A double-hybrid finite element formulation for general compressible-incompressible elasticity problems using de Rham compatible spaces

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Abstract. This work proposes a novel double-hybrid finite element formulation for general elasticity problems, combining $H(\text{div}, \Omega)$ conforming vector functions for displacement and $L^2(\Omega)$ discontinuous scalar functions for pressure. This pair is De Rham compatible, which means that within the incompressible regime, the divergence-free constrain will hold strongly at element level. As the $H(\text{div})$ spaces only present continuity of the normal displacements across elements boundaries, the tangential displacement continuity can be weakly imposed performing a hybridization of the shear stresses using $H^{-1/2}(\partial\Omega_e)$ functions. From past researches conducted at LabMeC, this approach has demonstrated to pose some numerical difficulties as it leads to a saddle point problem with two constraint variables - the pressure and the shear-stresses. In this work, a second hybridization is done by approximating the tangential component of the primal displacement variable using the dual $H^{1/2}(\partial\Omega_e)$ space. Two benchmarks are used to verify the developed numerical scheme - the classical Cook's membrane and a tridimensional cantilever beam subjected to an end shear load. Optimum convergence rate is achieved under compressible, quasi-incompressible and full incompressible scenarios.

Keywords: Hybrid finite element method, Locally conservative formulation, Incompressible elasticity

1 Introduction

The application of classical Galerkin method in elasticity problems using continuous $H^1(\Omega)$ functions may lead to spurious energy modes under bending, originating the well known shear-locking phenomenon. Another difficulty emerges when the material is treated as quasi or full incompressible, so the stresses tend to infinity and volume locking may happen. Examples of these situations can be seen in the works of Cervera et al. [1] and Belytschko et al. [2].

A way around is to employ a mixed formulation where displacements and stresses (or pressure) are approximated independently. The Taylor Hood elements, proposed by Taylor and Hood [3], is one example that fulfill the *inf-sup* condition and yield stable results within the incompressible regime. The inconvenience is that continuous approximations like this one is not locally conservative as the pair of functions used to approximate displacements and pressure are not de Rham compatible. Another possibility is to use hybrid methods where the interelement continuity of a given field is broken and weakly imposed by means of a Lagrange multiplier (see for instance Harder et al. [4]).

In this work we extend the semi-hybrid approach proposed in [5] for Stokes problems to develop a new primal hybrid finite element formulation for elasticity using De Rham compatible spaces. The $H(\text{div})$ functions are constructed using a systematic methodology described in details by [6, 7]. All the implementations are carried out using a C++ programming environment called NeoPZ[1].

2 Problem statement

Let $\Omega \in \mathbb{R}^3$ be an open domain with Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$ and $\partial\Omega_N$ stand for the boundary portion where the Dirichlet and Neumann boundary conditions are applied, respectively. The mixed form of elasticity problem can then be defined as:

[1]NeoPZ open source platform `https://github.com/labmec/neopz`

$$
\begin{cases}
-\nabla \cdot \boldsymbol{\sigma}'(\mathbf{u}) + \nabla p - \mathbf{b} = \mathbf{0} \ \text{ in } \Omega \\
-\nabla \cdot \mathbf{u} - \frac{1}{\kappa} p = 0 \ \text{ in } \Omega \\
\mathbf{u} = \mathbf{u}_D \ \text{ on } \partial\Omega_D \\
\boldsymbol{\sigma} \cdot \mathbf{n} = \boldsymbol{\sigma}_N \ \text{ on } \partial\Omega_N
\end{cases}
\tag{1}
$$

where $\mathbf{u} \in H(\text{div}, \Omega)$ is the displacement field, $p \in H^1(\Omega)$ is the pressure, $\mathbf{u}_D \in H^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\sigma}_N \in H^{-1/2}(\partial\Omega_N)$ are the prescribed displacements and surface tractions, $\mathbf{n}$ is the outward unit normal vector, $\mathbf{b} \in L^2(\Omega)$ is the body force and $\kappa = \frac{2\mu+3\lambda}{3}$ is the material's bulk modulus. The Cauchy stress tensor is split into its deviatoric and hydrostatic counterparts as:

$$
\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I}, \tag{2}
$$

where the deviatoric stress is computed as:

$$
\boldsymbol{\sigma}' = 2\mu \left(\boldsymbol{\varepsilon} - \frac{1}{3}\text{tr}(\boldsymbol{\varepsilon})\mathbf{I} \right). \tag{3}
$$

3 Double Hybrid finite element formulation

The weak form of Eqs. (1) is obtained through the Garlekin method, choosing appropriate test functions and integrating over the domain. Let $\mathcal{T} = \{\Omega_e, e = 1, \cdots, n_e\}$ be a partition of Ω in n_e finite elements with usual shapes. The set $\mathcal{E}$ contains all the element edges E and $\mathcal{E}_0 = \{E \in \mathcal{E} : E \subset \Omega\}$ denotes the internal edges or element interfaces. The finite subspaces are written in terms of a single discretization parameter $\gamma = (h, k)$, where h is the characteristic element length and k is the polynomial degree. In this work, divergence-compatible pair $(\mathbf{V}^\gamma, W^\gamma) \subset H(\text{div}, \Omega) \times L^2(\Omega)$ is used to approximate the displacement and pressure fields, respectively. Details on the construction of such spaces can be found in the works of De Siqueira et al. [7] and Devloo et al. [6].

Let $\mathbf{V}^\gamma$ and W^γ be finite subspaces of vector and scalar functions, defined respectively by:

$$
\mathbf{V}^\gamma = \left\{ \mathbf{u} \in H(\text{div}, \Omega) \cap H^1(\mathcal{T}) : \mathbf{u}|_{\Omega_e} \in \mathbf{V}_k(\Omega_e), \Omega_e \in \mathcal{T} \right\}, \tag{4}
$$

$$
W^\gamma = \left\{ p \in L^2(\Omega) : p|_{\Omega_e} \in W_k(\Omega_e), \Omega_e \in \mathcal{T} \right\}. \tag{5}
$$

For vector fields belonging to $H(\text{div}, \Omega)$, the normal component across element interfaces are continuous, so $[\![\mathbf{u} \cdot \mathbf{n}]\!]|_E = 0$, where $[\![\bullet]\!]$ refers to the jump operator. Carvalho et al. [5] propose a semi-hybrid method where the continuity of the tangential component of the displacement is weakly imposed by means of a Lagrange multiplier, namely the tangential traction. Introducing the space $\boldsymbol{\Lambda}^\gamma = \left\{ \boldsymbol{\sigma} \cdot \mathbf{t}|_E \in H^{-1/2}(\partial\Omega_e), \Omega_e \in \mathcal{T} \right\}$, the weak statement for the semi-hybrid formulation thus reads: find $\{\mathbf{u}^\gamma, p^\gamma, \boldsymbol{\lambda}^{t\gamma}\} \in \mathbf{V}^\gamma \times W^\gamma \times \boldsymbol{\Lambda}^\gamma$ such that for all $\{\mathbf{v}^\gamma, q^\gamma, \boldsymbol{\eta}^{t\gamma}\} \in \mathbf{V}^\gamma \times W^\gamma \times \boldsymbol{\Lambda}^\gamma$:

$$
\begin{aligned}
\int_{\Omega_e} \boldsymbol{\varepsilon}(\mathbf{v}^\gamma) : \mathcal{D}' : \boldsymbol{\varepsilon}(\mathbf{u}^\gamma) d\Omega_e - \int_{\Omega_e} p^\gamma (\nabla \cdot \mathbf{v}^\gamma) d\Omega_e + \int_{\mathcal{E}^0} \boldsymbol{\lambda}^{t\gamma} \cdot [\![\mathbf{v}^\gamma]\!] \, d\mathcal{E}^0 = \\
\int_{\Omega_e} \mathbf{v}^\gamma \cdot \mathbf{b} d\Omega_e + \int_{\partial\Omega_N} \mathbf{v}^\gamma \cdot \boldsymbol{\sigma}_N \, d\partial\Omega_e
\end{aligned}
\tag{6}
$$

$$
-\int_{\Omega_e} (\nabla \cdot \mathbf{u}^\gamma) \, q^\gamma \, d\Omega_e - \int_{\Omega_e} \frac{1}{\kappa} \, p^\gamma q^\gamma d\Omega_e = \mathbf{0} \tag{7}
$$

$$
\int_{\mathcal{E}^0} [\![\mathbf{u}^\gamma]\!] \cdot \eta^{t\gamma} \, d\mathcal{E}^0 = \mathbf{0}, \tag{8}
$$

where $\mathbf{v}^\gamma$, q^γ and $\boldsymbol{\eta}^{t\gamma}$ are test functions for displacement, pressure and tangential traction, respectively. Equation (8) plays the role of weakly enforcing the tangential displacement continuity over the element interfaces.

The displacement functions can be decomposed in internal and normal to the boundary counterparts. As the pressure functions are discontinuous, we can statically condense at element level all the internal variables and pressures except by one. Thus, this formulation results in a saddle-point problem with two constraints, i.e. pressure and tangential traction. This type of systems requires a specific pivoting strategy in order to avoid zero pivots during matrix decomposition, while using an iterative strategy demonstrated to be unstable. In this work, however, we perform a second hybridization of the tangential traction so the tangential displacements can be introduced as a variable acting on the element interfaces. This strategy aims to address some of the drawbacks associated with the semi-hybrid scheme mentioned before.

Introducing the finite subspace for the tangential velocity functions

$$\boldsymbol{\mathcal{L}}^{\gamma} = \left\{ \mathbf{u}^{t}|_{E} \in H^{1/2}(\partial\Omega_e), \Omega_e \in \mathcal{T} \right\}, \tag{9}$$

the double-hybrid form thus reads: find $\{\mathbf{u}^{\gamma}, p^{\gamma}, \boldsymbol{\lambda}_e^{t\,\gamma}, \mathbf{u}^{t\gamma}\} \in \mathbf{V}^{\gamma} \times W^{\gamma} \times \boldsymbol{\Lambda}^{\gamma} \times \boldsymbol{\mathcal{L}}^{\gamma}$ such that for all $\{\mathbf{v}^{\gamma}, q^{\gamma}, \boldsymbol{\eta}_e^{t\,\gamma}, \mathbf{v}^{t\gamma}\} \in \mathbf{V}^{\gamma} \times W^{\gamma} \times \boldsymbol{\Lambda}^{\gamma} \times \boldsymbol{\mathcal{L}}^{\gamma}$, the following equations are satisfied:

$$\begin{aligned} \int_{\Omega_e} \boldsymbol{\varepsilon}(\mathbf{v}^{\gamma}) : \mathcal{D}^{'} : \boldsymbol{\varepsilon}(\mathbf{u}^{\gamma}) \, d\Omega_e - \int_{\Omega_e} p^{\gamma}(\nabla \cdot \mathbf{v}^{\gamma}) d\Omega_e - \int_{\partial\Omega_e} \boldsymbol{\lambda}_e^{t\,\gamma} \cdot \mathbf{v}^{\gamma} \, d\partial\Omega_e = \\ \int_{\Omega_e} \mathbf{v}^{\gamma} \cdot \mathbf{b} \, d\Omega_e + \int_{\partial\Omega_N} \mathbf{v}^{\gamma} \cdot \boldsymbol{\sigma}_N \, d\partial\Omega_e \end{aligned} \tag{10}$$

$$-\int_{\Omega_e} (\nabla \cdot \mathbf{u}^{\gamma}) \, q^{\gamma} \, d\Omega_e - \int_{\Omega_e} \frac{1}{\kappa} \, p^{\gamma} q^{\gamma} \, d\Omega_e = \mathbf{0} \tag{11}$$

$$\int_{\partial\Omega_e} (\mathbf{u}^{t\gamma} - \mathbf{u}^{\gamma}) \cdot \boldsymbol{\eta}_e^{t\,\gamma} \, d\partial\Omega_e = \mathbf{0} \tag{12}$$

$$\int_{\partial\Omega_e} \boldsymbol{\lambda}_e^{t\,\gamma} \cdot \mathbf{v}^{t\gamma} \, d\partial\Omega_e = \mathbf{0}, \tag{13}$$

where Eq. (13) was introduced to impose the continuity of tangential tractions across $\mathcal{E}^0$. Even though an additional constraint is introduced, tangential tractions $\boldsymbol{\lambda}^{t\gamma}$ is now associated to a single element so it can be statically condensed and eliminated from the global system. For a compressible solid, a symmetric positive-definite matrix is obtained with two unknowns, namely $\mathbf{u}^{\gamma} \in H(\text{div}, \Omega)$ and $\mathbf{u}^{t\gamma} \in H^{1/2}(\partial\Omega_e)$. For the incompressible case where pressures cannot be computed explicitly from the displacements, a piecewise constant pressure associated to each element appears in the global system.

4 Examples

4.1 Uniform stretch of a non-homogeneous material

The first example consists on a unitary cube $\Omega = [0,1] \times [0,1] \times [0,1]$ subjected to an uniform stretch. The following boundary conditions are considered: $u_x = 0.5$ at $x = 1$, $u_x = 0$ at $x = 0$, $u_z = 0$ at $z = 0$, $u_z = 0$ at $z = 1$, $u_y = 0$ at $y = 0$ and $\sigma_N = 0$ at $y = 1$. The domain is composed of two distinct materials, i.e. the first one has young modulus $E_1 = 1$ and poisson $\nu_1 = 0.5$ whilst $E_2 = 2$ and $\nu_2 = 0.5$ are adopted for the second one, as depicted in Figure 1. The analytical solution for this example is written as:

$$\begin{cases} u_x = \frac{x}{2} \\ u_y = -\frac{y}{2} \\ u_z = 0 \end{cases} \quad ; \quad \begin{cases} \sigma_x x = \frac{E}{1+\nu} \\ \sigma_y y = 0 \\ \sigma_z z = -\frac{E(\nu-1)}{2\nu(1+\nu)} \\ p = -\frac{E}{6\nu} \\ \sigma_x y = \sigma_x z = \sigma_y z = 0 \end{cases} . \tag{14}$$

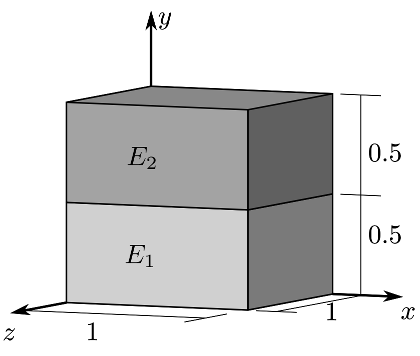
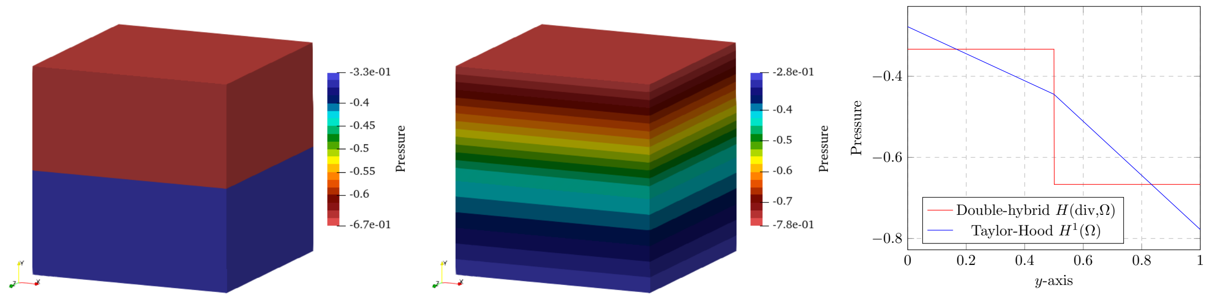


Figure 1. Uniform stretch of a non-homogeneous material - geometry

A comparison of the pressure field obtained with the double-hybrid $H(\text{div}, \Omega)$ and with the classical Taylor-Hood $H^1(\Omega)$ continuous approximation is shown in Figure 2. One can see that the proposed scheme is able to



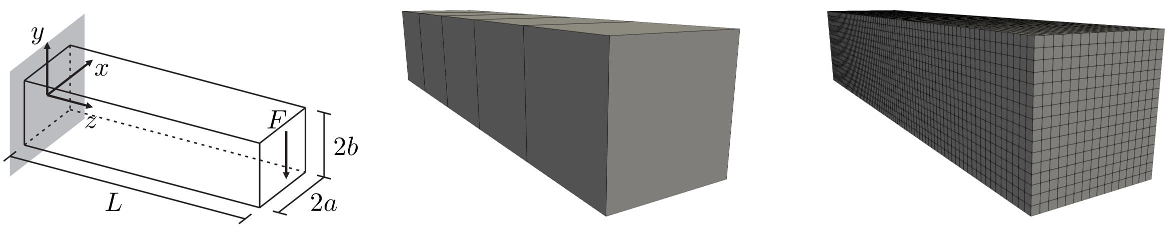
(a) (b) (c)

Figure 2. Uniform stretch of a non-homogeneous material - pressure field obtained with (a) the double-hybrid $H(\text{div}, \Omega)$, (b) Taylor-Hood and (c) pressure distribution over the $y-$axis

exactly reproduce the analytical solution (Figure 2a), whilst in the Taylor-Hood formulation a continuous pressure field is obtained (Figure 2b). In these scenarios with heterogeneous media, the H^1 approximations can lead to solutions with a considerable error as it can be noted in Figure 2c.

4.2 Cantilever beam subjected to an end shear load

A cantilever beam is shown in Figure 3a, where $L = 5$, $a = 0.5$ and $b = 0.5$. The beam is assumed to be fixed at $z = 0$ and subjected to an unitary shear-force $F = \int_{-b}^{b} \int_{-a}^{a} \sigma_{yz} dx dy = 1$ at $z = L$. The Young modulus is $E = 1$ and two values for the poisson coefficient are used: $\nu = 0.3$ and $\nu = 0.5$. The analytical solutions for the stresses and displacements are available in [8].



(a) (b) (c)

Figure 3. Cantilever beam subjected to an end shear load - geometry (at left), coarsest mesh (at center) and finest mesh (at right)

The mesh size is computed as $h_e = 1/2^N$, with $N = \{0, 1, 2, 3, 4\}$ being the number of division in the x direction. The coarsest ($h_e = 1$) and finest ($h_e = 0.0625$) meshes are depicted in Figures 3b-3c. A convergence test is performed for displacement, pressure, stress and mass conservation using the L^2 norm, and the results can be seen in Figures 4-5. Optimal convergence rates of $k + 1$ for the displacement and k for pressure and stresses are observed for both compressible and incompressible cases. For the incompressible case ($\nu = 0.5$), a divergence free displacement field is obtained even for the coarsest mesh thanks to the stable pair $H(\text{div}, \Omega)$-$L^2(\Omega)$.

Figure 6 plots the displacement, pressure, normal and shear stresses fields using the refined mesh with $k = 2$ over the deformed configuration of the beam for the compressible case. The results qualitatively agree with the reference solution of [8].

5 Conclusion

In this work a double-hybrid formulation for tridimensional elasticity problems was presented, where the main advantages and contributions are summarized bellow.

- It leads to a locally conservative scheme as a compatible de Rham complex is used, that is, displacements and pressures are approximated using a stable pair $H(\text{div}, \Omega) - L^2(\Omega)$;
- By performing a hybridization of the tangential stresses, a positive semi definite element matrices are obtained after statically condensing internal displacements, pressure and tangential tractions. This leads to a

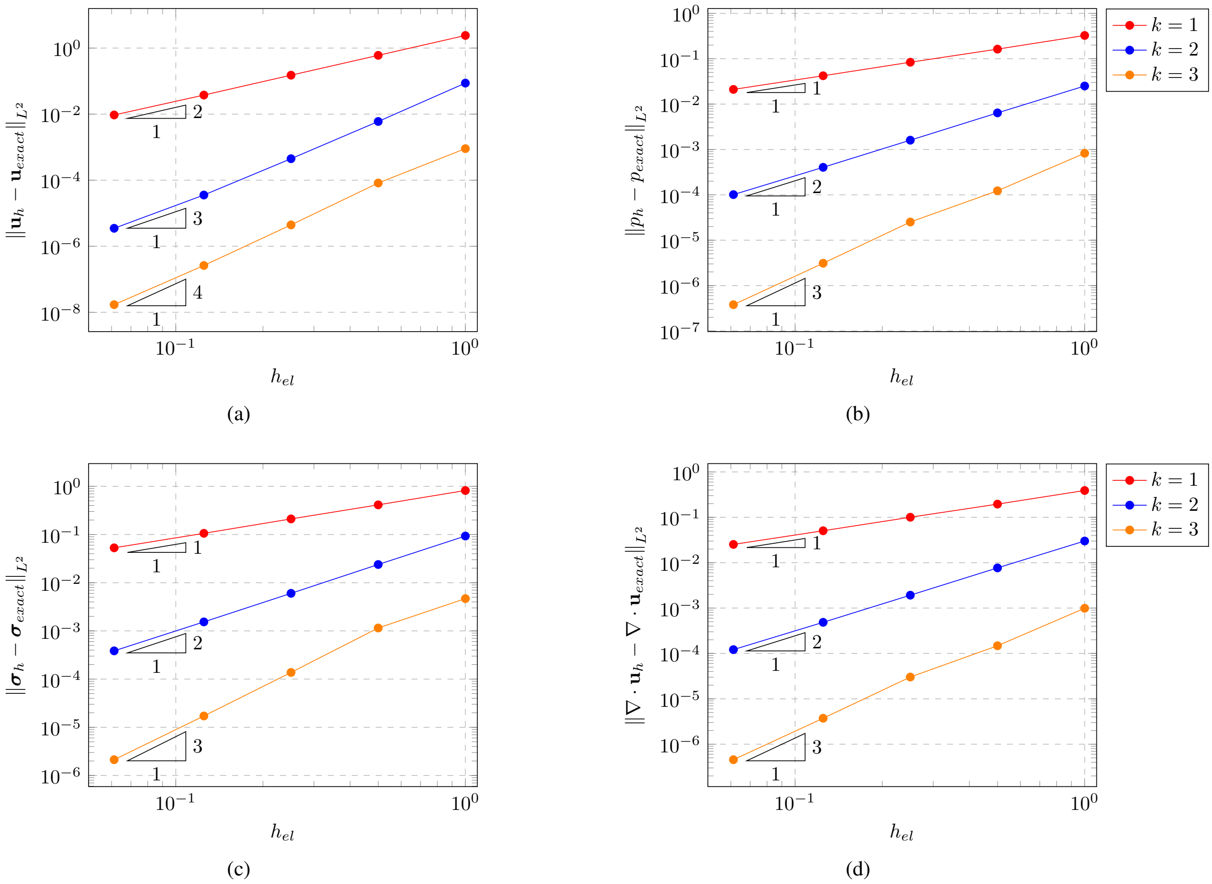


Figure 4. Cantilever beam subjected to an end shear load - convergence analysis for the compressible case ($\nu = 0.30$)

symmetric positive-definite matrix when simulating compressible solids and a saddle point problem with normal and tangential displacements and single elemental pressure when approaching incompressibility;

- One of the advantages of using the proposed scheme is the ability of precisely capturing discontinuities in the material properties, as demonstrated in the first example. When stretching a domain composed of two different materials, the double-hybrid formulation recovered the exact solution with discontinuos pressure;
- A cantilever beam subjected to an end shear load was also simulated and the results showed optimal convergence rates of $k + 1$ for the displacement and k for the remaining variables for all the compressibility ranges.

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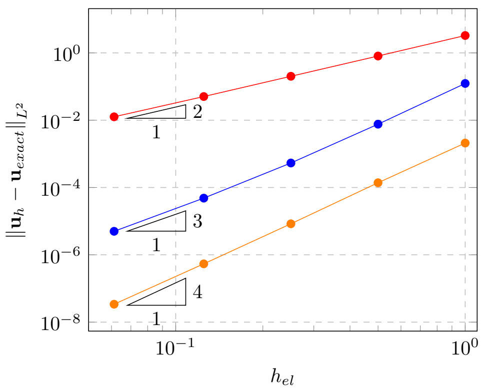
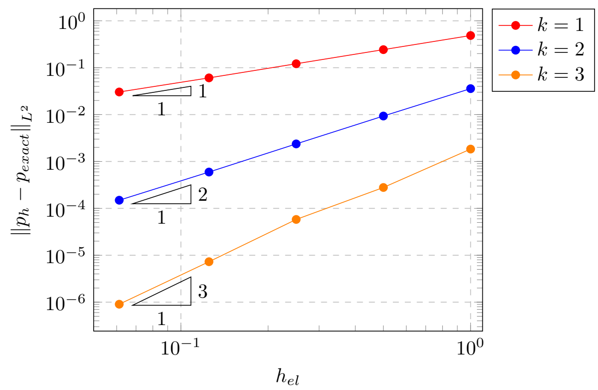
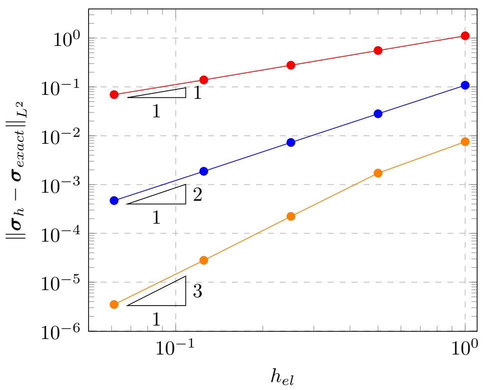


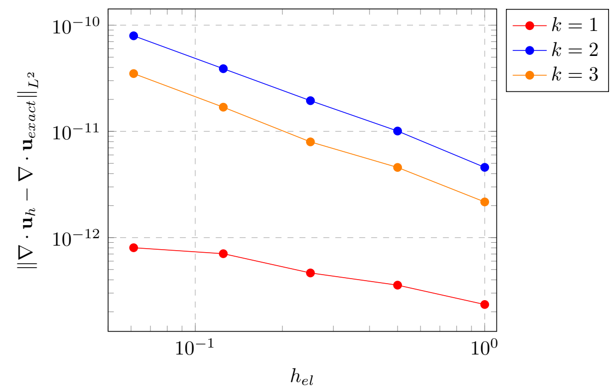
(a) Displacement



(b) Pressure



(c) Stress



(d) Divergence

Figure 5. Cantilever beam subjected to an end shear load - convergence analysis for the incompressible case ($\nu = 0.5$)

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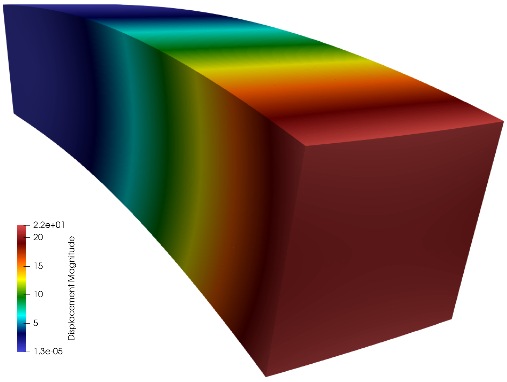
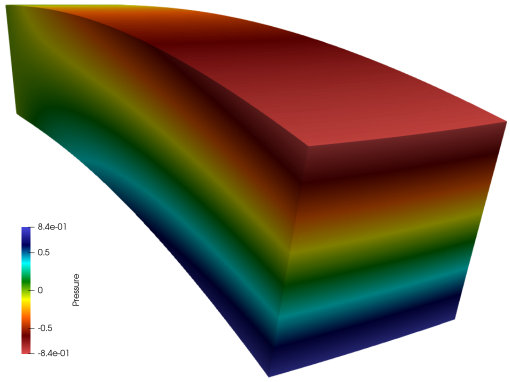
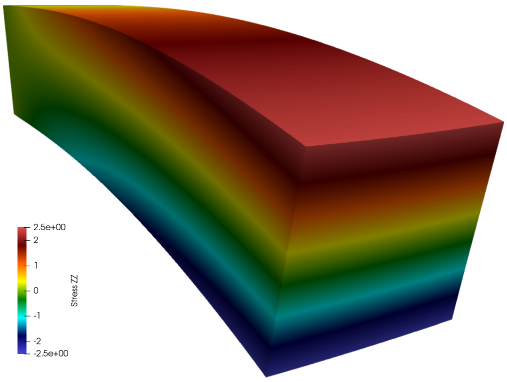


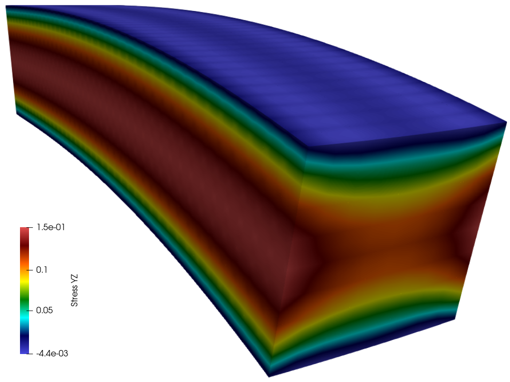
(a) Displacement



(b) Pressure



(c) Normal stress σ_{zz}



(d) Shear stress σ_{yz}

Figure 6. Cantilever beam subjected to an end shear load - snapshots for $\nu = 0.3$