Finite-volume theory formulation for the analysis of three-dimensional continuum elastic structures

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Abstract. The finite-volume theory is a powerful numerical technique for structural analysis in solid mechanics and has emerged as an alternative to the finite-element method. The finite-volume theory is an equilibrium-based approach that employs surface-averaged tractions and displacements acting on the faces of a subvolume. In addition, this theory employs the equilibrium equations at the subvolume level and continuity conditions between adjacent subvolumes along subvolume faces. So far, the structural analyses performed by this theory have been in the context of two-dimensional problems in solid mechanics for linear elastic materials. This contribution proposes the stress analysis of a cantilever beam based on the three-dimensional finite-volume theory, which has provided excellent adherence with the analytical solution.

Keywords: finite-volume theory, three-dimensional structures, continuum elastic structures.

1 Introduction

The finite-volume method is a well-established numerical technique for solving boundary-value problems in fluid mechanics governed by parabolic and hyperbolic equations [1,2]. The satisfaction of governing equations in a volumetric sense is a feature of the finite-volume method that distinguishes it from variational numeric techniques, such as the finite element method [3]. The simplicity and stability of the finite-volume method applied to fluid mechanics problems have motivated the implementation of this technique in solid mechanics problems. The finite-volume method for solid mechanics analysis has been developed in different ways during the past 35 years, with the application of the finite difference methods, computational fluid mechanics algorithms, and the finite element method, which has been a source of inspiration [4].

Unlike the former versions of the finite-volume method for structural and solid mechanics analysis, the finite-volume theory has been developed independently by modeling materials with heterogeneous microstructures, including periodic and functionally graded materials. The finite-volume theory has its origin in the higher-order theory for functionally graded materials, developed in a sequence of papers in the 1990's and summarized in Aboudi et al. [5]. Later, the structural and homogenized versions of this theory were reconstructed by Bansal and Pindera [6,7] and Zhong et al. [8], who suggested simplifying the design domain discretization and implementing an efficient local/global stiffness matrix approach. This reconstruction has been extended by Cavalcante et al. [9,10], Marques et al. [11], Gattu et al. [12], and Khatam and Pindera [13,14] by incorporating parametric mapping, which allows the modeling of curved structures and more complex geometries. However, these reconstructed approaches reveal that the higher-order approaches are finite-volume methods, which has motivated the nomenclature change to adequately incorporate the fundamental character of the reconstructed theory [3].

This contribution addresses extending the standard finite-volume theory to analyze 3D continuum elastic structures. Two points can also be highlighted in this new approach: the implementation of a modified stiffness matrix, which directly relates resultant forces and surface-averaged displacements along the subvolume faces (whether than surface-averaged tractions and displacements as in previous implemented approaches), and the implementation of a practical iterative solution to the singularity of the global stiffness matrix that can appear in structural analyzes based on the finite-volume theory and has shown to be a fundamental issue in the 3D analysis. One example is performed to check the proposed approach's numerical stability and verify it with an analytical solution. The results demonstrated the efficiency of the new approach for the analysis of 3D continuum elastic structures.

2 Theoretical Framework

The proposed formulation is based on the standard (or zeroth-order) finite-volume theory for rectangular discretizations presented in Cavalcante and Pindera [15]. This contribution presents the generalized finite-volume theory, where the zeroth-order version involves static variables associated with surface-averaged tractions and kinematic variables associated with surface-averaged displacements. A three-dimensional extension of this theory can be obtained by considering a regular prismatic structure discretized in N_q smaller prismatic elements, called subvolumes, in the context of the finite-volume theory. Figure 1 shows the analysis domain, where x_i represent the global coordinate system, $x_i^{(q)}$ represent the local coordinate system, and l_q, h_q and b_q represent the subvolume dimensions for $q = 1, \dots, N_q$. The components of the displacement field can be extended to 3D problems by considering the incomplete quadratic representation modeled by second-order Legendre polynomials defined in the local coordinate system as follows

$$u_i^{(q)} = W_{i(000)}^{(q)} + x_1^{(q)} W_{i(100)}^{(q)} + x_2^{(q)} W_{i(010)}^{(q)} + x_3^{(q)} W_{i(001)}^{(q)} + \frac{1}{2}\left(3\left(x_1^{(q)}\right)^2 - \frac{l_q^2}{4}\right) W_{i(200)}^{(q)} + \frac{1}{2}\left(3\left(x_2^{(q)}\right)^2 - \frac{h_q^2}{4}\right) W_{i(020)}^{(q)} + \frac{1}{2}\left(3\left(x_3^{(q)}\right)^2 - \frac{b_q^2}{4}\right) W_{i(002)}^{(q)} \quad (1)$$

where $i = 1,2,3$, and $W_{i(mno)}^{(q)}$ represents the unknown coefficients of the displacement field.

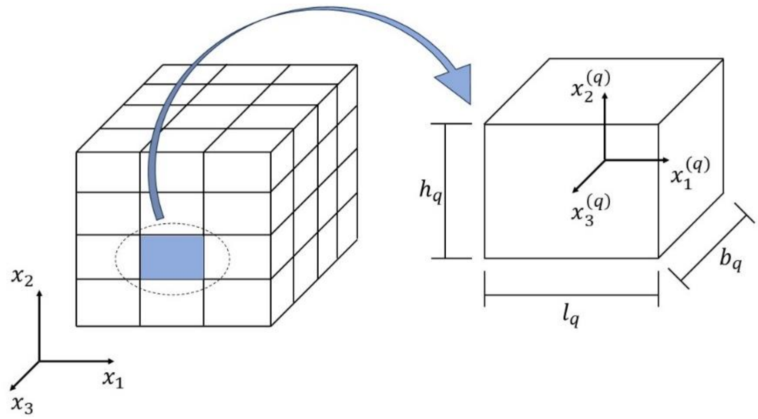

Figure 1. Analysis domain and global coordinate system (left) and subvolume and local coordinate system (right).

In the standard finite-volume theory, the kinematic variables are evaluated in terms of surface-averaged displacements in the subvolume faces, as illustrated in Figure 2. The subvolume face index proposed for this approach assumes the following order: 1, for the bottom face; 2, for the front face; 3, for the right face; 4, for the back face; 5, for the left face; and 6, for the upper face. Therefore, the surface-averaged displacements can be evaluated by the following matricial expression:

$$\bar{\mathbf{u}}^{(q)} = \boldsymbol{A}_{(18\times18)}^{(q)} \boldsymbol{W}^{(q)} + \boldsymbol{a}_{(18\times3)}^{(q)} \boldsymbol{W}_{(00)}^{(q)} \quad (2)$$

where $\bar{\mathbf{u}}^{(q)} = \left[\bar{\mathbf{u}}^{(q,1)}, \bar{\mathbf{u}}^{(q,2)}, \bar{\mathbf{u}}^{(q,3)}, \bar{\mathbf{u}}^{(q,4)}, \bar{\mathbf{u}}^{(q,5)}, \bar{\mathbf{u}}^{(q,6)}\right]^T$ is the local surface-averaged displacements vector, with $\bar{\mathbf{u}}^{(q,p)} = \left[\bar{u}_1^{(q,p)}, \bar{u}_2^{(q,p)}, \bar{u}_3^{(q,p)}\right]$ for $p = 1, \dots, 6$, $\mathbf{W}^{(q)} = \left[\mathbf{W}_1^{(q)}, \mathbf{W}_2^{(q)}, \mathbf{W}_3^{(q)}\right]^T$ is the vector containing the first and second order unknown coefficients, with $\mathbf{W}_i^{(q)} = \left[W_{i(100)}^{(q)}, W_{i(010)}^{(q)}, W_{i(001)}^{(q)}, W_{i(200)}^{(q)}, W_{i(020)}^{(q)}, W_{i(002)}^{(q)}\right]$, $\mathbf{W}_{(0)}^{(q)} = \left[W_{1(000)}^{(q)}, W_{2(000)}^{(q)}, W_{3(000)}^{(q)}\right]^T$ is the vector containing the zeroth order unknown coefficients, and $\boldsymbol{A}_{(18\times18)}^{(q)}$ and $\boldsymbol{a}_{(18\times3)}^{(q)}$ are matrices that depend on the geometric features of the subvolumes.

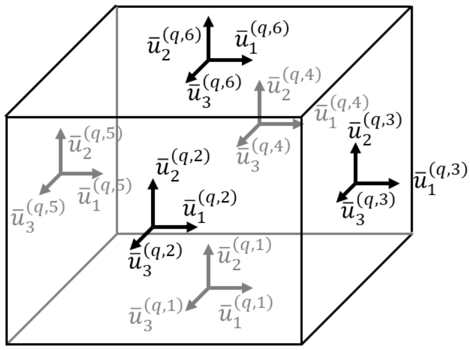

Figure 2. Surface-averaged displacements for a generic subvolume q.

Similarly to the kinematic variables, the static variables are also evaluated in a surface-averaged sense, as illustrated in Figure 3, where the degrees of freedom are indexed as in the surface-averaged displacements. For the standard finite-volume theory, the static variables are evaluated in terms of surface-averaged tractions acting on the faces of a generic subvolume q. The local surface-averaged traction vector $\bar{\mathbf{t}}^{(q)} = \left[\bar{\mathbf{t}}^{(q,1)}, \bar{\mathbf{t}}^{(q,2)}, \bar{\mathbf{t}}^{(q,3)}, \bar{\mathbf{t}}^{(q,4)}, \bar{\mathbf{t}}^{(q,5)}, \bar{\mathbf{t}}^{(q,6)}\right]$, with $\bar{\mathbf{t}}^{(q,p)} = \left[\bar{t}_1^{(q,p)}, \bar{t}_2^{(q,p)}, \bar{t}_3^{(q,p)}\right]^T$ for $p = 1, \dots, 6$, can be evaluated in terms of the vector containing the first and second order unknown coefficients, as follows

$$\bar{\mathbf{t}}^{(q)} = \boldsymbol{B}_{(18\times18)}^{(q)} \boldsymbol{W}^{(q)} \quad (3)$$

where $\boldsymbol{B}_{(18\times18)}^{(q)}$ is composed by elements that depends on the geometric and material properties of the subvolume.

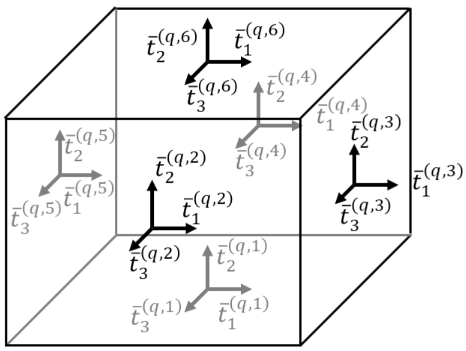

Figure 3. Surface-averaged tractions for a generic subvolume q.

Eq. (4) presents the local equilibrium conditions considering the absence of body forces

$$\sum_{p=1}^{6} \bar{\mathbf{t}}^{(q,p)} S_p^{(q)} = \mathbf{0}_{(3\times1)} \quad (4)$$

where $S_p^{(q)}$ represents the area of the face p associated with the subvolume q, and $\bar{\mathbf{t}}^{(q,p)}$ can be expressed as

$$\bar{\mathbf{t}}^{(q,p)} = \boldsymbol{B}_{(3\times18)}^{(q,p)} \left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \bar{\boldsymbol{u}}^{(q)} - \boldsymbol{B}_{(3\times18)}^{(q,p)} \left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \boldsymbol{a}_{(18\times3)}^{(q)} \boldsymbol{W}_{(0)}^{(q)} \quad (5)$$

where $\boldsymbol{B}_{(3\times18)}^{(q,p)}$ relates the local surface-averaged tractions acting on the face p in the subvolume q with the vector containing the unknown coefficients $\boldsymbol{W}^{(q)}$. Substituting Eq. (5) in Eq. (4), the following expression can be obtained

$$\left(\sum_{p=1}^{6} \boldsymbol{B}_{(3\times18)}^{(q,p)} Q_p^{(q)}\right)\left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \bar{\mathbf{u}}^{(q)} - \left(\sum_{p=1}^{6} \boldsymbol{B}_{(3\times18)}^{(q,p)} Q_p^{(q)}\right)\left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \boldsymbol{a}_{(18\times3)}^{(q)} \boldsymbol{W}_{(0)}^{(q)} = \mathbf{0}_{(3\times1)} \quad (6)$$

From Eq. (6), the vector $\boldsymbol{W}_{(0)}^{(q)}$ can be simplified to

$$\boldsymbol{W}_{(0)}^{(q)} = \bar{\boldsymbol{a}}_{(3\times18)}^{(q)} \bar{\boldsymbol{u}}^{(q)} \quad (7)$$

where $\bar{\boldsymbol{a}}_{(3\times18)}^{(q)} = \left[\left(\sum_{p=1}^{6} \boldsymbol{B}_{(3\times18)}^{(q,p)} Q_p^{(q)}\right)\left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \boldsymbol{a}_{(18\times3)}^{(q)}\right]^{-1} \left(\sum_{p=1}^{4} \boldsymbol{B}_{(3\times18)}^{(q,p)} Q_p^{(q)}\right)\left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1}$. In a similar procedure, the following expression can be obtained for the $\boldsymbol{W}^{(q)}$ vector:

$$\boldsymbol{W}^{(q)} = \bar{\boldsymbol{A}}_{(18\times18)}^{(q)} \bar{\boldsymbol{u}}^{(q)} \quad (8)$$

where $\bar{\boldsymbol{A}}_{(18\times18)}^{(q)} = \left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} - \left(\boldsymbol{A}_{(18\times18)}^{(q)}\right)^{-1} \boldsymbol{a}_{(18\times3)}^{(q)} \bar{\boldsymbol{a}}_{(3\times18)}^{(q)}$.

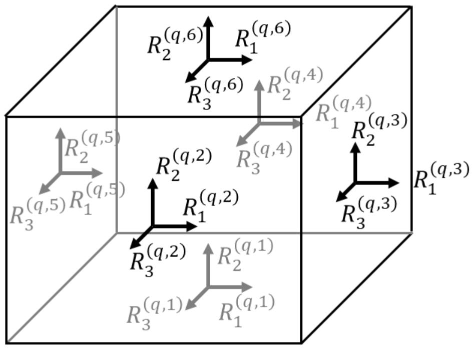

Figure 4. Resultant forces acting on the faces of a generic subvolume q.

By substituting Eq. (8) in Eq. (3), the following local system of equations for a generic subvolume can be expressed by

$$\bar{\mathbf{t}}^{(q)} = \boldsymbol{K}_{(18\times18)}^{(q)} \bar{\mathbf{u}}^{(q)} \quad (9)$$

where $\boldsymbol{K}_{(18\times18)}^{(q)} = \boldsymbol{B}_{(18\times18)}^{(q)} \bar{\boldsymbol{A}}_{(18\times18)}^{(q)}$ is the local stiffness matrix. However, it is unusual to have a system of equations that relates tractions and displacements. Additionally, the matrix $\boldsymbol{K}_{(18\times18)}^{(q)}$ is not symmetric, which requires an additional computational cost to solve the global system of equations. The resultant forces acting on the subvolume faces can be evaluated as

$$R_i^{(q,p)} = \int_{S_p^{(q)}} t_i^{(q,p)}\left(x_1^{(q)}, x_2^{(q)}, x_3^{(q)}\right) dS_p^{(q)} = S_p^{(q)} \bar{t}_i^{(q,p)} \quad (10)$$

Then, the local system of equations can be modified and rewritten as

$$\boldsymbol{R}^{(q)} = \mathbf{S}_{(18\times18)}^{(q)} \bar{\mathbf{t}}^{(q)} = \mathbf{S}_{(18\times18)}^{(q)} \boldsymbol{K}_{(18\times18)}^{(q)} \bar{\mathbf{u}}^{(q)} = \bar{\boldsymbol{K}}_{(18\times18)}^{(q)} \bar{\mathbf{u}}^{(q)} \quad (11)$$

where $\boldsymbol{R}^{(q)} = \left[R_1^{(q,1)}, R_2^{(q,1)}, R_3^{(q,1)}, \dots, R_3^{(q,2)}, \dots, R_3^{(q,3)}, \dots, R_3^{(q,4)}, \dots, R_3^{(q,5)}, \dots, R_3^{(q,6)}\right]^T$ is the local force vector acting on the faces of the subvolume q, with the components illustrated in Figure 4, $\bar{\boldsymbol{K}}_{(18\times18)}^{(q)} = \mathbf{S}_{(18\times18)}^{(q)} \boldsymbol{K}_{(18\times18)}^{(q)}$ is the modified local stiffness matrix, which is a symmetric matrix that relates the local force and displacement vectors, and $\boldsymbol{S}^{(q)}$ is a matrix that can be written as follows

$$\mathbf{S}^{(q)}_{(18\times18)} = \begin{bmatrix} \mathbf{S}^{(q,1)}_{(3\times3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{(q,2)}_{(3\times3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}^{(q,3)}_{(3\times3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}^{(q,4)}_{(3\times3)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}^{(q,5)}_{(3\times3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}^{(q,6)}_{(3\times3)} \end{bmatrix} \quad \text{for} \quad \mathbf{S}^{(q,p)}_{(3\times3)} = \begin{bmatrix} S_p^{(q)} & 0 & 0 \\ 0 & S_p^{(q)} & 0 \\ 0 & 0 & S_p^{(q)} \end{bmatrix} \tag{12}$$

3 Numerical Results

The proposed new formulation can be used in the analysis of solid mechanics problems considering linear isotropic materials. One example is analyzed to verify the stability of the new three-dimensional approach of the finite-volume theory, which consists of a cantilever beam, whose analysis domain and boundary conditions are illustrated in Figure 5. The beam dimensions can be described as $L = 500$ mm, $H = 100$ mm, and $B = 100$ mm, while the material properties are taken as $E = 150$ GPa (Young's modulus) and $\nu = 0.3$ (Poisson ratio), and the applied load is considered as $P = 2$ kN. The analytical solution for a cantilever deep beam with a rectangular transversal section is presented by Saad (2009) with the following stress expressions:

$$\begin{aligned} \sigma_{11} &= \frac{P}{I} x_2 (L - x_1) \\ \sigma_{21} &= -\frac{P}{2I}(a^2 - x_2^2) - \frac{\nu P}{6(1+\nu)I}\left[3x_3^2 - b^2 - \frac{12b^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cos\left(\frac{n\pi x_3}{b}\right)\cosh\left(\frac{n\pi x_2}{b}\right)}{\cosh\left(\frac{n\pi a}{b}\right)}\right] \\ \sigma_{31} &= -\frac{2\nu b^2 P}{(1+\nu)\pi^2 I}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\sin\left(\frac{n\pi x_3}{b}\right)\sinh\left(\frac{n\pi x_2}{b}\right)}{\cosh\left(\frac{n\pi a}{b}\right)} \end{aligned} \tag{13}$$

where I is the beam moment of inertia, $a = H/2$ is the half height, and $b = B/2$ is the half thickness.

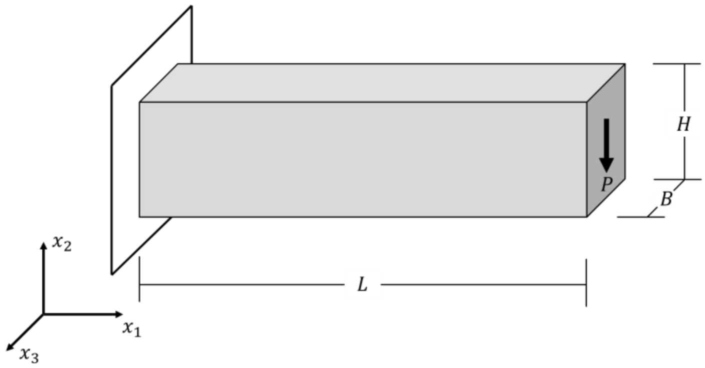

Figure 5. Cantilever beam with a rectangular cross-section.

The structure is discretized in 13 subvolumes in vertical direction along the axis x_2, also 13 subvolumes along x_3 axis, and 45 subvolumes in the length direction along axis x_1. Figure 6 shows the obtained discretized structure after deformation with a displacement amplification of 500 times. In the numerical solution, the applied load is distributed uniformly along the beam right border. Figure 9 presents the overall convergence between the three-dimensional finite-volume theory formulation (FVT) and the analytical solution, in terms of stresses σ_{11}, σ_{12}, and σ_{13}. Considering the Saint Venant principle, these stresses are analyzed in a cross-section at the middle of the beam length to avoid some instability problems that occur in the subvolumes close to the beam support or to the right border where the uniform load is applied. For the finite-volume theory, the stresses are evaluated in a volumetric-averaged sense and located at the subvolume center. As a result, the analytical expressions are evaluated at the central position of each subvolume and compared with the numerical values. In general, the results demonstrate that the proposed numerical technique can reproduce the values obtained by the analytical solution.

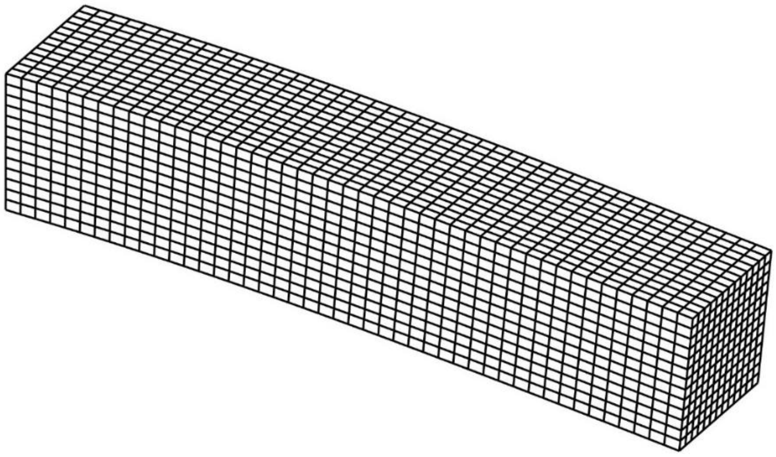

Figure 6. Deformed mesh of the cantilever beam for the flexure problem.

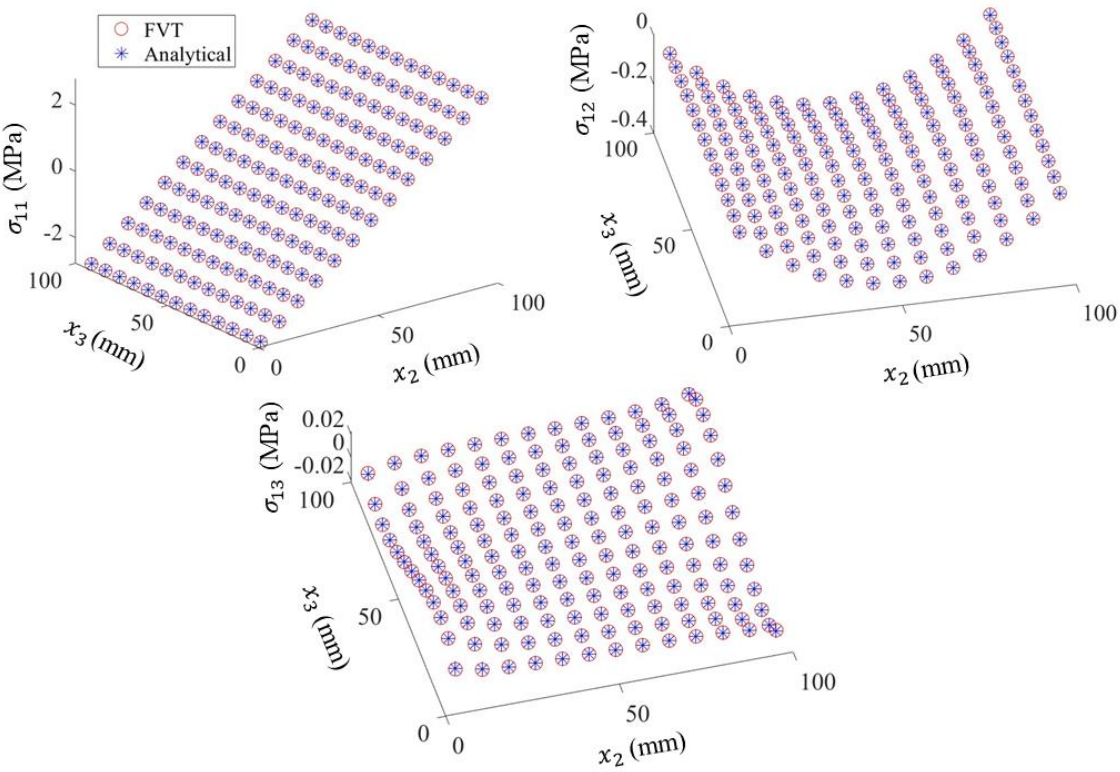

Figure 7. Stresses in a cross-section at the middle of the beam length for the flexure problem.

4 Conclusions

The proposed three-dimensional formulation of the finite-volume theory offers a new source for evaluating elastic stress in continuum structures, considering the use of structured prismatic meshes. A cantilever beam subjected to a concentrated load on the right border is analyzed, where the finite-volume theory formulation has provided excellent adherence to the analytical solution.

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