Isogeometric Boundary Element Method for 2D analysis of elastostatic problems

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Abstract. In structural engineering, there is a growing trend towards using elements that better conform to geometric characteristics. This is particularly true when leveraging surface (or line, in 2D cases) generation in CAD environments, directly benefiting from more precise geometry. In this context, the need for Isogeometric elements arises. Through their utilization, the analysis of more complex stress and strain distributions is conducted with greater accuracy. This paper presents an alternative for analyzing the structural behavior of elastostatic structural cases through 2D modeling, employing an approximate method with Isogeometric elements. A computational implementation of the Boundary Element Method (BEM) was developed using the Python programming language. The "2D BEM" code was adapted, employing Kelvin's fundamental solution with the use of continuous and discontinuous linear elements, as well as higher-order elements, for more accurate boundary discretization. The formulation of BEM thus enables the modeling of the domain, in this case, the analysis of 2D bodies in the presence of mechanical damage, with iterative monitoring. The behavior is assessed through the displacements obtained for the boundary and internal points, as well as the stresses, conveniently evaluated in graphical representations. Applications are presented to test the implemented modeling and to provide an alternative for analyzing an important area within structural systems in civil construction projects, among others.

Keywords: Isogeometric elements; Python; discretization.

1 Introduction

In the field of Engineering, it is common to encounter scenarios that require the application of differential equations for their solution. However, it is recurrent to find equations of this nature whose analytical solution is not feasible. In this manner, the application of numerical methods emerges as an essential alternative nowadays. Thus, Bacarji [1] states that the Boundary Element Method (BEM), which originated from Betti [2] with the use of equivalent integral equations for problem solving, replaces the integral equations by discretized integral equations, in which unknowns of the boundary are used in a finite set of nodes.

As discussed by Ribeiro and Vicentini [3], the BEM demonstrates remarkable advantages in situations that involve infinite extension media or that demand a frequent update of the domain, illustrated exemplarily by the analysis of crack propagation. However, the authors point out that the method works with a less intuitive formulation, when compared to other methods, such as the finite element method.

Isogeometric analysis, according to Peixoto [4], is an advanced numerical method whose use has been gaining increasing interest in Engineering. The method was developed by Hughes *et al.* [5] and proposes a differentiated and more sophisticated alternative to the traditional method, based on the use of NURBS (Non-Uniform Rational B-Splines) curves for problem representation. Isogeometric analysis improves computational simulation by minimizing mesh approximation errors, as it preserves the exact geometry of the model throughout the process.

This is possible through to the use of the same class of functions to represent both the geometry and the problem variables.

2 Boundary Elements Method Linear Formulation

2.1 Boundary integral equations

It is necessary to use an integral form representation for applying the boundary element method to a given body. This representation can be achieved using weighted residual methods.

Lima Junior [6] adopted the methodology proposed by Somigliana [7], based on Betti's reciprocity theorem, which is grounded in the principle of conservation of energy, resulting in the following expression:

$$u_i(s) = \int_{\Gamma} p_k(Q) u_{ik}^*(s,Q) d\Gamma - \int_{\Gamma} p_k(s,Q) u_k(Q) d\Gamma + \int_{\Omega} b_k(q) u_{ik}^*(s,q) d\Omega, \tag{1}$$

defining the displacement field at a point s within the domain, based on the displacements and forces measured at boundary points.

2.2 Boundary Elements Method

The formulation based on boundary integral equations, which were developed from Betti's reciprocity theorem proposed by Somigliana [7], was presented. This allows the analysis of two-dimensional, isotropic, and homogeneous elastic solids. For practical use, it is necessary to transform the boundary integral formulation into algebraic equations, which consolidates its discretized form, resulting in a linear system that will be solved after applying the boundary conditions of the problem.

The representation of the boundary in a finite dimension is achieved through the definition of nodes that delimit the so-called boundary elements. This boundary parametrization can be exact or approximate, depending on the geometry of the domain under analysis and the type of parametrization adopted. Figure 1 illustrates these two situations using linear elements. Furthermore, the geometric characterization of the element should approximate the variables of interest in the problem, based on approximating functions and their nodal values at specific points. Polynomial functions are commonly used.

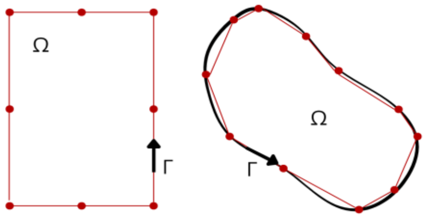


Figure 1. Boundary discretization: adjusting the elements to the shape

Equation (1) can be rewritten according to eq. (2), as a function of nodal values, defined in the discretization of the boundary only, ignoring the part of the domain forces.

$$c_{lk}^i u_{lk}^i + \left(\int_{\Gamma} p_{lk}^* \cdot \phi \, d\Gamma \right) \cdot u_k^j = \left(\int_{\Gamma} u_{lk}^* \cdot \phi \, d\Gamma \right) \cdot p_k^j \ . \tag{2}$$

In fact, eq. (2) can be reduced to the form of eq. (3), in matrices and vectors:

$$\mathbf{HU} = \mathbf{GP} \ . \tag{3}$$

Determining the displacements and forces for each node in both directions leads to the global matrices G and H, which contain the core of the p^*_{lk} and u^*_{lk} integrals. With the above matrices and the vector of prescribed values, a system of linear equations with unknowns and prescribed variables is formed. When solving the linear system, the unknown values of the problem boundary are determined, with which displacements and stresses at internal points in the body can be determined. Applying the boundary conditions gives the system of linear equations as shown in eq. (4),

$$\mathbf{AX} = \mathbf{B}. \tag{4}$$

The system of linear equations, eq. (4), provides the unknown values of surface forces and displacements at points along the boundary.

2.3 Fundamental solution

Considering a fundamental problem in the infinite domain Ω^* subjected to a unit load applied at a given point *s*, referred to as a source point along direction *i*, it is possible to characterize the fundamental problem. Assuming a solid body in linear elastic regime within a domain Ω bounded by the boundary, and imposing the equilibrium of an infinitesimal representative element at any point of the solid, the equilibrium reactions are condensed as written below, eq.(5),

$$\sigma_{kj,j}(q) + b_{ik}(q) = 0, \tag{5}$$

where $\sigma_{kj}(q)$ corresponds to the stress tensor component and $b_k(q)$ o the component of the volume force vector acting on the body.

Thus, with the equilibrium reaction (1), it is possible to represent the unit loading by substituting the term $b_k(q)$ with a Dirac delta distribution, weighted by a Kronecker delta, relating directions *i* and *k*.

Thus, the equilibrium equation (1) can be written as follows,

$$\sigma^*{}_{ikj,j}\,(s,q) + \delta(s,q)\delta_{ik} = 0, \tag{6}$$

where σ^*(s,q) corresponds to the stress tensor in the fundamental state. The presence of an asterisk (*) in quantities refers to the fundamental state.

In the formulation used in this work, the fundamental Kelvin solution is used, which is considered to be the most widespread and widely used in technical circles. The expressions for displacements and surface forces for the 2D Kelvin problem are shown in eq. (7) and (8), respectively.

$$u^*_{ij} = \frac{1}{8\pi G(1-\upsilon)}\left[(3-4\upsilon)\,ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r_{,j}\right]; \tag{7}$$

$$p^*_{ij} = -\frac{1}{4\pi(1-\upsilon)r}\left\{\frac{\partial r}{\partial n}\left[(1-2\upsilon)\delta_{ij} + 2r_{,i}r_{,j}\right] + (1-2\upsilon)(n_i r_{,j} - n_j r_{,i})\right\}, \tag{8}$$

where G is the transverse modulus of elasticity and υ is Poisson's ratio. In addition, δ_{ij} is Kronecker's delta function, $r = \sqrt{r_i * r_i}$; $r_i = X_i(q) - X_i(s)$; $r_{,i} = \frac{r_i}{r}$.

3 Isogeometric Formulation

The isogeometric formulation of the Boundary Element Method (BEM) combines geometric modeling concepts (usually used in CAD) with numerical methods to solve engineering and applied science problems.

Isogeometric formulation (IGA) was proposed to unify the representation of geometry and the discretization of domains used in numerical analyses. Traditionally, numerical methods use simple shape functions, such as polynomials, to approximate both the geometry and the solutions. However, these geometric approximations do not correspond exactly to the original geometry, which can introduce errors. IGA uses more sophisticated shape functions derived from CAD geometric representations (Kapturczak et al. [8]), such as NURBS (Non-Uniform Rational B-Splines). These functions make it possible to represent geometry accurately and also to approximate the solutions of differential equations. This is particularly advantageous when dealing with problems involving complex geometry.

NURBS (Non-Uniform Rational B-Splines) are a flexible mathematical representation for curves and surfaces that allow you to accurately represent geometric shapes, from simple curves to complex geometries, including circles, spheres and organic surfaces (Loyola [9]). Second-degree NURBS make it possible to represent smooth curves and arcs of circles with great precision, and to represent segments of circles, which facilitates the analysis of symmetrical domains. Second-degree NURBS strike a balance between geometric precision and computational simplicity, making them ideal for many engineering applications. A second-degree NURBS is defined by the following components:

- Control Points (Pi): these are the points that define the shape of the curve. Each segment of the curve is influenced by three control points;
- Node vector U: the node vector $U = \{u_0, u_1, \dots, u_m\}$ defines the parameterization of the curve. The vector of nodes must have $m = n + p + 1$, where n is the number of control points and p is the degree of the curve;
- Weights w_i: the weights control the influence of each control point on the curve. Different weights allow for the exact representation of rational shapes, such as arcs of a circle.

The second-degree NURBS curve $C(u)$ can be determined using the equation (9):

$$C(u) = \frac{\sum_{i=0}^{2} N_{i,2}(u) w_i P_i}{\sum_{i=0}^{2} N_{i,2}(u) w_i} \tag{9}$$

where $u \in [0,1]$, and B-Spline basis functions are:

$$N_{i,0}(u) = \begin{cases} 1, if\ u_i \le u \le u_{i+1} \\ 0, \quad otherwise \end{cases} \tag{10}$$

$$N_{i,1}(u) = \frac{u - u_i}{u_{i+1} - u_i} N_{i,0}(u) + \frac{u_{i+2} - u}{u_{i+2} - u_{i+1}} N_{i+1,0}(u) \tag{11}$$

$$N_{i,2}(u) = \frac{u - u_i}{u_{i+2} - u_i} N_{i,1}(u) + \frac{u_{i+3} - u}{u_{i+3} - u_{i+1}} N_{i+1,1}(u) \tag{12}$$

Applications processed in this work have a circular boundary, as well as in the work of SIMPSON [11]. In view of these statements, therefore, the expression *C(u)* can be presented for a radius *R*, with 3 (three) control points and a representative arch segment for the first trigonometric quadrant (eq.(13)):

$$C(u) = \left(\frac{R(1-u)^2 + Ru(1-u)}{(1-u)^2 \sqrt{2} u(1-u) + u^2} \middle| \frac{Ru^2 + Ru(1-u)^2}{(1-u)^2 \sqrt{2} u(1-u) + u^2} \right) \tag{13}$$

4 Applications

4.1 Circular Hole

The first example involves a circular hole with a radius of 10 (Kapturczak et al. [10]), as shown in Fig. 2. Within the cylinder, it is defined in a plane stress state, with an applied force p = 15. The material parameters are: E = 21 and ν = 0.1.

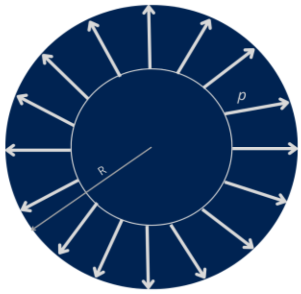


Figure 2: Circular Hole with internal pressure.
Source: Authors.

In this context, the example was initially implemented using the linear BEM formulation program, with 64 boundary nodes used for the calculations. The following Tab. 1 presents a comparative analysis of the results from the analytical solution and those obtained from the program developed in this study.

Table 1. Results of stresses at internal points for comparison with other works (Y=0, X=12, 15 and 20)

X	Analytical		BEM ([10])		MEC 2D				*ISO*MEC 2D	
	σ_x	σ_y	σ_x	σ_y	σ_x	σ_y	σ_1	σ_2	σ_x	σ_y
12	-10.417	10.417	-10.22	10.268	-10.268	10.268	-10.268	10.273	-10.414	10.414
15	-6.667	6.667	-6.525	6.651	-6.651	6.651	-6.667	6.528	-6.665	6.665
20	-3.750	3.750	-3.670	3.702	-3.702	3.702	-3.702	3.702	-3.749	3.749

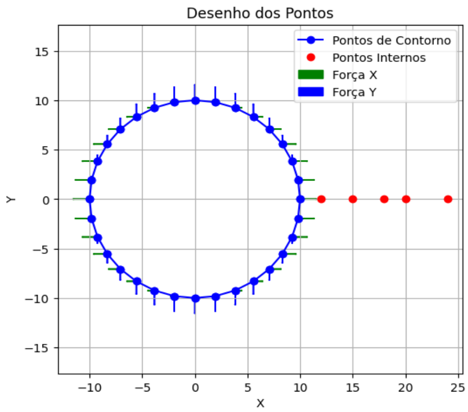


Figure 3: Circular Hole - boundary points, internal points and surface forces in x and y.
Source: Authors, 2024.

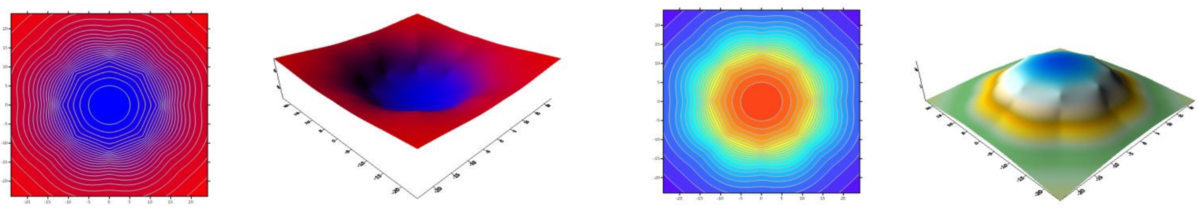


Figure 4: Graphs of principal stress distribution, σ_1 and σ_2.
Source: Authors, 2024.

4.2 Multiply Connected Domain

The second example is a Lamé problem with a = 10 cm and b = 25 cm (Kapturczak et al. [10]), as shown in Fig. 5. Inside the cylindrical tube, it is subjected to an internal hydrostatic pressure of p = 100 MPa, in a plane stress state. The material parameters are E = 2 x 10^5 MPa e v = 0.25.

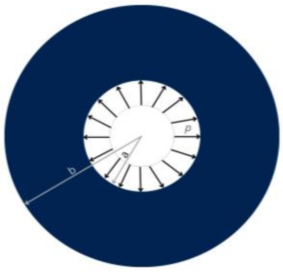


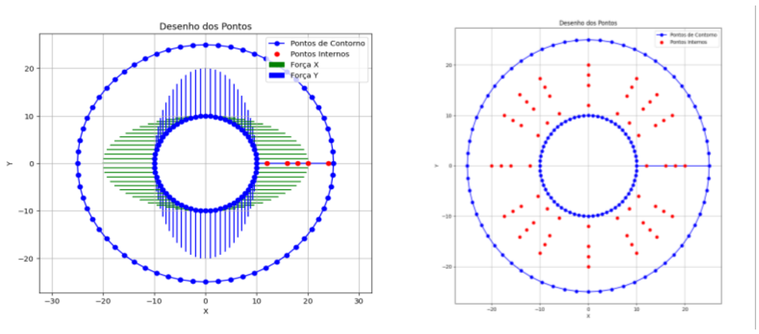
Figure 5: Cylindrical tube with internal pressure.
Source: Authors

In this context, the example was initially implemented using the linear BEM formulation program, employing 64 boundary nodes for the calculations. The following Tab. 3 presents a comparison of the results from the analytical solution, the reference article, and the program developed in this study.

Table 3. Results of stresses at internal points for comparison with other works (X=12, 18, 20 and 24; Y=0).

X	Analytical		NURBS ([10])		MEC 2D	
	σ_x	σ_y	σ_x	σ_y	σ_x	σ_y
12	-63.624	101.720	-63.681	101.681	-63.782	99.423
18	-17.695	55.791	-17.706	55.775	-18.425	54.096
20	-10.700	48.810	-10.724	48.798	-11.536	47.200
24	-1.620	39.716	-1.569	39.695	-1.606	38.214

Source: Authors, 2024



(a) (b)

Figure 6: (a) Boundary points, internal points and surface forces in x and y; (b) Mesh of internal points for graphics.
Source: Authors

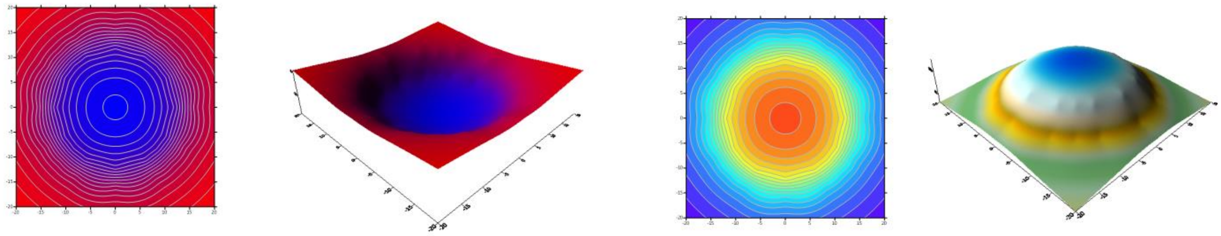


Figure 7: Graphs of principal stress distribution, σ_1 and σ_2.
Source: Authors, 2024

5 Conclusions

This paper presented an isogeometric implementation of BEM using NURBS equations. The results showed more agreement than those obtained with the traditional BEM formulation. It also showed positive results with the use of (discontinuous) linear elements.

However, the work involved in determining the NURBS basis functions in an application is significantly more than that required for linear discretization. In the examples processed, the analytical solutions are known, making it easier to perceive the convergence of the results. Their circular boundaries made it easier to obtain functions appropriate to the geometry. Other examples not so well behaved in their geometry need to be further investigated by this work to examine the suitability of the formulation for these cases as well. This will be future work.

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