Formulation of the fast boundary element method applied to the analysis of anisotropic materials subject to body forces

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Abstract. This work presents a formulation of the boundary element method with fast multipole expansion applied to the analysis of anisotropic elastic materials subject to body forces. Integral equations are obtained using the Somigliana identity. Integrals are divided into near field and far field. Near-field, when the source points and integration elements are close, are treated as in the standard boundary element method, that is, integrating along the element and considering the interaction between source points (nodes) and the elements. On the other hand, in the far field, when the source points and integration elements are far away, the fast multipole method is applied. In this case, the fundamental solution is expanded in a Laurent series, and the node-to-node interaction is replaced by a cell-to-cell interaction. Cells are generated by hierarchical decomposition of the domain using the quad-tree algorithm. Different fast multipole operations are used to take advantage of the hierarchical domain decomposition and expansions of the fundamental solutions. Influence matrices are never explicitly obtained and the matrix-vector product is carried out with linear complexity. The linear system is solved by an iterative method. A preconditioning matrix is used to reduce the number of iterations to obtain a result with a specified accuracy. The effectiveness and efficiency in solving large-scale problems are discussed. The treatment of problems involving body forces is taken into account using the modified boundary condition method. This approach entails augmenting the boundary condition with a specific solution tailored to the problem. Following the solution of the linear system, the particular solution is subsequently subtracted from both displacements and tractions. Importantly, this procedural step eliminates the need to generate additional vectors or matrices within the matrix equation. The formulation presented in this article is based on a representation of complex variables of the integrands, similar to the formulation previously developed for potential (scalar) problems. Validation is carried out by comparing the results obtained by the two formulations: the standard boundary element method and the boundary element method with fast multipole expansion. Numerical examples are presented to demonstrate the efficiency, accuracy, and potential of the boundary element method with fast multipole expansion to solve large-scale problems, i.e., with tens of thousands of degrees of freedom.

Keywords: Boundary element method, fast multipole, anisotropic materials, body forces.

1 Introduction

One of the main drawbacks of the boundary element method is related to the linear algebraic system produced by the method. The matrix of the system is fully populated and nonsymmetrical due to the interaction between the source points and the elements. Thus, this method becomes limited to solving problems with a few thousand degrees of freedom (DOFs), because it requires $\mathcal{O}(N^2)$ operations to compute the coefficients of the matrix and the memory required to store is $\mathcal{O}(N^2)$, where N is the number of DOFs. If the solution of the linear system is found with the help of direct solvers, such as Gaussian elimination, it is necessary $\mathcal{O}(N^3)$ operations. This complexity can be reduced to $\mathcal{O}(N^2)$ with iterative solvers. However, it is still unsuitable for solving large-scale problems.

Therefore, these issues have prevented the BEM from solving these types of problem.

This article introduces the Fast Multipole Boundary Element Method (FMBEM) as a robust approach to addressing 2-D anisotropic elasticity problems subject to body forces (centrifugal loads, gravity, electromagnetic forces, etc.). To this end, a modified boundary condition approach is adopted to account for body forces. The key advantage here is the absence of additional integrals within the boundary integral formulation. This circumvents the need for new multipole expansions, a development that could otherwise undermine the Fast Multipole Boundary Element Method's advancement. Given that the explicit assembly of the influence matrix is avoided, an iterative technique becomes imperative to solve the linear system. The Generalized Minimum Residual (GMRES) method is used for this purpose. For a comprehensive assessment of accuracy and computational efficiency in relation to the standard boundary element method (BEM), numerical examples, including one involving large-scale problems, are scrutinized using both FMBEM and standard BEM approaches. As anticipated, the analysis of the results obtained demonstrates the superior computational efficiency of FMBEM in comparison to conventional BEM, particularly for large-scale problems.

2 Fast Multipole BEM formulation for 2-D anisotropic elasticity

The boundary integral equation for 2-D anisotropic problem is given by [1]:

$$c_{ij}(\mathbf{z_o})u_i(\mathbf{z_o}) + \int_\Gamma T_{ij}(\mathbf{z},\mathbf{z_o})u_j(\mathbf{z})d\Gamma(\mathbf{z}) = \int_\Gamma U_{ij}(\mathbf{z},\mathbf{z_o})t_j(\mathbf{z})d\Gamma(\mathbf{z}) + \int_\Omega U_{ij}(\mathbf{z},\mathbf{z_o})b_j(\mathbf{z})d\Omega(\mathbf{z}) \ \forall\, z \,\epsilon\, \Gamma \quad (1)$$

where the coefficient $c_{ij}(\mathbf{z_o})$ is given by $\delta_{ij} + A_{ij}(\mathbf{z_0})$, in which δ_{ij} is the Kronecker's delta. At a smooth boundary point, $c_{ij}(z_o) = \delta_{ij}/2$. Fundamental solutions for displacement $U_{ij}(\mathbf{z},\mathbf{z_o})$ and traction $T_{ij}(\mathbf{z},\mathbf{z_o})$ are:

$$U_{ij}(\mathbf{z},\mathbf{z_o}) = 2\Re[q_{i1}A_{j1}\log(z_{o_1} - z_1) + q_{i2}A_{j2}\log(z_{o_2} - z_2)] \quad (2)$$

$$T_{ij}(\mathbf{z},\mathbf{z_o}) = 2\Re\left[\frac{g_{j1}(\mu_1 n_1 - n_2)A_{i1}}{(z_{o_1} - z_1)} + \frac{g_{i2}(\mu_2 n_1 - n_2)A_{j2}}{(z_{o_2} - z_2)}\right] \quad (3)$$

where the terms q_{ij}, g_{ji} and A_{ij} are complex material constants, $\Re$ stands for the real part of a complex number and $\log$ is the natural logarithm. Constants μ_k are complex numbers that are the roots of a characteristic polynomial as given by [1]. The field point $\mathbf{z}$ and the source point $\mathbf{z_o}$ are written in complex form as:

$$\mathbf{z} = \left\{ \begin{array}{c} z_1 \\ z_2 \end{array} \right\} = \left\{ \begin{array}{c} x_1 + \mu_1 x_2 \\ x_1 + \mu_2 x_2 \end{array} \right\} \quad (4)$$

$$\mathbf{z}_o = \left\{ \begin{array}{c} z_{o_1} \\ z_{o_2} \end{array} \right\} = \left\{ \begin{array}{c} x_{o_1} + \mu_1 x_{o_2} \\ x_{o_1} + \mu_2 x_{o_2} \end{array} \right\} \quad (5)$$

(x_1, x_2) is the Cartesian coordinate of the field point (x_{o1}, x_{o2}) is the Cartesian coordinate of the source point.

The fundamental solution given by equation (2) and equation (3) can be rewritten by introducing functions $G(z, z_o)$ and its derivative $G'(z, z_o)$. They are given as:

$$G(z_{o_i}, z_i) = \log(z_{o_i} - z_i) \quad \text{for i} = 1, 2 \quad (6)$$

$$U_{ij}(z_o, z) = 2\Re\left[q_{i1}A_{j1}G(z_{o_1}, z_1) + q_{i2}A_{j2}G(z_{o_2}, z_2)\right] \quad (7)$$

$$G'(z_{o_i}, z_i) = \frac{\partial G(z_{o_i}, z_i)}{\partial z} = \frac{1}{(z_{o_i} - z_i)} \quad (8)$$

$$T_{ij}(z_o, z) = 2\Re[G'(z_{0_1}, z_1)g_{i1}(\mu_1 n_1 - n_2)A_{j1} + G'(z_{o_2}, z_2)g_{i2}(\mu_2 n_1 - n_2)A_{j2}] \quad (9)$$

Note that G and G' are very similar to those functions presented by [2] for the Laplace equation. As constant boundary elements are used, all integrations are performed analytically, as given in [2]. Provided we write the fundamental solutions in terms of G and G', the expansion of these fundamental solutions is straight forward, following the work of [2] on potential scalar problems. In fact, a code similar to that provided in [2] can be utilized to develop an FMBEM code for anisotropic elastic problems. Only modification of certain functions within the code is required. In the next section, details of the expansion and multipole operations are given. Note that fast multipole operations have the same names as those given by [2].

2.1 Multipole Expansion

The first operation is multipole expansion, where fundamental solutions are written in series form. An intermediate point $\mathbf{z}_c$ is introduced between the source point $\mathbf{z}_0$ and the field point $\mathbf{z}$. This point is near the field point. So, it is assumed that: $|z_1 - z_{c_1}| << |z_{o_1} - z_{c_1}|$ and $|z_2 - z_{c_2}| << |z_{o_2} - z_{c_2}|$. Thus, the function $G(z_{o_i}, z_i)$ can be rewritten as

$$G(z_{o_i}, z_i) = \sum_{k=0}^{\infty} O_k(z_{o_i} - z_{c_i}) I_k(z_i - z_{c_i}) \tag{10}$$

or
where $O(z)$ and $I(z)$ are given by:

$$O_o(z) = -\log(z) \ \text{ and } \ O_k(z) = \frac{(k-1)!}{z^k}, \ \text{ for } \ k \geq 1 \tag{11}$$

$$I_k(z) = \frac{z^k}{k!}, \ \text{ for } \ k \geq 0, \tag{12}$$

Function $G'(z_o, z)$ is written with the use of derivatives of functions $I(z)$:

$$I'_k(z) = I_{k-1}(z) = \frac{k(z^{k-1})}{k!} \ \text{ for } \ k \geq 1 \ \text{ and } \ I'_o(z) = 0 \tag{13}$$

as

$$G'(z_o, z) = \sum_{k=1}^{\infty} O_k(z_o - z_c) I_{k-1}(z - z_c) \tag{14}$$

All other multipole operations (moment-to-momeent, moment-to-local, local-to-local, and local expansion) are also similar to those presented by [2]. Due to the lack of space, these operations are not shown here. Full details of them can be found in [3].

3 Treatment of body forces

It is commonly understood that including body forces in the governing equation within the standard boundary element method leads to the introduction of domain integrals. Many methods exist to avoid the need for domain discretization [4]. However, to simplify the implementation of fast multipole expansion and reduce computational overhead, this paper presents a novel approach that eliminates the need for extra integrals beyond those already present in the standard boundary element method formulation without body forces. This is achieved by decomposing the solutions to the differential equation, u_i and t_i, into distinctive components: particular solutions, u_i^p and t_i^p, and homogeneous solutions, u_i^h and t_i^h, i.e.:

$$\begin{aligned} u_i &= u_i^p + u_i^h \\ t_i &= t_i^p + t_i^h \end{aligned} \tag{15}$$

The homogeneous solution is the solution of the differential equation without any body force term, i.e.:

$$K_{kl} u_l^h = 0, \qquad k, l = 1, 2 \tag{16}$$

Boundary conditions of this equation are given by:

$$\bar{u}_i^h = \bar{u}_i - \bar{u}_i^p \qquad \text{on } \Gamma_{u_i} \tag{17}$$

$$\bar{t}_i^h = \bar{t}_i - \bar{t}_i^p \qquad \text{on } \Gamma_{t_i} \tag{18}$$

Therefore, the first step in solving the non-homogeneous problem involves matching the boundary conditions with those of the corresponding homogeneous problem. This is done by calculating $\bar{u}_i^h$ and $\bar{t}_i^h$ using equations (17) and (18). Next, equation (16) is solved using the FMBEM, yielding u_i^h and t_i^h. Finally, u_i and t_i can be obtained by subtracting the particular solutions, as outlined in equation (15).

3.1 Centrifugal body forces

The body force field in this case is the one due to a uniform rotatory motion. If ω is the constant angular speed and ρ is the density of the medium, the body force due to centrifugal load is given by:

$$f(x_k) = \rho\omega^2 x_k \tag{19}$$

The following particular solution for displacement can be considered [5]:

$$u_1 = \frac{1}{3}\left[(b_{11}+b_{12})c_1 + b_{11}c_2\right]x_1^3 + \frac{1}{2}b_{13}c_2x_1^2x2 + \left[(b_{11}+b_{12})c_1 + b_{12}c_2\right]x_1x_2^2 + \frac{1}{3}\left[(b_{13}+b_{23})c_1 + \frac{1}{2}b_{23}c_2\right]x_2^3 \tag{20}$$

$$u_2 = \frac{1}{3}\left[(b_{13}+b_{23})c_1 + \frac{1}{2}b_{23}c_2\right]x_1^3 + \frac{1}{2}b_{23}c_2x_1x_2^2 + \left[(b_{22}+b_{1,2})c_1 + b_{12}c_2\right]x_1^2x_2 + \frac{1}{3}\left[(b_{22}+b_{12})c_1 + b_{22})c_2\right]x_2^3 \tag{21}$$

where

$$c_1 = \frac{4b_{12} - b_{33}}{2(3b_{11} + 2b_{12} + 3b_{22} + b_{33})}\rho\omega^2 \tag{22}$$

$$c_2 = -\frac{b_{11} + 2b_{12} + b_{22}}{(3b_{11} + 2b_{12} + 3b_{22} + b_{33}}\rho\omega^2 \tag{23}$$

and b_{ij} is the flexibility matrix element.

3.2 Gravitational body forces

The body force owing to the gravitational load is given by:

$$f(x_k) = \gamma_k \tag{24}$$

where $\gamma_1 = 0$ and $\gamma_2 = \rho g$, g is the gravity acceleration.

The following particular solution for displacement can be considered [5]:

$$u_1 = -\frac{1}{2}\gamma_1 b_{11}x_1^2 - \gamma_2 b_{12}x_1x_2 + \frac{1}{2}(\gamma_1 b_{12} - \gamma_2 b_{23})x_2^2, \tag{25}$$

$$u_2 = \frac{1}{2}(\gamma_2 b_{12} - \gamma_1 b_{13})x_1^2 - \gamma_1 b_{12}x_1x_2 - \frac{1}{2}\gamma_2 b_{22}x_2^2. \tag{26}$$

Particular solutions for stresses can be computed using constitutive equations, considering that strains are linear combinations of displacement derivatives.

4 Numerical example

4.1 Cantilever beam

This study analyzes a cantilever beam with a rectangular cross-section, where the width is significantly smaller than the depth, under plane stress conditions. The lenght of the beam is 100 mm and its height is 5 mm. The analysis considers the beam's own weight as a body force and employs inertial particular integrals and isoparametric quadratic elements. Figure 1 illustrates the beam's geometric configuration (dimensions in millimeters) and its boundary element discretization pattern, with 88 elements total (4 along each vertical side and 40 along each horizontal side). The material properties are as follows: density = 1.534 $\times$ kg/mm^3, E_l = 1.31 $\times$ 10^5 MPa, E_2 = 0.13 $\times$ 10^5 MPa, G_{12} = 0.064 $\times$ 10^5 MPa, and ν_{12} = 0.038.

The material's major principal axis is oriented with beam axis (x_l), indicating a orthotropic material behavior.

To assess the accuracy of the proposed formulation, this problem was analyzed using various mesh densities (from 42 to 5166 constant elements). The results were then compared to the analytical solutions presented by [5] (maximum displacement equals to 0.00068925 mm). Figure 1 illustrates the second coarsest mesh used, consisting of 84 elements.

Figure 2 demonstrates the convergence of the numerical results towards the analytical solution. It is evident that the results approach the analytical solution as the mesh refinement increases. The error in the most refined mesh is 3.9 %.

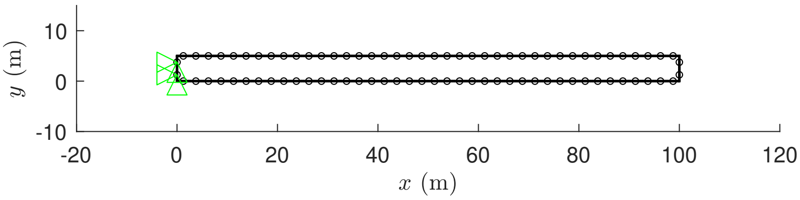

Figure 1. Mesh for the beam problem.

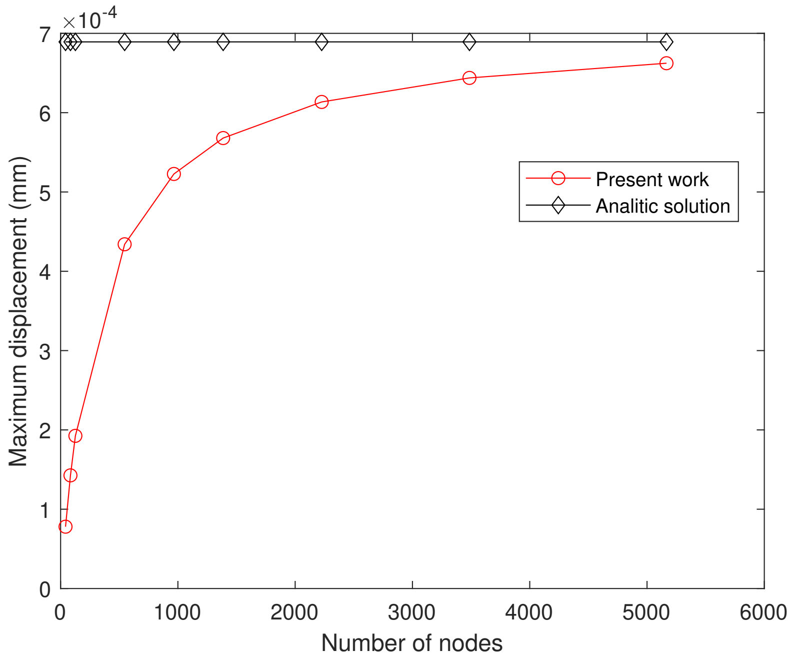

Figure 2. Covergency of the maximum displacement.

4.2 Turbine propeller

To test the proposed formulation on a large-scale problem, this section analyses turbines with different numbers of blades. Figure 3 shows the representation of the turbine blades, with R_i, R_m, and R_o being the internal, medium, and external radii, respectively, which are provided to build the model. The number of blades is n and $\theta = 2\pi$. As a boundary condition, the internal radius is clamped. The load is given only by the centrifugal forces as a result of rotation.

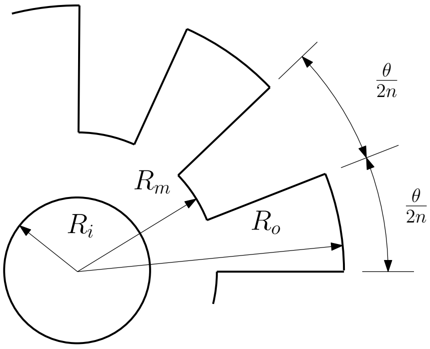

Figure 3. Turbine geometry.

Based on mesh refinement, the number of tree levels `nivT`, and them maximum number of boundary elements per leaf `maxEL`, it is possible to create a hierarchical tree of the turbine model, as shown in Figure 4. In this Figure, it is possible to observe that five blades were created, and the total model was discretized by 380 constant boundary elements. In this case, a maximum of two elements were used per leaf, as can be seen in the zoom highlighted in the Figure. Therefore, the number of blades, as well as the number of boundary elements, can vary in the propeller.

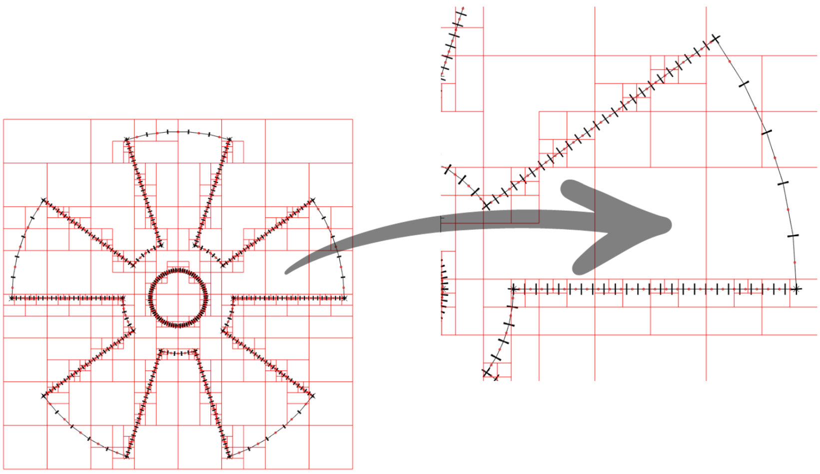

Figure 4. Turbine blades inscribed in a hierarchical tree.

The time calculations are shown in Figure 5, revealing that FMBEM initially does not show a significant advantage over conventional BEM until the boundary elements exceed approximately 1,500. Beyond this threshold, the efficiency of FMBEM compared to standard BEM becomes evident. In the final scenario, employing 12,160 boundary elements, standard BEM consumed approximately 25,668 seconds, while FMBEM required only 2,632 seconds, resulting in a notable time savings of over 6 hours.

In summary, the computational cost efficiency illustrated in Figure 5 underscores the advantages of employing FMBEM to solve large-scale 2D anisotropic elasticity problems.

5 Conclusions

This paper presented the development of the Fast Multipole Boundary Element Method (FMBEM) for solving 2-D anisotropic elasticity problems and its application to systems under body forces. The numerical examples demonstrate the accuracy and efficiency of FMBEM in the resolution of large-scale 2-D anisotropic elasticity

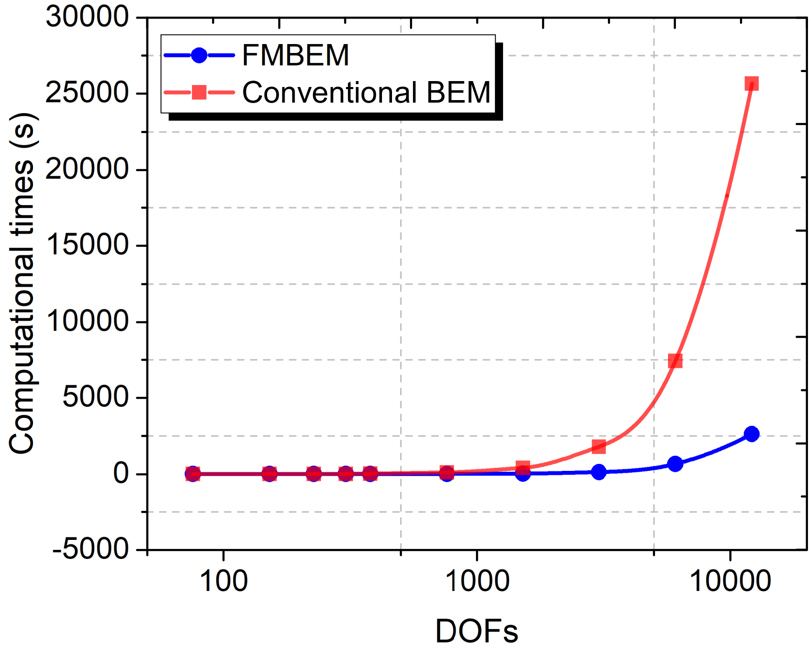

Figure 5. Computational time for turbine propeller.

problems. The proposed approach for handling centrifugal loads is shown to be highly compatible with FMBEM, as it avoids the introduction of new integrals into the boundary integral equations. Instead, centrifugal loads are incorporated by modifying the boundary conditions. The same method can be easily extended to other body force problems, such as self-weight loads.

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