Finite elements formulation for non-prismatic frames

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Abstract. Non-prismatic structural elements constitute a special class of slender structures. These types of structures capture the interest of engineers and architects due to their ability to optimize geometry to meet specific needs, such as weight reduction, material consumption, environmental impact, and costs. Despite the advantages that engineers can gain from using non-prismatic structural elements, modeling these structures poses non-trivial challenges, resulting in inaccuracies that may compromise the benefits offered by such structures. Taking this into consideration, this work presents an innovative approach to obtain the displacement solution for non-prismatic beams, obtaining the elastic stiffness matrix through the principle of virtual work, using the kinematics of Timoshenko's theory. This proposed formulation is independent of bar discretization to achieve an analytical result. This is due to the absence of considering any additional approximations beyond those already contained in the analytical idealization of bar behavior, resulting in a nearly natural discretization of the structure.

Keywords: Non-prismatic beams; principle of virtual work; finite element method.

1 Introduction

Non-prismatic structural elements constitute a special class of slender structures that cause great interest among engineers and architects. This is due to their geometric optimization capability, which can meet specific needs such as reducing weight, material consumption, environmental impact, and costs. These types of structures are used in various engineering applications, such as aircraft wings, helicopter blades and turbines, and bridge beams. Despite the advantages, the modelling of these elements presents complex challenges, resulting in inaccuracies that can compromise their benefits.

Therefore, many studies employ a simplified modelling approach, disregarding the variation of the axis, as seen in Gesteira [1]. However, such simplifications result in inaccuracies both in the calculation of displacements and internal forces. Recently, however, some studies, such as those by Giuseppe Balduzzi et al. [2], and Vo et al. [3] , have started to consider complete modeling, taking into account both the variation in cross-sectional geometry and the axis.

Furthermore, most studies use numerical methods that require many discretizations to solve problems involving non-prismatic elements and variable cross-sections, which lead to high computational costs. Among these methods, isogeometric analysis proposed by Vo et al. [3] and the power series method presented by Masoumeh Soltani and Asgarian [4] stand out. When opting for an analytical solution, it generally becomes more complex due to the adoption of a reduced-dimension methodology, such as in the case of Mercuri et al. [5] using the Hellinger-Reissner functional.

In this context, this work stands out from others by developing the elastic stiffness matrix through the principle of virtual work using the kinematics of Timoshenko's theory, improving upon the approach presented by Vo et al. [3]. The proposed formulation is independent of the discretization of the beam to achieve the analytical result. This is due to the absence of any additional approximations beyond those already contained in the analytical idealization of the beam’s behavior, resulting in a nearly natural discretization of the structure.

2 Analytical Idealization of Non-Prismatic Beam

To investigate the influence of considering the axis in the analysis of non-prismatic structures, this work modifies the kinematics of Timoshenko's theory by incorporating the effects of sections oblique to the beam axis in the formulation deduction. Thus, this section summarizes the main mathematical concepts involved in the idealization of non-prismatic beam behavior. This idealization is based on simplifications of the continuum kinematics through stress and strain expressions, compatibility conditions between displacements and strains, equilibrium conditions, and material constitutive laws.

Figure 1 (a) illustrates the undeformed configuration for the non-prismatic element. In this representation, the undeformed beam axis is depicted as a curve. All sections of the beam are initially considered vertical, and the height of the sections varies along the axis of the structure. Additionally, the director vector corresponds to the unit vector aligned with the section.

All relevant kinematic quantities for describing the non-prismatic beam are schematized in Figure 1 (b). It is important to emphasize that these quantities in the undeformed and updated configurations are denoted by uppercase and lowercase letters, respectively. A generic point on the beam can be identified through the parameters S and Q. Where, S represents the arc length parameter and Q corresponds to the position parameter along the sectional height, so $-0.5h \leq Q \leq 0.5h$. In Figure 1 (b), the tangent vectors to the axis at points G_0 and g_0 are written as:

$$A_1 = \frac{dR_0(S)}{dS} = R_0'(S) \tag{1}$$

$$a_1 = \frac{\partial r_0(S,t)}{\partial S} = A_1 + \frac{du_0(S,t)}{dS} \tag{2}$$

Furthermore, the derivative is with respect to the arc length of the undeformed axis S, and, A_1 corresponds to the unit vector. The director vector A_2 can be directly described through the undeformed configuration, while a_2 corresponding to the deformed configuration, is calculated through the relationship between the rotational operator $\Lambda\ \theta$ and the director vector A_2. However, it is worth noting that in this study, the magnitudes of displacement and rotation are considered very small, thus, $u_0(S,t) \ll 1$,$du_0(S,t)/dS \ll 1$,$|\theta| \ll 1$. As a result, $\Lambda(\theta)$is simplified, so the expression for the director vector a_2 and its derivative are written as:

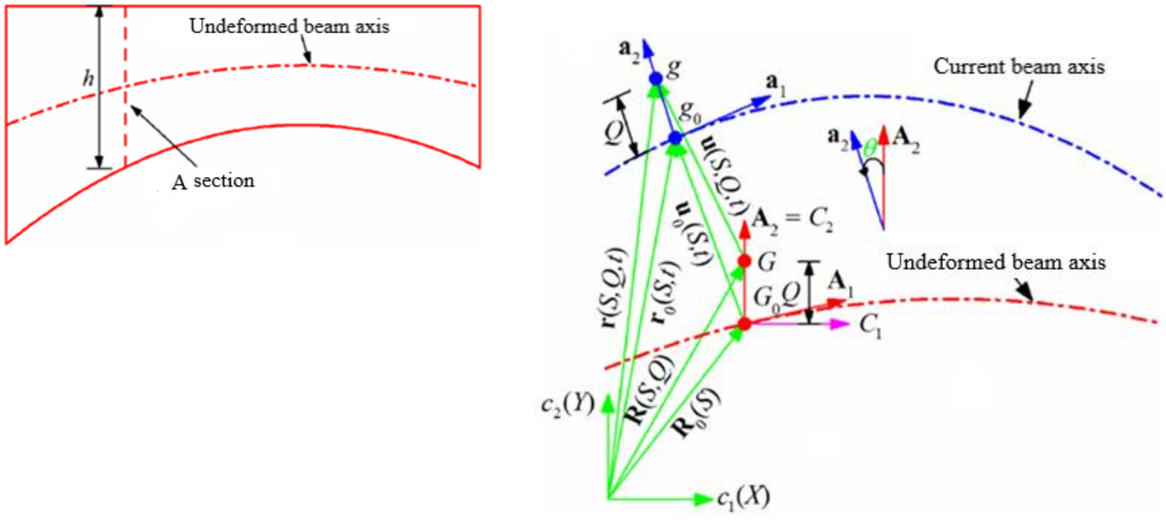
$$a_2 = A_2 + \Lambda\left(\frac{\pi}{2}\right) A_2 \theta \tag{3}$$

$$\frac{da_2}{dS} = \frac{dA_2}{dS} + \Lambda\left(\frac{\pi}{2}\right)\frac{dA_2}{dS}\theta + \Lambda\left(\frac{\pi}{2}\right) A_2 \frac{d\theta}{dS} \tag{4}$$

On the other hand, the expressions for the position vectors G_0 and g_0 are formulated as follows, according to Equations (5) e (6):

$$R(S,Q) = R_0(S) + QA_2 \tag{5}$$

$$r(S,Q,t) = r_0(S,t) + Qa_2 \tag{6}$$



(a) (b)

Figure 1 –a) A possible undeformed configuration of the beam with an oblique section and the beam axis b) Kinematic quantities in the undeformed and updated configurations [3].

Thus, the displacement vector is expressed through points G and g, as indicated by Equation(7):

$$u(S,Q,t) = r(S,Q,t) - R(S,Q) = u_0(S,t) + Q\Lambda\left(\frac{\pi}{2}\right)A_2\theta \tag{7}$$

It is worth noting that the displacement vector at a generic point is determined through the vectors u_0 S,t and θ, which are the kinematic unknowns in this study. The nonlinear terms of the kinematic unknowns are obtained by substituting the tangent vectors and director vectors from Equations (1) and (2) into the non-zero components of the Green-Lagrange strain tensor. However, since the magnitudes of the kinematic unknowns are considered small, higher-order terms can be neglected. Therefore, the equations can be linearized as follows:

$$\varepsilon_x^a = A_1\frac{du_0}{dS} \qquad \gamma^s = A_2\frac{du_0}{dS} + A_1\Lambda\left(\frac{\pi}{2}\right)A_2\theta \tag{8}$$

$$\kappa^f = \frac{dA_2}{dS}\frac{du_0}{dS} + A_1\Lambda\left(\frac{\pi}{2}\right)\frac{dA_2}{dS}\theta + A_1\Lambda\left(\frac{\pi}{2}\right)A_2\frac{d\theta}{dS} \tag{9}$$

It is important to highlight that the expression for shear distortion in an element with a varying axis, γ^s, depends on both the displacement vector u_0 which corresponds to axial and transverse displacements, and the rotation. This contrasts with the shear distortion in prismatic beams and those with variable cross-sections, where the dependence was only on transverse displacement and rotation.

The differential relationship between the strain components of an infinitesimal beam element and the corresponding internal resultants, known as axial force N^{11}, shear force Q^{12} and bending moment M^{11}, based on the director vectors $A_\alpha \otimes A_\beta$, as illustrated in Figure 1 (b), is characterized by the internal relative displacements. This relationship is calculated by integrating the stress components over the cross-section of the element:

$$N^{11} = EA\left(A_1\frac{du_0}{dS}\right) \tag{10}$$

$$Q^{12} = ZA\left(A_1\frac{du_0}{dS}\right) + \kappa_s GA\left(A_2\frac{du_0}{dS} + A_1\Lambda\left(\frac{\pi}{2}\right)A_2\theta\right) \tag{11}$$

$$M^{11} = EI\left(\frac{dA_2}{dS}\frac{du_0}{dS} + A_1\Lambda\left(\frac{\pi}{2}\right)\frac{dA_2}{dS}\theta + A_1\Lambda\left(\frac{\pi}{2}\right)A_2\frac{d\theta}{dS}\right) \tag{12}$$

where the coefficient Z is given by:

$$Z = (A_1 A_2)(2G - E) \tag{13}$$

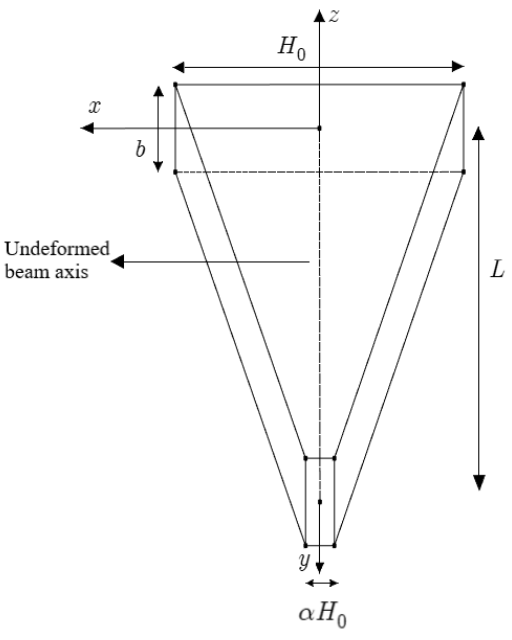
In Eqs (10)-(13) A, I and κ_s correspond to the cross-sectional area, the moment of inertia, and the shape factor, respectively. This factor defines the effective shear area of the cross-section and has a value of 5/6 for rectangular sections, which is the type of section used in this study.

3 Formulation of the discrete problem by finite elements in non-prismatic beams

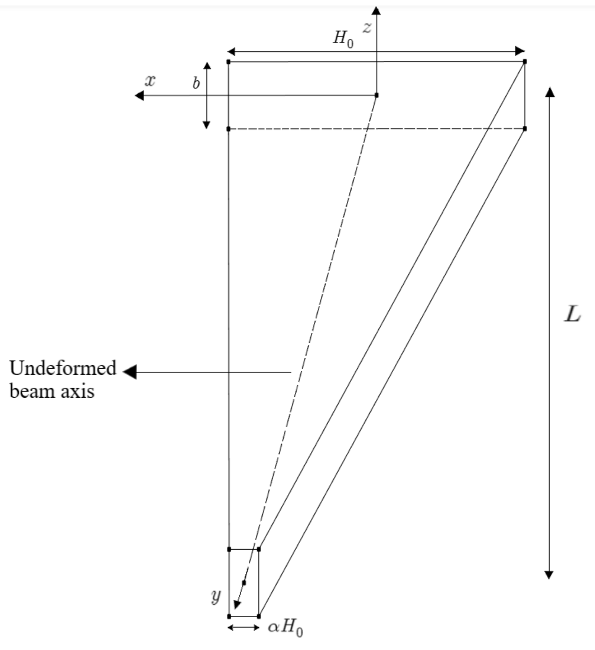
In this context, with the aim of representing non-prismatic structures in a way that requires minimal and nearly immediate discretization, it is proposed to use the principle of virtual work combined with the new deduction of the kinematics of Timoshenko's theory. This new approach considers sections oblique to the beam axis in deriving the expression, as presented in the previous section. Thus, it is possible to represent non-prismatic elements without the need for additional approximations beyond those already included in the analytical idealization of beam behavior.

In this section, with the aim of obtaining the elastic stiffness matrix for elements with a tapered beam, considering and not considering that the beam axis is aligned with the cartesian system, the height $H(x)$ is assumed to vary linearly along the element. Therefore, $H(x)$, can be written as a function of the known values at the beginning, H_0, and end, H_L, of each element of length L and thickness b. Since the variation is linear, it is possible to express H_L as a function of H_0, such that $H_L = \alpha H_0$:

$$H(x) = H_0 + (H_1 - H_0)\frac{x}{L} = H_0\left(1 + (\alpha - 1)\frac{x}{L}\right) \tag{14}$$



(a)



(b)

Figure 2 –a) Tapered beam without variation in the beam axis b) Tapered beam considering that the beam axis is aligned with the cartesian system.

From Equation (14), one can derive the functions for the cross-sectional area, $A(x)$, bending inertia, $I(x)$, and the dimensionless parameter $\Omega(x)$, which relate the bending inertia , $EI(x)$, to the shear inertia, $G\kappa_s A(x)$. This parameter is used to account for the beam length L:

$$\Omega(x) = \frac{EI(x)}{G\kappa_s A(x)L^2} \tag{15}$$

The stiffness matrix of a variable cross-sectional element using Timoshenko's formulation is determined according to the force method, where the fundamental parameters are the forces or moments, and the solution to the continuous problem is obtained through the flexibility coefficients. These coefficients are defined by the principle of virtual work. From these coefficients, it becomes possible to calculate the local flexibility matrix of the element. Considering that, in determining these coefficients, we use the expressions for internal forces calculated through the new deduction of Timoshenko's kinematics, and then the stiffness matrix coefficients can be obtained by inverting the resulting flexibility matrix.

Subsequently, the elastic stiffness matrix of a tapered beam structure was derived using the principle of virtual work, considering the variation of the beam axis. For this, Equation (14) were used again to obtain the expressions for height, area, inertia, and the dimensionless parameter, respectively. In this case, unlike the previous one, the expression representing the beam axis is employed:

$$c(x) = \frac{(H_0 - \alpha H_0)x}{2L} \tag{16}$$

Indeed, the expressions used to describe the height and axis of the structure allow for precise analysis. This is because they are used both in the expression of height, for calculating the area, inertia, and dimensionless parameters, and in the expression of the axis, for determining the director vectors, which are essential for deriving the kinematic formulation of Timoshenko's theory. Therefore, the resulting elastic stiffness matrices for both cases of tapered beams, that is, considering or not considering the axis, are available as open-source code in the files TaperedBeamTimoshenko [6] and TaperedBeamObliqueAxisTimoshenko [7].

4 Numerical application

This section analyses the displacements and internal efforts of bars with a tapered beam, considering two cases: one with a constant axis and the other with a variable axis. It is worth noting that all bars have a value of 0.1 for the dimensionless parameter α, in order to represent one of the most critical situations of the structure composed of tapered beams, as it corresponds to a large difference between the final height H_L and the initial height H_0 of the element in Figure 2. It is important to emphasize that, given the absence of an available analytical solution in this specific context, the error was quantified through highly refined modelling. This modelling was developed by applying plane stress elements, using the bilinear quadrilateral element, in the Robot software [8].

Table 1. Properties of the Structure

Parameter.	E (kN/m^2)	G (kN/m^2)	ν	H_0 (m)	b (m)	L (m)	χ
Beam	20.10^7	$76.9230.10^6$	0.3	0.5	0.5	4	5/6

The results of the displacements for the models developed in this study are presented in Figure 3 and Table 2. Comparisons between the two-dimensional model and the numerical models developed in this study, shown in Table 2, demonstrate that the proposed formulation yields satisfactory results regarding displacements. The internal forces are shown in Table 3. In structures where the axis follows a non-constant function as it passes through the centroids of the cross-sections, the geometry of the beam is characterized by defining the equation of this axis, as the cross-section of the structure, is defined at any point along this axis. Consequently, the internal forces are also described according to the equation of the structure's axis: the normal force represents the internal axial force in the direction of the x axis expression, the shear force corresponds to the internal transverse forces in the direction of the y axis expression, and the bending moment refers to the bending around the z axis expression. Thus, it is possible to consider that the internal forces are provided in the generalized local directions and coordinates in the local coordinate system of the beam, that is, coordinates defined through the variable axis.

Considering this, it is observed that the values obtained for displacements and forces differ between the case where the axis varies and the case where it remains constant. This difference occurs for both internal forces and displacements because, when the axis varies, there is a projection of shear and normal forces relative to the variable axis, i.e., the local axis. This causes a change in values compared to those obtained with a constant axis.

Table 2. Results of the displacements

	Constant axis			Varible axis		
	Proposed Model	FEM	Relative Error (%)	Proposed Model	FEM	Relative Error (%)
u(m)	-	-	-	0.0477268	0.0474710	0.538855
v(m)	-0. 8471489	-0.8453530	0.21244	-0.84848806	-0.850669	0.256560
θ	-0.768	-	-	-0.7692140404	-	-

Table 3. Results of internal force

	Constant axis		Varible axis	
Beam points	1	2	1	2
Axial force (kN)	0	0	0.561612214	-0.561612214
Shear force (kN)	10	-10	9.98421714	-9.98421714
Bending moment (kNm)	40	0	40	0

These second analysis studies the displacement of structure for varying values of the dimensionless parameter α, which corresponds to the ratio between the final base H_L and the initial base H_0 of the element, as shown in Figure 2 .To achieve a more precise understanding of the influence of α , its value is varied from 0.1 to 1 . This range allows for a comprehensive evaluation of the impact of α on the calculation of nodal displacement. The results presented in Figure 3 highlight the influence of α on the structure, utilizing a solution based on the principle of virtual work and the kinematics of Timoshenko's theory. Based on the results presented in the graph, it is possible to see that, especially in the case of α =0.1, where the difference between the final and initial base is more pronounced, there is a significant discrepancy in the horizontal displacement when considering the variation along the axis. This highlights the importance of incorporating the variation along the axis in the analysis, providing greater accuracy in the results, especially when dealing with non-prismatic structures.

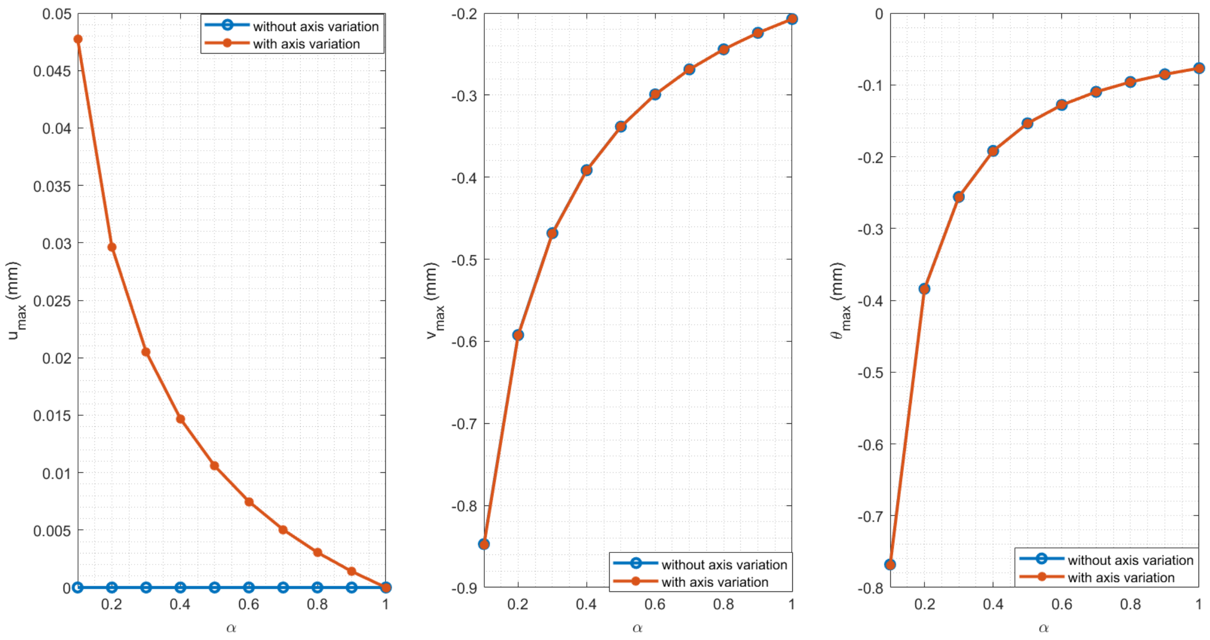


Figure 3 – Results of the element with a tapered beam varying from α -0.1 to 1.

5 Conclusion

In this work, a formulation for the elastic stiffness matrix of non-prismatic bars was developed using the principle of virtual work and the kinematics of Timoshenko's beam theory. The objective was to obtain an exact formulation for these bars, so that they do not depend on the discretization of the structure to achieve an accurate result. As a result, it was possible to determine the displacements and internal forces of a non-prismatic beam. The displacement results, when compared with an extremely refined finite element model, were consistent with the proposed model. Additionally, the internal forces in the bar with a variable axis and cross-section differ from those obtained for a constant axis, as the internal forces are described according to the equation of the structure's axis.

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