A boundary element solution for free vibration of FGM beams

J.W.F. de Souza[1], P.C. Nascimento Jr[2], A.C.A Pereira[1], W.K.F. de Brito[1], A.V. Mendonca[1]

[1]*LAMFIC, Federal University of Paraiba*
Cidade Universitaria, 58059-900, João Pessoa, Paraiba, Brazil
johnwilliams.engcivil@gmail.com, acppereira@gmail.com,
klefsonbrito@hotmail.com, mendonca@ct.ufpb.br
[2]*University of Pernambuco at Garanhuns, R. Cap. Pedro Rodrigues,*
55294-902, Pernambuco, Brazil
paulocavalcante_jr@hotmail.com

Abstract. The main purpose in this article is to derive a free vibration solution based on Boundary Element Method (BEM) for Euler-Bernoulli beams made of Functionally Graded Materials (FGM) that have properties that vary continuously through the thickness direction according to the volume fraction of constituents. A boundary element solution for any problem is obtained using integral equations and fundamental solutions defined on continuum, and algebraic equations for the discretized problem. In this paper, original discussions on deriving both integral equations and fundamental solutions for dynamic composite beam problems are properly made. Numerical results of different cases of mechanical properties and boundary conditions are considered and compared to other published works.

Keywords: Boundary elements, Vibration, Integral Equations, Fundamental solutions.

1 Introduction

Nonhomogeneous materials such as laminated composites and Functionally Graded Materials have been attracted attentions of designers due to excellent performance in terms of high stiffness and strength-to-weight ratios. However, laminated composed materials suffer inherent problems associated with de-bonding and de-lamination phenomena caused mainly by larger inter-laminar stresses. Functionally Graded Materials (FGM) are alternative composites where those undesirable effects are reduced. FGM composites are made of two or more distinct constituents and have mechanical properties that vary continuously with respect to spatial coordinates according to the volume fraction of constituents. Many works have been published on FGM problems since topics on design, processing, and applications until mathematical theories and their modeling [3,4]. Most of FGM beam analysis has been done using analytical and finite element solutions for Euler-Bernoulli and Timoshenko models [6-10]. For many engineering problems Boundary Element Method (BEM) is an alternative numerical technique to FEM, but this has not been verified for FGM beam theories. In fact, BEM solutions have been applied to the analysis of beams and frames made of homogeneous materials, for instance, Banerjee [12], Antes [13], Beskos and Providakis [14], Antes et al. [15] and laminated composite beam theories Cavalcanti and Mendonca [16]. In this paper, a direct BEM formulation is established for Euler-Bernoulli FGM so that integral equations, fundamental solutions, and algebraic systems are properly derived. Only in-plane bending problem is taken into account and the BEM results are compared to analytical solutions.

2 Real and Fundamental Problems

The Euler-Bernoulli beam theory is based on the following hypotheses: plane sections initially normal to the mid-surface remain plane and normal to the mid-surface during the bending; the transverse normal stress is small compared to the axial normal stress; displacement, rotation, and strain are assumed to be smooth (i.e., small) fields; damping response is assumed to be negligible. The assumed kinematic relations associated with the axial and in-plane transverse displacements of the beam can be written as follows:

$$U(x,z,t) = u(x,t) - z\,\partial w(x,t)/\partial x\,, \quad W(x,z,t) = w(x,t)\,, \tag{1}$$

where u, v are in-plane and w transverse displacements, respectively. z is the depth of point with respect to mid-plane. The mid-plane kinematic relations are:

$$\varepsilon_x = \partial u/\partial x, \varepsilon_y = \partial v/\partial y,\ \gamma_{xy} = \partial u/\partial y + \partial v/\partial x, k_x = -\partial^2 w/\partial x^2, k_y = -\partial^2 w/\partial y^2, k_{xy} = -2\partial^2 w/\partial x\partial y. \tag{2}$$

In this study is considered the case of a functionally graded beam made of a combination of ceramic and metal. According to the rule of mixtures, the effective material properties (such as Young's modulus E, Poisson's ratio ν and the mass density ρ) are assumed to be:

$$E(z) = E_m V_m + E_c V_c,\ \nu(z) = \nu_m V_m + \nu_c V_c\ and\, \rho(z) = \rho_m V_m + \rho_c V_c\,, \tag{3}$$

where E_m, ν_m, ρ_m , E_c, ν_c and ρ_c are material properties associated with metal and the ceramic, respectively. V_m and V_c are their volume fractions. The volume relations between the two constituents and the ceramic volume fraction in terms of power law distribution across the height of the beam are:

$$V_m + V_c = 1,\ V_c = (z/h + 1/2)^n, \tag{4}$$

where the power law exponent n is defined on range $0 \leq n\ \leq \infty$, with $n = 0$ for a fully ceramic material.

Substituting Eq. (4) into Eq. (3), results:

$$E(z) = E_m + (E_c - E_m)(z/h + 1/2)^n, \nu(z) = \nu_m + (\nu_c - \nu_m)(z/h + 1/2)^n, \rho(z) = \rho_m + (\rho_c - \rho_m)(z/h + 1/2)^n, \tag{5}$$

In FGM beams, the values for mass coefficient and stiffness coefficients associated with extensional, coupling and bending problems can be respectively written as:

$$I_1\ = b\int_{-h/2}^{h/2} \rho(z)dz\ = bh[\rho_m + (\rho_c - \rho_m)/(n+1)],\ (A_{11}, B_{11}, D_{11}) = b\int_{-h/2}^{h/2} (1, z, z^2)E(z)/[1 - \nu^2(z)]dz\,. \tag{6}$$

The evaluation of the integrals in Eq. (6) can be done numerically. If Poisson's ratio is assumed constant $\nu(z) = \nu$ across the beam height, closed forms for stiffness coefficients can be written as in refs. [1] and [2].

$$A_{11}\ = bh[E_m + (E_c - E_m)/(n+1)]/(1-\nu^2),\ B_{11}\ = bh^2(E_c - E_m)n/[2(1-\nu^2)(n+1)(n+2)]$$
$$D_{11}\ = bh^3\{E_m/12 + (E_c - E_m)(n^2 + n + 2)/[4(n+1)(n+2)(n+3)]\}/(1-\nu^2). \tag{7}$$

Normal force and bending moment of the beam Eq. (5) can be written in terms of displacement as follows:

$$N(x,t) = A_{11}\,\partial u(x,t)/\partial x - B_{11}\,\partial^2 w(x,t)/\partial x^2,\ M(x,t) = B_{11}\,\partial u(x,t)/\partial x - D_{11}\,\partial^2 w(x,t)/\partial x^2. \tag{8}$$

Applying the equilibrium conditions, the following relationships can be written:

$$\partial N(x,t)/\partial x + p_x(x,t) = f_{Ix}(x,t), \partial V(x,t)/\partial x + p_z(x,t) = f_{Iz}\,(x,t), \partial M(x,t)/\partial x + V(x,t) = 0 \tag{9}$$

where $f_{Ix}(x,t) = I_1\,\partial^2 u(x,t)/\partial t^2$ and $f_{Iz}(x,t) = I_1\,\partial^2 w(x,t)/\partial t^2$ are inertial forces acting on opposing beam displacements. $V(x,t)$ is the shear force.

Substituting Eq. (8) into Eq. (9), the motion equations in terms of displacements can be finally written:

$$A_{11}\,\partial^2 u(x,t)/\partial x^2 - B_{11}\,\partial^3 w(x,t)/\partial x^3 = I_1\,\partial^2 u(x,t)/\partial t^2 - p_x(x,t)$$
$$B_{11}\,\partial^3 u(x,t)/\partial x^3 - D_{11}\,\partial^4 w(x,t)/\partial x^4 = I_1\,\partial^2 w(x,t)/\partial t^2 - p_z(x,t) \tag{10}$$

If axial and transverse loads act harmonically in time $p_x(x,t) = \breve{p}_x(x)e^{i\omega t}$and $p_z(x,t) = \breve{p}_z(x)e^{i\omega t}$, the beam responses are harmonic as well. Thus, motion equations given in Eq. (10) should be changed to:

$$[L]\{u\} = \{f\} \tag{11}$$

where ω denotes circular frequency and and operator matrix [L] and vectors {u}, {f} are:

$$[L] = \begin{bmatrix} A_{11}\frac{d^2}{dx^2} + S_1 & -B_{11}\frac{d^3}{dx^3} \\ B_{11}\frac{d^3}{dx^3} & -D_{11}\frac{d^4}{dx^4} + S_1 \end{bmatrix}, \{u\} = \begin{Bmatrix} \breve{u}(x) \\ \breve{w}(x) \end{Bmatrix}, \{f\} = -\begin{Bmatrix} \breve{p}_x(x) \\ \breve{p}_z(x) \end{Bmatrix}, S_1 = I_1\omega^2 \tag{12}$$

The fundamental problem of Euler-Bernoulli FGM beam is associated with an infinite domain member under point loads ($\breve{p}_x^*$,$\breve{p}_z^*$) and is governed by same relations applied to the real problem. Therefore, the fundamental governing equations are analogous to Eq. (11), resulting in:

$$[L][u^*] = [f^*] \tag{13}$$

where $[u^*]$ and $[f^*]$ is the fundamental solution matrix given by:

$$[u^*] = \begin{bmatrix} \breve{u}_F^*(x,\hat{x}) & \breve{u}_P^*(x,\hat{x}) \\ \breve{w}_F^*(x,\hat{x}) & \breve{w}_P^*(x,\hat{x}) \end{bmatrix}, [f^*] = -\begin{bmatrix} \breve{p}_x^*(x,\hat{x}) & 0 \\ 0 & \breve{p}_z^*(x,\hat{x}) \end{bmatrix} = -\begin{bmatrix} \delta(x,\hat{x}) & 0 \\ 0 & \delta(x,\hat{x}) \end{bmatrix} \tag{14}$$

With ($\breve{u}_F^*$,$\breve{w}_F^*$) denoting axial and transverse displacement fundamental solutions when only the axial load $\breve{p}_x^*(x,\hat{x}) = \delta(x,\hat{x})$ is applied, while ($\breve{u}_P^*$,$\breve{w}_P^*$) denote the solution counterparts due to the transverse load $\breve{p}_z^*(x,\hat{x}) = \delta(x,\hat{x})$ only. Field and source points are labelled as x and $\hat{x}$. Dirac's delta is represented by $\delta(x,\hat{x})$.

The solution of Eq. (13) is established using method of Hormander [17], which is a decoupling technique where the solution is written in terms of a scalar parameter ψ, yielding to

$$[u^*] = [L^{cof}]^T\psi \tag{15}$$

Combining Eq. (14) and Eq. (12) gives:

$$det[L]\,\psi = -\delta(x,\hat{x}) \tag{16}$$

After evaluating the determinant of [L], Eq. (16) can be written as follows:

$$d^6\psi/dx^6 + C_1\, d^4\psi/dx^4 + C_2\, d^2\psi/dx^2 + C_3\psi = -\delta(x,\hat{x})/(B_{11}^2 - A_{11}D_{11}), \tag{17}$$

where C_1, C_2 and C_3 are coefficients of the differential equation that can be expressed by:

$$C_1 = -D_{11}S_1/(B_{11}^2 - A_{11}D_{11}), C_2 = A_{11}S_1/(B_{11}^2 - A_{11}D_{11}), C_3 = {S_1}^2/(B_{11}^2 - A_{11}D_{11}), \tag{18}$$

For Eq. (17), the characteristic equation can be written as follows:

$$y^3 + C_1y^2 + C_2y + C_3 = 0. \tag{19}$$

The nature of the roots from cubic equation in Eq. (19) affects directly the choice of the set of solutions for the uncoupled ordinary differential equation given in Eq. (16). A positive sign of the discriminant of the reduced cubic equation given in Eq. (19) results in on real positive root and two complex roots. On the other hand, the negative discriminant provides three distinct real roots. If the discriminant is written in terms of frequencies and set to zero, the intervals between the frequency roots can provide information for the sign behavior of the discriminant. Making this study for Eq. (19), the frequency value ω_{ref} is given by:

$$\omega_{ref} = \sqrt{1/I_1}\sqrt{q_1 + \sqrt{{q_1}^2 + q_2}} \tag{20}$$

where:

$$q_1 = (8\,{A_{11}}^2{D_{11}}^2 - 36A_{11}\,D_{11}{B_{11}}^2 + 27{B_{11}}^4)/(8{D_{11}}^3), q_2 = (\tfrac{A_{11}}{D_{11}})^3({B_{11}}^2 - A_{11}D_{11}). \tag{21}$$

For frequencies lower than ω_{ref} give discriminant negative, resulting in three distinct real roots for Eq. (20). Moreover, majority of frequency intensity and materials used in FGM beam give two negative roots. Under this root conditions, a solution proposed in this paper for decoupled ODE in Eq. (16) is:

$$\psi = [a_1 f_1(r)/\sqrt{-y_1} + a_2 g_1(r)/\sqrt{y_2} + a_3 h_1(r)/\sqrt{-y_3}]/K \qquad (19)$$

where $a_1 = (y_2 - y_3)/\lambda$, $a_2 = (y_1 - y_3)/\lambda$, $a_3 = (y_1 - y_2)/\lambda$, $K = 2(B_{11}^2 - A_{11}D_{11})$, $f_1(r) = \sin(r\sqrt{-y_1})$, $g_1(r) = \sinh(r\sqrt{y_2})$, $h_1(r) = \sin(r\sqrt{-y_3})$, $\lambda = -(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)$.

Using Eq. (15), displacement fundamental solutions can be written in terms of ψ, resulting in:

$$\breve{u}_F^*(x,\hat{x}) = -D_{11}d^4\psi/dr^4 + S_1\psi, \breve{w}_F^*(x,\hat{x}) = -sgn(x,\hat{x})B_{11}d^3\psi/dr^3$$
$$\breve{u}_P^*(x,\hat{x}) = sgn(x,\hat{x})B_{11}d^3\psi/dr^3 \text{ and } \breve{w}_P^*(x,\hat{x}) = A_{11}d^2\psi/dr^2 + S_1\psi, \qquad (203)$$

where $d^n\psi/dr^n$ is higher order differentiation of ψ with respect to r.

Other displacement fundamental solutions of interest are:

$$\theta_F^*(x,\hat{x}) = d\breve{w}_F^*(x,\hat{x})/dx = -B_{11}d^4\psi/dr^4,$$
$$\theta_P^*(x,\hat{x}) = d\breve{w}_P^*(x,\hat{x})/dx = sgn(x,\hat{x})(A_{11}d^3\psi/dr^3 + S_1 d\psi/dr),$$
$$d\theta_P^*(x,\hat{x})/d\hat{x} = \theta_{P,\hat{x}}^*(x,\hat{x}) = -(A_{11}d^4\psi/dr^4 + S_1 d^2\psi/dr^2), \qquad (214)$$
$$\breve{u}_{P,\hat{x}}^*(x,\hat{x}) = d\breve{u}_P^*(x,\hat{x})/d\hat{x} = d\breve{w}_F^*(x,\hat{x})/dx, \breve{w}_{P,\hat{x}}^*(x,\hat{x}) = d\breve{w}_P^*(x,\hat{x})/d\hat{x} = -\theta_P^*(x,\hat{x}),$$

and $sgn(x,\hat{x})$ is the sign function and defined by $sgn(x,\hat{x}) = 1\ if x > \hat{x}$ or $sgn(x,\hat{x}) = -1\ if x < \hat{x}$.

Force fundamental solutions can be obtained substituting Eq. (15) into Eq. (8), resulting:

$$\breve{N}_F^*(x,\hat{x}) = sgn(x,\hat{x})[(B_{11}^2 - A_{11}D_{11})d^5\psi/dr^5 + A_{11}S_1 \mathrm{d}\psi/\mathrm{d}r], \quad \breve{M}_F^*(x,\hat{x}) = sgn(x,\hat{x})B_{11}S_1\mathrm{d}\psi/\mathrm{d}r$$
$$\breve{V}_F^*(x,\hat{x}) = B_{11}S_1 d^2\psi/dr^2, \breve{M}_P^*(x,\hat{x}) = (B_{11}^2 - A_{11}D_{11})d^4\psi/dr^4 - A_{11}S_1 d^2\psi/dr^2, \qquad (225)$$
$$\breve{N}_P^*(x,\hat{x}) = -\breve{V}_F^*(x,\hat{x}) , \breve{V}_P^*(x,\hat{x}) = sgn(x,\hat{x})B_{11}[(B_{11}^2 - A_{11}D_{11})d^5\psi/dr^5 - A_{11}S_1 d^3\psi/dr^3].$$

Other force fundamental solutions of interest are:

$$\breve{M}_{P,\hat{x}}^*(x,\hat{x}) = -\breve{V}_P^*(x,\hat{x}), \quad \breve{N}_{P,\hat{x}}^*(x,\hat{x}) = -d\breve{N}_P^*(x,\hat{x})/dx = -B_{11}S_1 d^4\psi/dr^4,$$
$$\breve{V}_{P,\hat{x}}^*(x,\hat{x}) = -d\breve{V}_P^*(x,\hat{x})/dx = (B_{11}^2 - A_{11}D_{11})d^6\psi/dr^6 - A_{11}S_1 d^4\psi/dr^4. \qquad 236)$$

3 Integral equations and algebraic equations

If Eq. (11) is weighted by corresponding fundamental solutions $[u^*]$ associated with $\breve{p}_x^*$ or $\breve{p}_z^*$ loading, the method of weighted residuals states:

$$\int_0^L \left\{ \begin{bmatrix} S_1 + A_{11}\dfrac{d^2}{dx^2} & -B_{11}\dfrac{d^3}{dx^3} \\ B_{11}\dfrac{d^3}{dx^3} & S_1 - D_{11}\dfrac{d^4}{dx^4} \end{bmatrix} \begin{Bmatrix} \breve{u}(x) \\ \breve{w}(x) \end{Bmatrix} + \begin{Bmatrix} \breve{p}_x \\ \breve{p}_z \end{Bmatrix} \right\}^T \begin{bmatrix} u_F^*(x,\hat{x}) & u_P^*(x,\hat{x}) \\ w_F^*(x,\hat{x}) & w_P^*(x,\hat{x}) \end{bmatrix} dx = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}^T \qquad (247)$$

After applying conveniently successive integrations by parts of Eq. (27) and then with help of Eq. (8), integral equations for axial and transverse displacements can be written as follows:

$$\breve{u}(\hat{x}) + [\breve{N}_F^*(x,\hat{x})\breve{u}(x)]_0^L + [\breve{V}_F^*(x,\hat{x})\breve{w}(x)]_0^L - [\breve{M}_F^*(x,\hat{x})\theta(x)]_0^L = [\breve{N}(x)\breve{u}_F^*(x,\hat{x})]_0^L +$$
$$[\breve{V}(x)\breve{w}_F^*(x,\hat{x})]_0^L - [\breve{M}(x)\theta_F^*(x,\hat{x})]_0^L + \int_0^L [\breve{p}_x\breve{u}_F^*(x,\hat{x}) + \breve{p}_z\breve{w}_F^*(x,\hat{x})]dx, \qquad (258)$$

$$\breve{w}(\hat{x}) + [\breve{N}_P^*(x,\hat{x})\breve{u}(x)]_0^L + [\breve{V}_P^*(x,\hat{x})\breve{w}(x)]_0^L - [\breve{M}_P^*(x,\hat{x})\theta(x)]_0^L = [\breve{N}(x)\breve{u}_P^*(x,\hat{x})]_0^L +$$
$$[\breve{V}(x)\breve{w}_P^*(x,\hat{x})]_0^L - [\breve{M}(x)\theta_P^*(x,\hat{x})]_0^L + \int_0^L [\breve{p}_x\breve{u}_P^*(x,\hat{x}) + \breve{p}_z\breve{w}_P^*(x,\hat{x})]dx. \qquad (269)$$

The Euler-Bernoulli FGM beam problems require three unknowns at boundary to be determined, so that an additional equation can be obtained by differentiation of Eq. (29) with respect to source point $\hat{x}$, $\theta(\hat{x}) = d\breve{w}(\hat{x})/d\hat{x}$, yielding to slope integral equation:

$$\theta(\hat{x}) + [\breve{N}_{P,\hat{x}}^*(x,\hat{x})\breve{u}(x)]_0^L + [\breve{V}_{P,\hat{x}}^*(x,\hat{x})\breve{w}(x)]_0^L - [\breve{M}_{P,\hat{x}}^*(x,\hat{x})\theta(x)]_0^L = [\breve{N}(x)\breve{u}_P^*(x,\hat{x})]_0^L + \qquad (30)$$
$$[\breve{V}(x)\breve{w}_{P,\hat{x}}^*(x,\hat{x})]_0^L - [\breve{M}(x)\theta_{P,\hat{x}}^*(x,\hat{x})]_0^L + \int_0^L [\breve{p}_x\breve{u}_{P,\hat{x}}^*(x,\hat{x}) + \breve{p}_z\breve{w}_{P,\hat{x}}^*(x,\hat{x})]dx. \qquad (27)$$

The algebraic representation in terms of boundary quantities for displacements and for forces can be written by collocating the source at the edges of the beam, that is, for $\hat{x} = 0 + \xi$ and $\hat{x} = L - \xi$ with $\xi \to 0$, resulting:

$$\{\breve{u}\} + [\hat{H}]\{\breve{u}\} = [G]\{\breve{p}\} + \{\breve{b}\}, \qquad (28)$$

where $\{\breve{b}\}$ is a null vector for free vibration problems, while boundary vectors $\{\breve{u}\}$ and $\{\breve{p}\}$ shown in Fig.1 are:

$$\{\breve{u}\} = [\breve{u}_i \quad \breve{w}_i \quad \theta_i \quad \breve{u}_j \quad \breve{w}_j \quad \theta_j]^T, \; \{\breve{p}\} = [\breve{N}_i \quad \breve{V}_i \quad \breve{M}_i \quad \breve{N}_j \quad \breve{V}_j \quad \breve{M}_j]^T.$$

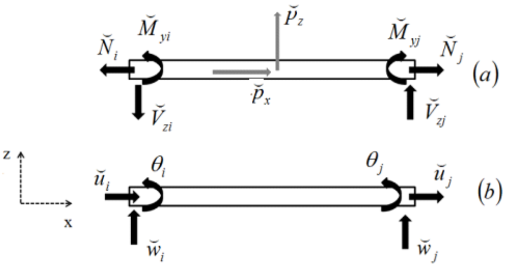

Figure 1- Boundary quantities: a) stress resultants, b) displacements

The influence matrices $[\hat{H}]$ and [G] in Eq. (32) have integral- free elements so that it is required the evaluation of their limits only, resulting in closed forms of influence matrices as follows:

$$[\hat{H}] = \begin{bmatrix} -1/2 & 0 & 0 & \alpha_1(L) & -\alpha_2(L) & \alpha_3(L) \\ 0 & -1/2 & 0 & -\alpha_4(L) & \alpha_5(L) & -\alpha_6(L) \\ 0 & 0 & -1/2 & \alpha_7(L) & -\alpha_8(L) & \alpha_5(L) \\ \alpha_1(L) & \alpha_2(L) & \alpha_3(L) & -1/2 & 0 & 0 \\ \alpha_4(L) & \alpha_5(L) & \alpha_6(L) & 0 & -1/2 & 0 \\ \alpha_7(L) & \alpha_8(L) & \alpha_5(L) & 0 & 0 & -1/2 \end{bmatrix},$$

$$[G] = \begin{bmatrix} 0 & 0 & 0 & -\beta_1(L) & \beta_2(L) & -\beta_3(L) \\ 0 & 0 & 0 & -\beta_2(L) & -\beta_4(L) & \beta_5(L) \\ 0 & 0 & 0 & \beta_3(L) & \beta_5(L) & -\beta_6(L) \\ \beta_1(L) & \beta_2(L) & \beta_3(L) & 0 & 0 & 0 \\ -\beta_2(L) & \beta_4(L) & \beta_5(L) & 0 & 0 & 0 \\ -\beta_3(L) & \beta_5(L) & \beta_6(L) & 0 & 0 & 0 \end{bmatrix} \qquad (33)$$

where:

$\alpha_1(z) = (e_1 cos_1 + e_2 ch_2 + e_3 cos_3)/K,\quad e_1 = A_{11}b_1 - B_{11}y_1c_1,\ e_2 = A_{11}b_2 - B_{11}y_2c_2$

$\alpha_2(z) = -\left(-g_1\sqrt{-y_1}sin_1 + g_2\sqrt{y_2}sh_2 - g_3\sqrt{-y_3}sin_3\right)/K,\quad e_3 = A_{11}b_3 - B_{11}y_3c_3,$

$\alpha_3(z) = -\,(g_1 cos_1 + g_2 ch_2 + g_3 cos_3)/K,\quad g_1 = B_{11}b_1 - D_{11}y_1c_1,$

$\alpha_4(z) = -\left(j_1 sin_1/\sqrt{-y_1} + j_2\, sh_2/\sqrt{y_2} + j_3 sin_3/\sqrt{-y_3}\right)/K,\quad g_2 = B_{11}b_2 - D_{11}y_2c_2,$

$\alpha_5(z) = (m_1 cos_1 + m_2 ch_2 + m_3 cos_3)/K,\ g_3 = B_{11}b_3 - D_{11}y_3c_3,\ j_1 = A_{11}c_1(-y_1) - B_{11}y_1d_1,$

$\alpha_6(z) = \left(m_1 sin_1/\sqrt{-y_1} + m_2\, sh_2/\sqrt{y_2} + m_3 sin_3/\sqrt{-y_3}\right)/K,\ j_2 = -A_{11}c_2(y_2) - B_{11}y_2d_2,$

$\alpha_7(z) = -\,(j_1 cos_1 + j_2 ch_2 + j_3 cos_3)/K,\ j_3 = A_{11}c_3(-y_3) - B_{11}y_3d_3,\ m_1 = -B_{11}c_1y_1 - D_{11}y_1d_1$

$\alpha_8(z) = \left(m_1\sqrt{-y_1}sin_1 + m_2\sqrt{y_2}sh_2 + m_3\sqrt{-y_3}sin_3\right)/K,\quad m_2 = -B_{11}c_2y_2 - D_{11}y_2d_2$

$\beta_1(z) = -\left(b_1 sin_1/\sqrt{-y_1} + b_2 sh_2/\sqrt{y_2} + b_3\, sin_3/\sqrt{-y_3}\right)/K,\quad m_3 = -B_{11}c_3y_3 - D_{11}y_3d_3$

$\beta_2(z) = (c_1 cos_1 + c_2 ch_2 + c_3 cos_3)/K,\quad b_1 = (-D_{11}y_1^2 + S_1)a_1$

$\beta_3(z) = \left(-c_1\sqrt{-y_1}sin_1 + c_2\sqrt{y_2}sh_2 - c_3\sqrt{-y_3}sin_3\right)/K,\quad b_2 = -(-D_{11}y_2^2 + S_1)a_2$

$\beta_4(z) = -\left(d_1 sin_1/\sqrt{-y_1} + d_2 sh_2/\sqrt{y_2} + d_3\, sin_3/\sqrt{-y_3}\right)/K,\quad b_3 = (-D_{11}y_3^2 + S_1)a_3$

$\beta_5(z) = -\,(d_1 cos_1 + d_2 ch_2 + d_3 cos_3)/K,\ c_1 = -B_{11}y_1a_1,\ c_2 = B_{11}y_2a_2,\quad c_3 = -B_{11}y_3a_3,$

$\beta_6(z) = -\left(d_1\sqrt{-y_1}sin_1 + d_2\sqrt{y_2}sh_2 + d_3\sqrt{-y_3}sin_3\right)/K,\quad d_1 = (A_{11}y_1 + S_1)a_1,$

$d_2 = -(A_{11}y_2 + S_1)a_2,\ d_3 = (A_{11}y_3 + S_1)a_3,\quad sin_1 = sin\left(L\sqrt{-y_1}\right),\quad sh_2 = sinh\left(L\sqrt{y_2}\right),$

$sin_3 = sin\left(L\sqrt{-y_3}\right),\quad cos_1 = cos\left(L\sqrt{-y_1}\right),\quad ch_2 = cosh\left(L\sqrt{y_2}\right),\quad cos_3 = cos\left(L\sqrt{-y_3}\right)$

4 Numerical results

Consider a functionally graded beam made of Alumina (Al_2O_3) and Aluminium (Al) having aspect ratio of L/h = 20. Material properties of Alumina are E = 380 GPa, $\rho = 3960\ kg/m^3$ and $\nu = 0.3$, while for Aluminium they are $E_{Al} = 70\ GPa$, $\rho_{Al} = 2702\ kg/m^3$ and $\nu_{Al} = 0.3$. The results from BEM and validation analyses are presented in terms of the fraction volume index n and a dimensionless frequency parameter given by $\bar{\Omega} = \omega L^2 \sqrt{\rho_{Al}/E_{Al}}/h$.

Example 4.1 Consider a simply supported beam, where the first four dimensionless bending frequency results from BEM and analytical solutions are presented in Table 1. It should be noted that a poison ratio of 0.3 is used to calculated stiffness coefficients A_{11}, B_{11} and C_{11} given in Eq. (7). The analytical solution for a simply supported beam can be obtained prescribing the boundary conditions ($N = M = w = 0$) at beam ends. Consequently, if axial and transverse displacements are represented by $u = \sum_1^\infty A_m \sin(m\pi x/L)$ and $w = \sum_1^\infty B_m \sin(m\pi x/L)$ these boundary conditions are automatically satisfied. Inserting u and w into Eq. (11), results in:

$$\begin{bmatrix} S_1 + A_{11}\frac{d^2}{dx^2} & -B_{11}\frac{d^3}{dx^3} \\ B_{11}\frac{d^3}{dx^3} & S_1 - D_{11}\frac{d^4}{dx^4} \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \qquad (294)$$

Natural frequencies can be found by setting the determinant of Eq. (34) equal to zero, resulting in:

$$\omega_{1m} = \sqrt{\Upsilon_1 + \frac{X_m{}^2}{2I_1}\sqrt{\Upsilon_2}},\ \omega_{2m} = \sqrt{\Upsilon_1 - \frac{X_m{}^2}{2I_1}\sqrt{\Upsilon_2}}\ , \qquad (305)$$

where $X_m = m\frac{\pi}{L}$, $\Upsilon_1 = (A_{11}\, X_m{}^2 + D_{11} X_m{}^4)/(2I_1)$, and $\Upsilon_2 = (A_{11} - D X_m{}^2)^2 + 4B_{11}^2 X_m{}^2$.

Table 1. Nondimensional *frequency $\bar{\Omega}$ of a simply supported FGM beam with L/h = 20*

Mode		*n=0*	*n=0.2*	*n=0.5*	*n=1.0*	*n=2.0*	*n=5.0*
1	*Exact*	*5.748*	*5.348*	*4.894*	*4.424*	*4.037*	*3.843*
	BEM	*5.748*	*5.348*	*4.894*	*4.424*	*4.037*	*3.843*
2	*Exact*	*22.993*	*21.392*	*19.572*	*17.688*	*16.133*	*15.361*
	BEM	*22.993*	*21.392*	*19.572*	*17.688*	*16.133*	*15.361*
3	*Exact*	*51.733*	*48.127*	*44.022*	*39.764*	*36.250*	*34.511*
	BEM	*51.733*	*48.127*	*44.022*	*39.764*	*36.250*	*34.511*
4	*Exact*	*91.970*	*85.550*	*78.222*	*70.608*	*64.316*	*61.226*
	BEM	*91.970*	*85.550*	*78.222*	*70.608*	*64.316*	*61.226*

From the results in Table 1, it can be seen that both analytical and BEM results are in good agreement.

Example 4.2 Consider FGM beams having different boundary conditions such as S-S and C-F where S, C, F denote respectively simply, clamped, and free supported ends. In addition, the coefficients A_{11}, B_{11} and C_{11} are evaluated setting null Poisson ratio in Eq. (7). The fundamental bending frequencies for each boundary conditions (B.C.) of the beams are presented in Table 2. The results from BEM are compared to Differential Transformation Method (DTM) given in Ref [18]. It should be noted that a poison ratio of 0.3 is used to calculated stiffness coefficients A_{11}, B_{11} and C_{11} given in Eq. (7).

Table 2. Nondimensional *frequency $\bar{\Omega}$ of FGM beams with L/h = 20 and various boundary conditions*

B.C.		*n=0*	*n=0.2*	*n=0.5*	*n=1.0*	*n=2.0*	*n=5.0*
S-S	*DTM*	*5.483*	*5.102*	*4.669*	*4.221*	*3.852*	*3.668*
	BEM	*5.483*	*5.102*	*4.669*	*4.22*	*3.851*	*3.666*
C-F	*DTM*	*1.953*	*1.816*	*1.663*	*1.504*	*1.372*	*1.307*
	BEM	*1.953*	*1.816*	*1.663*	*1.504*	*1.372*	*1.307*

As it can been in Table 2, a good agreement between BEM and DTM responses have achieved. According to ref. [18], the DTM results were obtained using 15 terms in the series.

Conclusions

In this paper a boundary element modelling was established to FGM beam problems under the hypotheses of the classical laminated beam theory. Only in-plane bending is considered and the results suggest the correctness and effectiveness of the BEM formulation here presented for CBT.

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