Boundary element method for buckling analysis of elastically connected double-beam system

Moab B. Rodrigues[1], Welky K.F. Brito[1], Arthur C.A. Pereira[1], P. C. Nascimento Junior[2], Angelo V. Mendonça[1]

[1]*Dept. of Civil and Environmental Engineering, Federal University of Paraíba*
Cidade Universitária, 58051-900, João Pessoa/PB, Brasil
klefsonbrito@hotmail.com
[2]*University of Pernambuco, Campus Garanhuns*
R. Cap. Pedro Rodrigues – São José, 55294-902, Garanhuns/PE, Brasil

Abstract. Studies of Double-beam systems have received significant attention from researchers due to their wide applications in civil and mechanical engineering. Commonly, finite element method or exact solutions have been employed to solve double-beam systems, with few others considering buckling of axially loaded double-beam systems with classical boundary conditions. This paper presents a novel formulation of the Boundary Element Method (BEM) to determine the buckling load of the double-beam system elastically connected by a Winkler elastic layer with generalized boundary conditions. Based on the Euler-Bernoulli beam theory, the double-beam system is composed of two identical beams. This paper provides a detailed discussion of each step involved in the BEM, including the fundamental solution, boundary integral equations, and algebraic system. Examples considering different boundary conditions, material properties, and load cases are done and compared to corresponding analytical solution. The results show excellent agreement between the BEM approach and the analytical solution, confirming the accuracy and effectiveness of the technique.

Keywords: BEM, Buckling, DBS, Euler-Bernoulli, Winkler.

1 Introduction

The composite structures, which consist of multiple components connected by flexible interfaces, have gained significant attention due to their wide applications in various engineering fields, including civil, aerospace, and mechanical engineering. Among these structures, the double-beam system (DBS), which consists of two parallel solids joined by an elastic medium, has attracted considerable interest from the scientific community. Numerous studies, predominantly conducted over the past few decades, have addressed issues such as free and forced vibration phenomena [1–10], and bending analysis [11, 12], employing the Euler-Bernoulli beam theory and the Winkler elastic layer model.

Previous studies have investigated the buckling behavior of double-beam systems [13–16], aiming to determine the buckling load and understand the behavior of the system, which is introduced due to the elastic layer between the beams, affecting significantly the stress and strain distributions within the system. Typically, the Euler-Bernoulli beam theory is used in the DBS model linked by a Winkler layer, and analytical or numerical methods are used to solve these problems.

The use of the BEM for the study of DBS are limited to [12], since it presented the fundamental solution for bending analysis of DBS connected by Winkler and Pasternak foundation. The influence of the different boundary conditions, material properties, and layer parameters in the behavior of the system are discussed. Surprisingly, the buckling analysis by BEM for DBS connected by the Winkler elastic layer is not available in the literature.

This paper presents a comprehensive buckling analysis of a double-beam system elastically connected by a Winkler elastic layer using the BEM formulation. All the steps are discussed, including obtaining the fundamental solution and deriving the integral and algebraic equations for buckling. The study examines the effects of various parameters, such as the elastic layer's stiffness and different boundary conditions, on the buckling load of the system. The results of this analysis provide valuable insights into using the BEM formulation to study connected structures with elastic connections. The examples included demonstrate the accuracy and applicability of the technique.

2 Mathematical model

Consider the double-beam system shown in Figure 1, which consists of two prismatic, homogeneous beams of the same length L, subjected to axial loads and continuously joined by a Winkler elastic layer with rigidity K. Based on Euler-Bernoulli beam theory, the effects of shear strain are neglected, i.e., the plane cross-sections, initially perpendicular to the axis of the beam, remain plane and perpendicular to the neutral axis during the bending process. Furthermore, the beams can have different mechanical properties and boundary conditions.

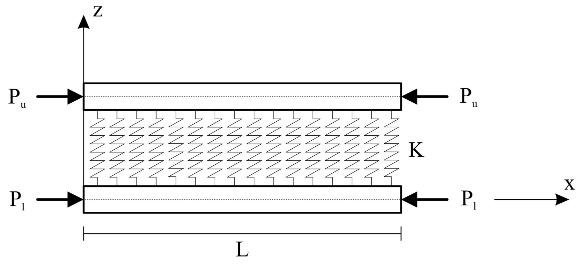


Figure 1. Double-beam system elastically connected by Winkler layer.

Following the Euler-Bernoulli beam theory, the equilibrium equation of the DBS with a Winkler layer in-between is given as follows:

$$\frac{d^4 w_u}{dx^4} + F\frac{d^2 w_u}{dx^2} + c\left(w_u - w_l\right) = g_u/D_u, \tag{1}$$

$$\beta\frac{d^4 w_l}{dx^4} + \chi F\frac{d^2 w_l}{dx^2} - c\left(w_u - w_l\right) = g_l/D_u, \tag{2}$$

where $w_i = w_i(x)$ is the beam's displacement, with the subscript $i = u, l$ denoting the upper and lower beams, respectively, $D_i = E_i I_i$, E_i is the modulus of elasticity, I_i is the moment of inertia of the cross-section, $F = P_u/D_u$, $c = K/D_u$, $\beta = D_l/D_u$, and $\chi = P_l/P_u$. Also, the bending moment and effective shear force are defined as:

$$M_i = -D_i\frac{d^2 w_i}{dx^2}, \tag{3}$$

$$V_i = -D_i\frac{d}{dx}\left(P_i + \frac{d^2 w_i}{dx^2}\right). \tag{4}$$

2.1 Fundamental solution

The fundamental problem can be understood as a virtual problem with an infinite domain subject to concentrated load in terms of Dirac's delta, acting in a source point $\widehat{x}$, governed by the same relations as the real problem. By the direct BEM, the differential governing equation of the fundamental problem will be analogous to real problem, eqs.(1) and (2), resulting in

$$D_u\begin{bmatrix} d^4/dx + Fd^2/dx^2 + c & -c \\ -c & \beta d^4/dx^4 + \chi F d^2/dx^2 - c \end{bmatrix}\begin{bmatrix} w_u^{*u} & w_u^{*l} \\ w_l^{*u} & w_l^{*l} \end{bmatrix} = \delta\left(x,\widehat{x}\right)[I], \tag{5}$$

where x is the field point and Dirac's delta is defined as:

$$\delta\left(x,\widehat{x}\right) = \begin{cases} \infty, \; if\; x = \widehat{x}, \\ 0, \; if\; x \neq \widehat{x}. \end{cases} \tag{6}$$

Note that the coupled governing equations (5) require additional steps to uncouple and derive their solution. Therefore, it is assumed that the fundamental solutions to upper and lower beam displacements can be expressed in uncoupled form as:

$$w_u^{*u}(x,\widehat{x}) = D_u\left(\beta\frac{d^4\psi}{dr^4} + \chi F\frac{d^2\psi}{dr^2} + c\psi\right), \tag{7}$$

$$w_l^{*u}(x,\widehat{x}) = w_u^{*l}(x,\widehat{x}) = cD_u\,\psi, \tag{8}$$

$$w_l^{*l}(x,\widehat{x}) = D_u\left(\frac{d^4\psi}{dr^4} + F\frac{d^2\psi}{dr^2} + c\psi\right), \tag{9}$$

where $r = |x - \widehat{x}|$, the superscript * denotes the fundamental solution and $\psi(r) = \psi$ is the uncouple fundamental solution. Replacing eqs.(7) and (8) into eq.(5), and subsequently making a change of variable $d^2\psi/dr^2 = y$, the homogeneous equation can be written as follows:

$$y\left\{y^3 + F\left(1+\frac{\chi}{\beta}\right)y^2 + \left[c\left(1+\frac{1}{\beta}\right) + F^2\frac{\chi}{\beta}\right]y + Fc\left(\frac{\chi}{\beta}+\frac{1}{\beta}\right)\right\} = 0. \tag{10}$$

where the root y_0 is equal to zero, and the others are given by:

$$y_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}} - \frac{1}{3}F\left(1+\frac{\chi}{\beta}\right), \tag{11}$$

$$y_2 = \frac{-1}{2}\left[F\left(1+\frac{\chi}{\beta}\right) + y_1\right] + \frac{1}{2}\sqrt{F^2\left(\frac{\chi}{\beta}-1\right)^2 - y_1\left[2F\left(1+\frac{\chi}{\beta}\right)+3y_1\right] - 4c\left(1+\frac{1}{\beta}\right)}, \tag{12}$$

$$y_3 = \frac{-1}{2}\left[F\left(1+\frac{\chi}{\beta}\right) + y_1\right] - \frac{1}{2}\sqrt{F^2\left(\frac{\chi}{\beta}-1\right)^2 - y_1\left[2F\left(1+\frac{\chi}{\beta}\right)+3y_1\right] - 4c\left(1+\frac{1}{\beta}\right)}, \tag{13}$$

where $p = \frac{1}{3}F^2\left[\frac{\chi}{\beta}\left(1-\frac{\chi}{\beta}\right)-1\right] + c\left(1+\frac{1}{\beta}\right)$, $q = \frac{F}{27}\left\{2F^2\left(\frac{1}{2}-\frac{\chi}{\beta}\right)\left(2-\frac{\chi}{\beta}\right)\left(1+\frac{\chi}{\beta}\right) + 9\frac{c}{\beta}\left[\chi\left(2-\frac{1}{\beta}\right)-\beta+2\right]\right\}$, and $\Delta = (p/3)^3 + (q/2)^2$.

The fundamental solution is dependent on the nature of the roots, and the proposed solutions can be seen as follows:

Case I. For $\Delta < 0$, the root y_1 is negative, while y_2, y_3 are complex conjugate roots. Thus, the proposed solution is expressed as follows:

$$\begin{aligned}\psi(r) &= -\alpha bs\left(s^2 + y_1^2 - 2ay_1\right)r + \alpha\frac{bs^3}{\sqrt{-y_1}}sin\left(r\sqrt{-y_1}\right)\\ &+ \alpha e^{-pr}\left[y_1 q\left(4mp^2 + s^2 + ny_1\right)cos\left(qr\right) + y_1 p\left(4nq^2 + s^2 - my_1\right)sin\left(qr\right)\right],\end{aligned} \tag{14}$$

where $\alpha = \left[2D_u^2\beta s^3 by_1\left(s^2 + y_1^2 - 3ay_1\right)\right]^{-1}$, $a = p^2 - q^2$, $b = 2pq$, $m = p^2 - 3q^2$, $n = q^2 - 3p^2$, $s = p^2 + q^2$, $p = Re\left(\sqrt{y_2}\right)$ and $q = Im\left(\sqrt{y_2}\right)$.

Case II. For $\Delta < 0$, the roots are distinct, and negative. Thus, the proposed solution is expressed as follows:

$$\begin{aligned}\psi(r) &= -\frac{\alpha}{y_1y_2y_3}\left(y_1-y_2\right)\left(y_1-y_3\right)\left(y_2-y_3\right)r + \alpha\frac{y_2-y_3}{y_1\sqrt{-y_1}}sin\left(r\sqrt{-y_1}\right)\\ &+ \alpha\frac{y_3-y_1}{y_2\sqrt{-y_2}}sin\left(r\sqrt{-y_2}\right) + \alpha\frac{y_1-y_2}{y_3\sqrt{-y_3}}sin\left(r\sqrt{-y_3}\right),\end{aligned} \tag{15}$$

where $\alpha = \left[2D_u^2\beta\left(y_1-y_2\right)\left(y_1-y_3\right)\left(y_2-y_3\right)\right]^{-1}$.

Case III. For $\Delta = 0$, the roots are negative, but y_2 and y_3 are identical. Thus, the proposed solution is expressed as follows:

$$\begin{aligned}\psi(r) &= 2\alpha y_2\sqrt{-y_1}\left(y_1-y_2\right)\left(3y_1-7y_2\right)r - 14\alpha y_2^3 sin\left(r\sqrt{-y_1}\right)\\ &+ 2\alpha\left(3y_1-10y_2\right)y_1\sqrt{-y_1}\sqrt{-y_2}sin\left(r\sqrt{-y_2}\right) - \alpha\left(y_1-y_2\right)y_1y_2\sqrt{-y_1}\sqrt{-y_2}r^2 sin\left(r\sqrt{-y_2}\right),\end{aligned} \tag{16}$$

where $\alpha = -\left[4D_u^2\beta y_1 y_2^3\sqrt{-y_1}\left(y_1-y_2\right)\left(7y_1-11y_2\right)\right]^{-1}$.

The explicit forms of the fundamental solutions of displacements can be found by substituting eqs.(14), (15), or (16) into eqs.(7) to (9). In addition, the fundamental solutions for bending moments and shear forces are analogous to eqs.(3) and (4), as shown below:

$$M_u^{*u} = -D_u^2 \left[\beta \frac{d^6\psi}{dr^6} + \chi F \frac{d^4\psi}{dr^4} + c\frac{d^2\psi}{dr^2}\right], M_u^{*l} = -cD_u^2 \frac{d^2\psi}{dr^2}, M_l^{*u} = -\beta c D_u^2 \frac{d^2\psi}{dr^2}, \tag{17}$$

$$M_l^{*l} = -\beta D_u^2 \left[\frac{d^6\psi}{dr^6} + F\frac{d^4\psi}{dr^4} + c\frac{d^2\psi}{dr^2}\right], \tag{18}$$

$$V_u^{*u} = -D_u^2 \frac{d}{dr}\left[\beta\frac{d^6\psi}{dr^6} + F(\beta+\chi)\frac{d^4\psi}{dr^4} + \left(\chi F^2 + c\right)\frac{d^2\psi}{dr^2} + Fc\psi\right] sgn\left(x, \widehat{x}\right), \tag{19}$$

$$V_u^{*l} = -cD_u^2 \frac{d}{dr}\left(F\psi + \frac{d^2\psi}{dr^2}\right) sgn\left(x,\widehat{x}\right), V_l^{*u} = -cD_u^2\frac{d}{dr}\left(\chi F\psi + \beta\frac{d^2\psi}{dr^2}\right) sgn\left(x,\widehat{x}\right), \tag{20}$$

$$V_l^{*l} = -D_u^2 \frac{d}{dr}\left[\beta\frac{d^6\psi}{dr^6} + F(\beta+\chi)\frac{d^4\psi}{dr^4} + \left(\chi F^2 + \beta c\right)\frac{d^2\psi}{dr^2} + \chi Fc\psi\right] sgn\left(x,\widehat{x}\right), \tag{21}$$

where the signum function is given by:

$$sgn\left(x,\widehat{x}\right) = \begin{cases} 1, if\ x > \widehat{x} \\ -1, if\ x < \widehat{x}. \end{cases} \tag{22}$$

Furthermore, the derivative forms of displacements and internal forces are necessary to complete the definition of integral equations. The source point derivative is defined as follows:

$$\frac{d^n}{d\widehat{x}^n}(\cdot) = -\frac{d^n}{dx^n}(\cdot) = -\frac{d^n}{dr^n}(\cdot)\, sgn\left(x,\widehat{x}\right). \tag{23}$$

Higher-order derivatives of Dirac's delta function are avoided.

2.2 Integral and algebraic equations

The solution to the problem can be derived by transforming the differential equilibrium equations into equivalent integral equations. By applying the method of weighted residuals to eqs.(1) and (2), where the weighted functions are the fundamental solutions presented in eqs.(7) to (9), the following expressions are obtained:

$$\int_\Omega \left\{\left[\frac{d^4w_u}{dx^4} + F\frac{d^2w_u}{dx^2} + c\left(w_u - w_l\right)\right] w_u^{*u} + \left[\beta\frac{d^4w_2}{dx^4} + \chi F\frac{d^2w_2}{dx^2} - c\left(w_u - w_l\right)\right] w_l^{*u}\right\} d\Omega = 0, \tag{24}$$

$$\int_\Omega \left\{\left[\frac{d^4w_u}{dx^4} + F\frac{d^2w_u}{dx^2} + c\left(w_u - w_l\right)\right] w_u^{*l} + \left[\beta\frac{d^4w_l}{dx^4} + \chi F\frac{d^2w_l}{dx^2} - c\left(w_u - w_l\right)\right] w_l^{*l}\right\} d\Omega = 0. \tag{25}$$

After performing integration by parts and applying convenient manipulations, the eqs.(24) and (25) are transformed into:

$$\begin{aligned} w_u(\widehat{x}) + \left[V_u^{*u}w_u\right]_0^L - \left[M_u^{*u}w_{u,x}\right]_0^L + \left[V_l^{*u}w_l\right]_0^L - \left[M_l^{*u}w_{l,x}\right]_0^L = \left[V_u w_u^{*u}\right]_0^L \\ - \left[M_u w_{u,x}^{*u}\right]_0^L + \left[V_l w_l^{*u}\right]_0^L - \left[M_l w_{l,x}^{*u}\right]_0^L, \end{aligned} \tag{26}$$

$$\begin{aligned} w_l(\widehat{x}) + \left[V_u^{*l}w_u\right]_0^L - \left[M_u^{*l}w_{u,x}\right]_0^L + \left[V_l^{*l}w_l\right]_0^L - \left[M_l^{*l}w_{l,x}\right]_0^L = \left[V_u w_u^{*l}\right]_0^L \\ - \left[M_u w_{u,x}^{*l}\right]_0^L + \left[V_l w_l^{*l}\right]_0^L - \left[M_l w_{l,x}^{*l}\right]_0^L, \end{aligned} \tag{27}$$

where for the sake of convenience, the derivatives are denoted here by subscripts preceded by a comma.

Equations (26) and (27) contain four unknowns at the boundary: w_u, w_l, $w_{u,x}$, and $w_{l,x}$. However, only two integral equations have been obtained, necessitating the derivation of two more equations. Thus, differentiating eq.(26) with respect to $\widehat{x}$ results in the following expression:

$$\begin{aligned} w_{u,\widehat{x}}(\widehat{x}) + \left[V_{u,\widehat{x}}^{*u}w_u\right]_0^L - \left[M_{u,\widehat{x}}^{*u}w_{u,x}\right]_0^L + \left[V_{l,\widehat{x}}^{*u}w_l\right]_0^L - \left[M_{l,\widehat{x}}^{*u}w_{l,x}\right]_0^L = \left[V_u w_{u,\widehat{x}}^{*u}\right]_0^L \\ - \left[M_u w_{u,x\widehat{x}}^{*u}\right]_0^L + \left[V_l w_{l,\widehat{x}}^{*u}\right]_0^L - \left[M_l w_{l,x\widehat{x}}^{*u}\right]_0^L \end{aligned} \tag{28}$$

The derivative of eq.(27) with respect to $\widehat{x}$ is given by:

$$w_{l,\widehat{x}}(\widehat{x}) + \left[V^{*l}_{u,\widehat{x}} w_u\right]_0^L - \left[M^{*l}_{u,\widehat{x}} w_{u,x}\right]_0^L + \left[V^{*l}_{l,\widehat{x}} w_l\right]_0^L - \left[M^{*l}_{l,\widehat{x}} w_{l,x}\right]_0^L = \left[V_u w^{*l}_{u,\widehat{x}}\right]_0^L - \left[M_u w^{*l}_{u,x\widehat{x}}\right]_0^L + \left[V_l w^{*l}_{l,\widehat{x}}\right]_0^L - \left[M_l w^{*l}_{l,x\widehat{x}}\right]_0^L \quad (29)$$

The integral equations (26) to (29) permits the evaluation of the displacements and slopes at a point in the interior of the DBS, if the values of the displacements w_i, slopes $w_{i,x}$, the bending moments M_i and effective shear force V_i of the upper and lower beams on the boundary are known. However, for the BEM formulation it is necessary derived the correspondent expressions to determine this quantities on the boundary. These equivalent integral equations to boundary values are derived collocating the source at the edges of the DBS, meaning $\widehat{x} = \lim_{\xi\to 0}(0+\xi)$ and $\widehat{x} = \lim_{\xi\to 0}(L-\xi)$. Thus, the transformation of integral equations into an algebraic representation is done by discretization in boundary elements, resulting in the matrix equation:

$$\{u\} + [H]\{u\} = [G]\{p\} + \{f\}, \quad (30)$$

where $[H]$ and $[G]$ are influence matrices, $\{f\}$ is the load vector, $\{u\}$ is the displacement vector and $\{p\}$ is the forces vector on the boundary. The boundary vectors u and p are done as follows:

$$\{u\} = \left\{ \begin{array}{cccccccc} w_u(0) & w_{u,x}(0) & w_l(0) & w_{l,x}(0) & w_u(L) & w_{u,x}(L) & w_l(L) & w_{l,x}(L) \end{array} \right\}^T, \quad (31)$$

$$\{p\} = \left\{ \begin{array}{cccccccc} V_u(0) & M_u(0) & V_l(0) & M_l(0) & V_u(L) & M_u(L) & V_l(L) & M_l(L) \end{array} \right\}^T, \quad (32)$$

The influence matrix $[G]$ in eq.(30) can be written as follows:

$$[G] = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, [Q(r,S)] = D_u \begin{bmatrix} L_1\psi(r) & -\frac{d}{dr}L_1\psi(r)S & c\psi(r) & -c\frac{d}{dr}\psi(r)S \\ -\frac{d}{dr}L_1\psi(r)S & \frac{d^2}{dr^2}L_1\psi(r) & -c\frac{d}{dr}\psi(r)S & c\frac{d^2}{dr^2}\psi(r) \\ c\psi(r) & -c\frac{d}{dr}\psi(r)S & L_2\psi(r) & -\frac{d}{dr}L_2\psi(r)S \\ -c\frac{d}{dr}\psi(r)S & c\frac{d^2}{dr^2}\psi(r) & -\frac{d}{dr}L_2\psi(r)S & \frac{d^2}{dr^2}L_2\psi(r) \end{bmatrix}, \quad (33)$$

where the matrix $G_{11} = -Q(0,-1)$, $G_{12} = Q(L,1)$, $G_{21} = -Q(L,-1)$, $G_{22} = Q(0,1)$, and the operators $L_1 = \beta d^4(\cdot)/dr^4 + \chi F d^2(\cdot)/dr^2 + c(\cdot)$, and $L_2 = d^4(\cdot)/dr^4 + F d^2(\cdot)/dr^2 + c(\cdot)$.

Also the influence matrix $[H]$ in eq.(30) is given by:

$$[H] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, [N(r,S)] = -D_u^2 \begin{bmatrix} L_3\psi(r)S & -L_4\psi(r) & cL_6\psi(r)S & -c\beta\frac{d^2\psi}{dr^2}(r) \\ -\frac{d}{dr}L_3\psi(r) & \frac{d}{dr}L_4\psi(r)S & -c\frac{d}{dr}L_6\psi(r) & c\beta\frac{d^3\psi}{dr^3}(r)S \\ cL_5\psi(r)S & -c\frac{d^2\psi}{dr^2}(r) & L_7\psi(r)S & -\beta L_8\psi(r) \\ -c\frac{d}{dr}L_5\psi(r) & c\frac{d^3\psi}{dr^3}(r)S & -\frac{d}{dr}L_7\psi(r) & \beta\frac{d}{dr}L_8\psi(r)S \end{bmatrix}, \quad (34)$$

where $H_{11} = -N(0,-1)$, $H_{12} = N(L,1)$, $H_{21} = -N(L,-1)$, $H_{22} = N(0,1)$, and the operators $L_3(\cdot) = \beta d^7(\cdot)/dr^7 + F(\beta+\chi)d^5(\cdot)/dr^5 + \left(c + F^2\chi\right)d^3(\cdot)/dr^3 + Fcd(\cdot)/dr$, $L_4(\cdot) = \beta d^6(\cdot)/dr^6 + \chi F d^4(\cdot)/dr^4 + cd^2(\cdot)/dr^2$, $L_5(\cdot) = d^3(\cdot)/dr^3 + Fd(\cdot)/dr$, $L_6(\cdot) = \beta d^3(\cdot)/dr^3 + \chi F d(\cdot)/dr$, $L_7(\cdot) = \beta d^7(\cdot)/dr^7 + F(\beta+\chi)d^5(\cdot)/dr^5 + \left(\beta c + F^2\chi\right)d^3(\cdot)/dr^3 + \chi F c d(\cdot)/dr$, $L_8(\cdot) = d^6(\cdot)/dr^6 + F d^4(\cdot)/dr^4 + cd^2(\cdot)/dr^2$.

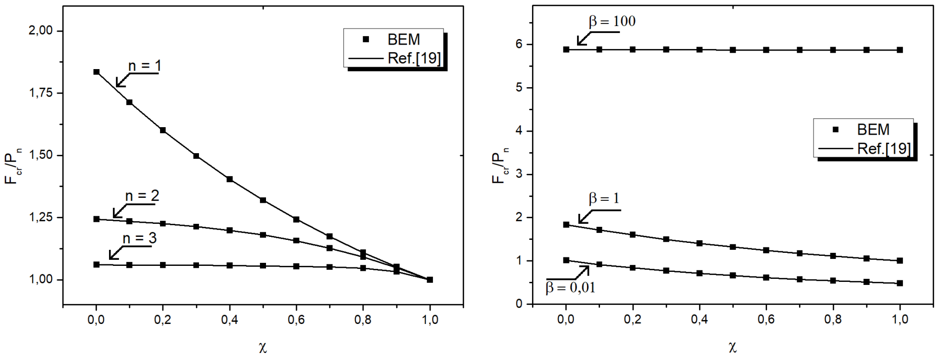
After applying the boundary conditions, it is necessary to study the determinant behavior of the matrix $[H]$, i.e., the instabilities are characterized by the nullity of the matrix.

3 Numerical examples

3.1 Example 1

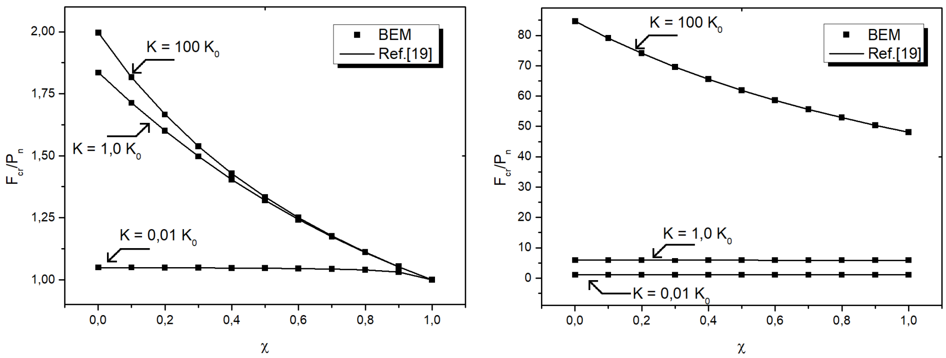
Consider the structural model analyzed by Zhang et al. [17], where the two beams have the same bending stiffness and cross-sectional area, and simply supported ends. The results are presents in terms of the ratio F_{cr}/P_n, where $P_n = EI\,(n\pi/L)^2$. The influence of the ratio χ of the axial load F_l to F_u on buckling load F_{cr} for the first three modes n can be seen in Figure 2(a). The critical buckling load is also dependent on the stiffness beam ratio β and modulus K of the Winkler elastic layer, as can be see in Figures 2(b) and 3, respectively.

It can be seen the accuracy of the technique and solutions proposed in this paper when compared with reference values.



(a) (b)

Figure 2. Buckling load with stiffness modulus $K = K_0$ for the (a) first three modes (with $\beta = 1$) and (b) varying β.



(a) (b)

Figure 3. Effect of the elastic layer stiffness on the buckling load, where (a) $\beta = 1$, and (b) $\beta = 100$.

3.2 Example 2

Consider the double-beam system given by Ref.[16] with ends under different boundary conditions, defined as: S – simply supported; C – clamped; and F – free ends. The following non-dimensional parameters are defined:

$$\bar{c} = cL^4, \;\; \overline{F} = FL^2 \tag{35}$$

The buckling loads obtained by BEM are compared with the three cases SS-SS, CS-CS, and CS-SS, as can be seen in Table 1. Both results are very close but not equal, as expected, since the reference paper considers the Timoshenko beam theory in the DBS model.

Table 1. Comparison of the first two buckling loads for different boundary conditions, where $\chi = 1$, $\beta = 1$ and $\bar{c} = 25$.

	SS-SS		CS-CS		CS-SS	
n	BEM	Ref.[16]	BEM	Ref. [16]	BEM	Ref.[16]
1	9.8696	9.6256	20.1907	19.1019	11.9194	11.6295
2	14.9357	14.6916	24.2852	23.2053	22.7066	21.6647

The results shows the versatility of the BEM solution, having a good performance even if having different properties, boundary conditions, and axial load cases.

4 Conclusions

In this paper, a Boundary Element Method formulation for the buckling analysis of a double-beam system elastically connected by a Winkler elastic layer is presented. The fundamental solution is proposed for each root nature case, and all the steps to obtain the integral equations are discussed. In addition, the algebraic equation is developed, and the influence matrices for the four-node double-beam boundary element with eight degrees of freedom are derived and explicitly shown. The validity of the BEM formulation is verified through examples considering different boundary conditions, geometrical and mechanical properties, and it is compared with those solutions available in the literature. The present BEM formulation is very attractive due to its versatility and efficiency in solving the buckling DBS problem with good agreement.

Acknowledgements. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] J. Seelig, I. Hoppmann, and others. Impact on an elastically connected double beam system. Technical report, RENSSELAER POLYTECHNIC INST TROY NY, 1963.

[2] J. Seelig and W. Hoppmann. Normal mode vibrations of systems of elastically connected parallel bars. *The Journal of the Acoustical Society of America*, vol. 36, n. 1, pp. 93–99, 1964.

[3] P. Kessel. Resonances excited in an elastically connected double-beam system by a cyclic moving load. *The Journal of the Acoustical Society of America*, vol. 40, n. 3, pp. 684–687, 1966.

[4] S. CHONAN. Dynamical behaviours of elastically connected double-beam systems subjected to an impulsive load. *Bulletin of JSME*, vol. 19, n. 132, pp. 595–603, 1976.

[5] T. R. Hamada, H. NAKAYAMA, and K. HAYASHI. Free and forced vibrations of elastically connected double-beam systems. *Bulletin of JSME*, vol. 26, n. 221, pp. 1936–1942, 1983.

[6] Z. Oniszczuk. Free transverse vibrations of elastically connected simply supported double-beam complex system. *Journal of sound and vibration*, vol. 232, n. 2, pp. 387–403, 2000.

[7] S. G. Kelly. Free and forced vibrations of elastically connected structures. *Advances in Acoustics and Vibration*, vol. 2010, 2010.

[8] Q. Mao. Free vibration analysis of elastically connected multiple-beams by using the adomian modified decomposition method. *Journal of Sound and Vibration*, vol. 331, n. 11, pp. 2532–2542, 2012.

[9] M. Huang and J. Liu. Substructural method for vibration analysis of the elastically connected double-beam system. *Advances in Structural Engineering*, vol. 16, n. 2, pp. 365–377, 2013.

[10] A. Mirzabeigy, R. Madoliat, and C. Surace. Explicit formula to estimate natural frequencies of a double-beam system with crack. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, vol. 41, n. 5, pp. 1–12, 2019.

[11] M. Chung and K. Lin. Model and analysis method for machine components in contact. In *Computational Mechanics' 88*, pp. 957–958. Springer, 1988.

[12] W. Brito, C. Maia, and A. Mendonca. Bending analysis of elastically connected euler–bernoulli double-beam system using the direct boundary element method. *Applied Mathematical Modelling*, vol. 74, pp. 387–408, 2019.

[13] Y. Zhang, Y. Lu, S. Wang, and X. Liu. Vibration and buckling of a double-beam system under compressive axial loading. *Journal of Sound and Vibration*, vol. 318, n. 1-2, pp. 341–352, 2008a.

[14] V. Stojanović, P. Kozić, R. Pavlović, and G. Janevski. Effect of rotary inertia and shear on vibration and buckling of a double beam system under compressive axial loading. *Archive of Applied Mechanics*, vol. 81, pp. 1993–2005, 2011.

[15] Q. Mao and N. Wattanasakulpong. Vibration and stability of a double-beam system interconnected by an elastic foundation under conservative and nonconservative axial forces. *International Journal of Mechanical Sciences*, vol. 93, pp. 1–7, 2015.

[16] B. Chen, Q. Xu, B. Zhu, Y. Yang, and Y. Li. Buckling and postbuckling behaviors of symmetric/asymmetric double-beam systems. *International Journal of Mechanical Sciences*, vol. 235, pp. 107712, 2022.

[17] Y. Zhang, Y. Lu, and G. Ma. Effect of compressive axial load on forced transverse vibrations of a double-beam system. *International Journal of Mechanical Sciences*, vol. 50, n. 2, pp. 299–305, 2008b.