcal{K}_{ij}^{(\Phi)}\nabla N_j^{(\Phi)pq} + \mathbf{k}_i^{(\Phi)pq}$, and problem 2 can be rewritten as follows:

Problem 3. Find $\mathbf{N}^{(\Phi)pq}(\mathbf{y})$ such that $\mathbf{N}^{(0)pq}(\mathbf{y})$ is $\varepsilon^{-1}\Omega_{\#}^{(0)}$-periodic, $\nabla\cdot\sigma_i^{(\Phi)} = 0$ in $\varepsilon^{-1}\Omega_{\#}^{(\Phi)}$ subjected to interface conditions eq. (8) and eq. (9) and uniqueness conditions $\langle\mathbf{N}^{(\Phi)pq}(\mathbf{y})\rangle_{\varepsilon^{-1}\Omega_{\#}^{(\Phi)}} = 0$.

Note that in problem 3 the system of partial differential equations are separated in domains but coupled through the interface conditions. If the solution in matrix domain $\varepsilon^{-1}\Omega_{\#}^{(0)}$ is known, then the solutions in fibers domains $\varepsilon^{-1}\Omega_{\#}^{(\alpha)}$ can be computed via FEM. Conversely, knowing the solutions in $\varepsilon^{-1}\Omega_{\#}^{(\alpha)}$ allows the computation of the solution in $\varepsilon^{-1}\Omega_{\#}^{(0)}$. This provides the idea for simple DDM iterative scheme below.

Given initial guesses $\mathbf{N}_0^{(\alpha)pq}$ in $\varepsilon^{-1}\Omega_{\#}^{(\alpha)}$ and an error tolerance $\epsilon \ll 1$ then, for $m \in \{0\} \cup \mathbb{N}$, do:

1. With $\mathbf{N}_m^{(\alpha)pq}$ in $\varepsilon^{-1}\Omega_{\#}^{(\alpha)}$, calculate $\mathbf{N}_m^{(0)pq}$ in $\varepsilon^{-1}\Omega_{\#}^{(0)}$.
2. With $\mathbf{N}_m^{(0)pq}$ in $\varepsilon^{-1}\Omega_{\#}^{(0)}$, calculate $\mathbf{N}_{m+1}^{(\alpha)pq}$ in $\varepsilon^{-1}\Omega_{\#}^{(\alpha)}$.
3. If $\max_\alpha\{\|\mathbf{N}_{m+1}^{(\alpha)pq} - \mathbf{N}_m^{(\alpha)pq}\|\} < \epsilon$, then stop and take $\{\mathbf{N}_m^{(0)pq}, \mathbf{N}_{m+1}^{(\alpha)pq}\}$ as the solution, else set $m = m+1$ and return to step 1.

The iterative scheme given above is trouble-free to realize with FEM software with the capabilities of FreeFEM. The variational formulation of the problems to solve in steps 1 and 2 is easy to obtain. Details about the norms used in step three depends of the FEM spaces used in the numerical implementation.

4 Numerical results and discussion

The strength of the AHM-DDF-FEM combination is illustrated with some examples, all with error tolerance $\epsilon = 10^{-10}$ and the $L_2(\Omega_{\#}^{(1)})$-norm in step 3 of the iterative scheme.

For the first example, consider a unit square periodic cell with a circular hole (so $k_n = k_t = k_a = 0$). The matrix is isotropic with Young's modulus of 70 GPa and Poisson's ratio of 0.3. In Table 1, the AHM-DDF-FEM results (labeled A-D-F) are compared with those of Otero et al. [10] (labeled O2013) for some values of the area fraction A of the hole. The components of the effective modulus are presented in Voigt notation.

Table 1. Components of the effective modulus of porous material

A	$\widehat{C}_{11}$ (O2013)	$\widehat{C}_{11}$ (A-D-F)	$\widehat{C}_{12}$ (O2013)	$\widehat{C}_{12}$ (A-D-F)	$\widehat{C}_{13}$ (O2013)	$\widehat{C}_{13}$ (A-D-F)
0.05	80.52537	80.57222	33.14981	33.15992	34.10255	34.11964
0.20	53.39001	53.41426	18.36599	18.37110	21.52680	21.53561
0.35	36.53604	36.54994	9.781059	9.783213	13.89513	13.89994
0.55	20.49858	20.50756	3.378678	3.378741	7.163179	7.165901
0.75	5.849942	5.859657	0.2891222	0.2881649	1.841719	1.844342
A	$\widehat{C}_{33}$ (O2013)	$\widehat{C}_{33}$ (A-D-F)	$\widehat{C}_{44}$ (O2013)	$\widehat{C}_{44}$ (A-D-F)	$\widehat{C}_{66}$ (O2013)	$\widehat{C}_{66}$ (A-D-F)
0.05	86.96153	86.97176	24.35897	24.36334	23.20944	23.23223
0.20	68.91608	68.92136	17.94506	17.94812	13.42208	13.44047
0.35	53.83708	53.83998	12.91530	12.91738	6.616327	6.629126
0.55	35.79791	35.79955	7.459089	7.460579	1.808771	1.816486
0.75	18.60503	18.60658	2.111431	2.113931	0.07890091	0.08675641

For the second example, consider the effect of the aspect ratio of an elliptical fiber imperfectly bonded to a unit square periodic cell. The matrix is as in the first example, and the isotropic fiber has Young's modulus of 450 GPa and Poisson's ratio of 0.17. Area fraction of the ellipse is 0.35 and the semi-major axis varies as $a \in [0.3, 0.4]$. The interface constants are $k_n = 10$, $k_t = k_a = 0.01$. Figure 1 shows the components of the effective modulus.

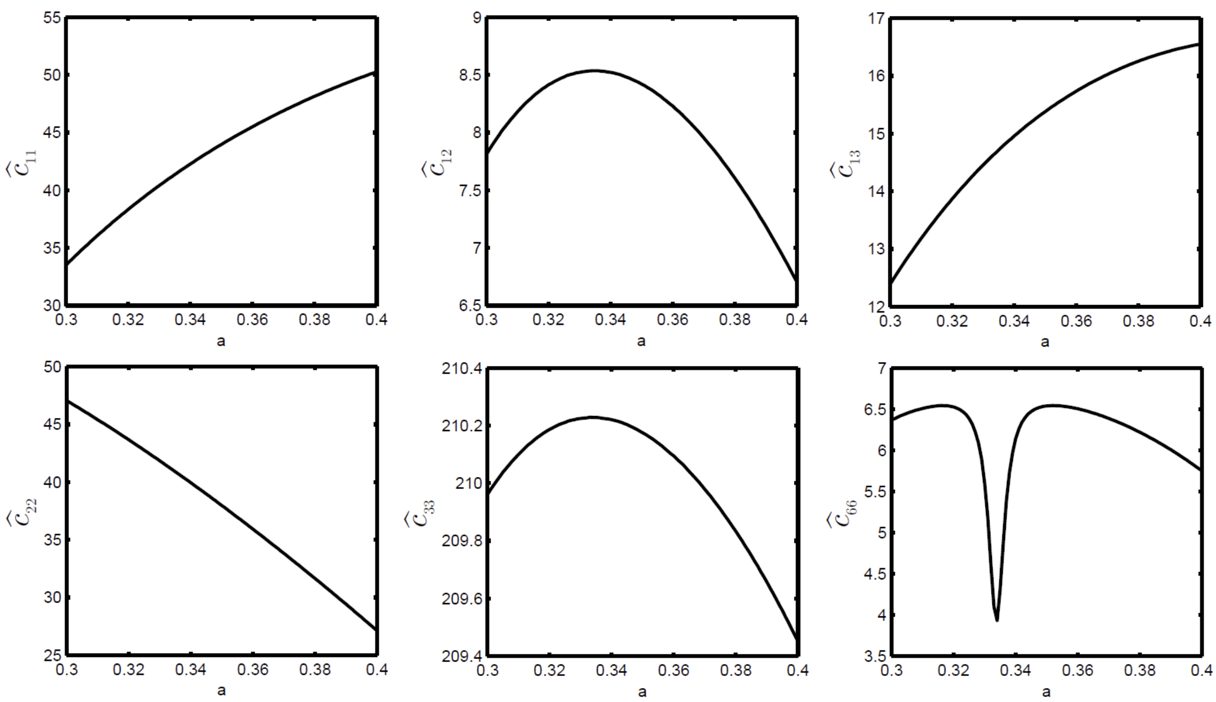

Figure 1. Components of the effective elastic modulus for elliptic fibers versus semi-major axis a.

For the third example, consider the effect of the in-plane rotation of an elliptical fiber imperfectly bonded to a unit square periodic cell. The setting is the same as in the second example, except for constant semi-major axis

$a = 0.4$ and interface constants are $k_n = k_t = k_a = 0.001$. The rotation angle varies as $\theta \in [0, \pi/2]$. Figure 2 shows the components of the effective modulus.

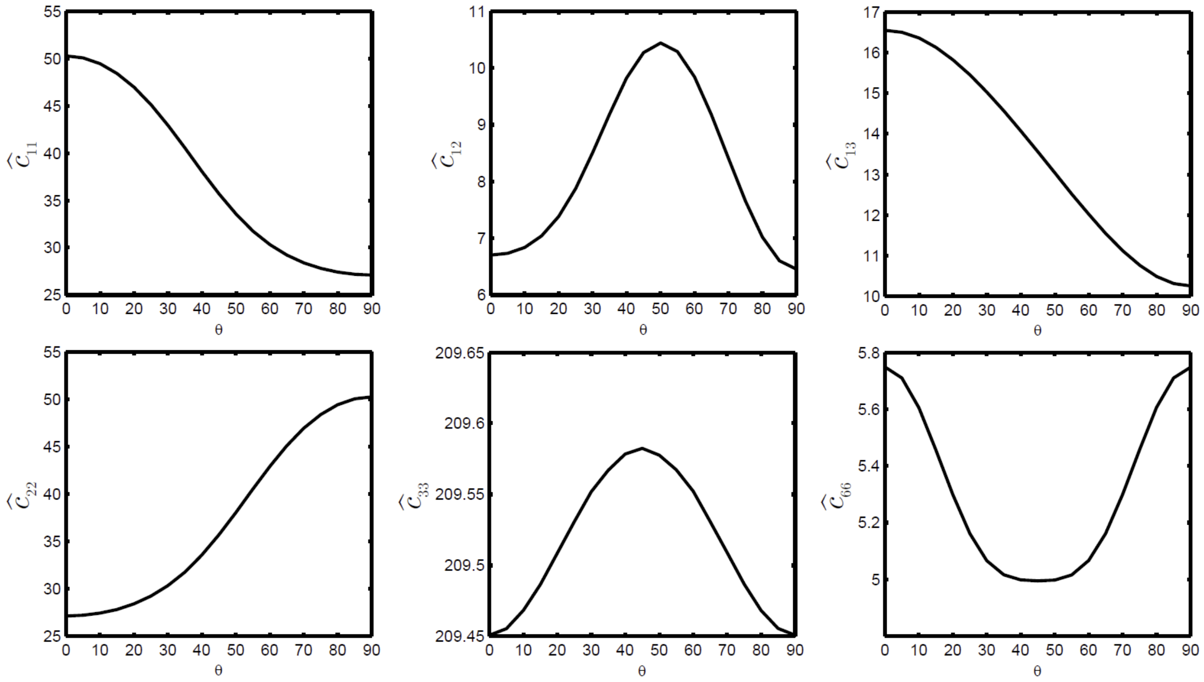

Figure 2. Components of the effective elastic modulus for elliptic fibers versus rotation angle θ.

The last example deals with the arbitrariness of the boundaries of the cross-sections of the periodic cell and the fibers. In particular, the periodic cell chosen among various possibilities is cross-shaped and contain a circular fiber and a T-shaped fiber. Such a choice of periodic cell is the only one to completely contain the two types of fibers and has the fewest subdomains, which facilitates decomposing the domain, its meshing and imposing adequate periodicity conditions. The mesh size of the last example is 0.02, resulting in a mesh with 3998 triangles and 2302 vertices. The matrix is made of isotropic epoxy resin LY558 with $C_{11}^{(0)} = 8.65$ GPa, $C_{12}^{(0)} = 4.75$ GPa and $C_{44}^{(0)} = 1.95$ GPa, whereas fibers are made of transversely isotropic type-I ($\alpha = 1$) and type-II ($\alpha = 2$) Morganite, respectively, with $C_{11}^{(\alpha)} = (12.1, 20.4)$ GPa, $C_{12}^{(\alpha)} = (6.49, 9.4)$ GPa, $C_{13}^{(\alpha)} = (6.5, 10.5)$ GPa, $C_{33}^{(\alpha)} = (410, 240)$ GPa, $C_{44}^{(\alpha)} = (13.7, 24)$ GPa, $C_{66}^{(\alpha)} = (2.8, 5.5)$ GPa. The interface constants are $k_n^{(\alpha)} = k_t^{(\alpha)} = k_a^{(\alpha)} = (1, 0.001)$. The components of the effective modulus obtained are $\widehat{C}_{11} = 6.34$ GPa, $\widehat{C}_{12} = 2.98$ GPa, $\widehat{C}_{13} = 3.29$ GPa, $\widehat{C}_{22} = 5.9$ GPa, $\widehat{C}_{23} = 3.14$ GPa, $\widehat{C}_{26} = 0.02$ GPa, $\widehat{C}_{33} = 102.83$ GPa, $\widehat{C}_{44} = 4.73$ GPa, $\widehat{C}_{55} = 4.88$ GPa, $\widehat{C}_{66} = 1.61$ GPa, so the effective behavior resembles that of a material with tetragonal or hexagonal crystalline structure. Figure 3 shows the corresponding solutions N_i^{pq} of the local problems with i and $pq \in \{11, 22, 33, 23, 13, 12\}$ corresponding to rows and columns, respectively.

The number of iterations of the iterative scheme was between 4 and 56 for all the pq problems in all the examples. For $k_{n,t,a} \ll 1$, the number of iterations was less than 10, with moderate mesh sizes in all the experiments.

5 Conclusions

This work presents a methodology for the calculation of the effective properties of microperiodic elastic composites with spring-type imperfect interfaces. The difference with other works is that the methodology presented here allows arbitrarily-shaped periodic cells as well as completely anisotropic constituents. This is achieved by using the DDM in a novel way that allows using FEM implementations without great complications. Im particular, we chose FreeFEM software because of its good 2D meshing capabilities and the way of assemble stiffness matrices directly from the variational formulation of the problem. The results presented validate the quality of the proposed methodology.

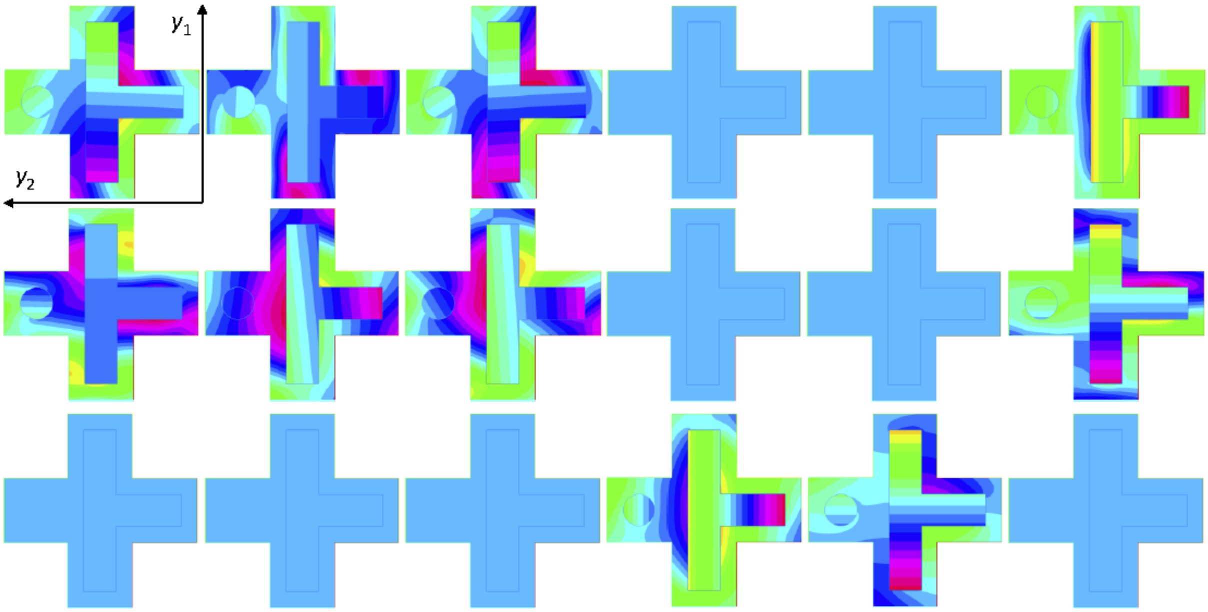

Figure 3. Local functions N_i^{pq} with i and $pq \in \{11, 22, 33, 23, 13, 12\}$ indicating rows and columns, respectively.

Acknowledgements. Support provided by Brazil's National Council for Scientific and Technological Development (CNPq) via Universal Project Nº 402857/2021-6 and Coordination of Superior Level Staff Improvement (CAPES) via PRAPG Program Nº 23038.003836/2023-39, and State of Sergipe's Foundation for Research and Technological Innovation (FAPITEC/SE) via Universal Project Nº 019203.01702/2024-6 is gratefully acknowledged.

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References

[1] N. Bakhvalov and G. Panasenko. *Homogenisation: averaging processes in periodic media*. Kluwer, 1989.

[2] I. Babuška. The finite element method for elliptic equations with discontinuous coefficients. *Computing*, vol. 5, n. 3, pp. 207–213, 1970.

[3] H. Duan, X. Yi, Z. Huang, and J. Wang. A unified scheme for prediction of effective moduli of multiphase composites with interface effects. part ii: application and scaling laws. *Mechanics of Materials*, vol. 39, n. 1, pp. 94–103, 2007.

[4] V. Dolean, P. Jolivet, and F. Nataf. *An introduction to domain decomposition methods: algorithms, theory, and parallel implementation*. SIAM, 2015.

[5] T. P. A. Mathew. *Domain decomposition methods for the numerical solution of partial differential equations*. Springer, 2008.

[6] F. Hecht. New development in FreeFEM++. *Journal of Numerical Mathematics*, vol. 20, n. 3-4, pp. 1570–2820, 2012.

[7] A. M. León-Mecías, J. A. Mesejo-Chiong, L. D. Pérez-Fernández, and J. Bravo-Castillero. Computation of the effective conductivity of fibrous composites with imperfect thermal contact by combination of asymptotic homogenization, domain decomposition and finite elements methods. *Defect and Diffusion Forum*, vol. 372, pp. 60–69, 2017.

[8] F. Jelassi, M. Azaez, and E. P. D. Barrio. A substructuring method for phase change modelling in hybrid media. *Computers & Fluids*, vol. 88, pp. 81–92, 2013.

[9] P.-L. Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. *International Symposium on Domain Decomposition Methods for Partial Differential Equations*, vol. 6, pp. 202–223, 1990.

[10] J. A. Otero, R. Rodríguez-Ramos, J. Bravo-Castillero, R. Guinovart-Díaz, F. J. Sabina, and G. Monsivais. Semi-analytical method for computing effective properties in elastic composite under imperfect contact. *International Journal of Solids and Structures*, vol. 50, n. 3-4, pp. 609–622, 2013.