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Multiscale Modelling of the Two-Dimensional Problem Using the Boundary Element Method

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Abstract. A full coupled multi-scale modelling using the Boundary Element Method for analysing the 2D problem of stretched plates composed of heterogeneous materials, where dissipative phenomena can be considered, is presented. Both the macro-scale and the micro-scale are modelled by BEM formulations where the consistent tangent operator (CTO) is used to achieve the equilibrium of the iterative procedures. The equilibrium equation of the plate (macro-continuum) is written in terms of in-plane strains while the equilibrium problem of the microstructure, which is defined by the RVE (Representative Volume Element), is solved in terms of displacements fluctuations. In this kind of modelling, the mechanical behaviour of the material is governed by the homogenized response of the RVE, obtained after solving its equilibrium problem. As this kind of modelling is expensive computationally, it is important to investigate other numerical methods to have faster formulations, but which are still accurate. To validate the presented model, the numerical results are compared to the ones where the material microstructure (RVE) is modelled by the FEM.

Keywords: Multi-scale modelling, RVE, boundary elements.

1 Introduction

In this work, the material microstructure is modelled by the RVE, where the materials properties of each phase of the microstructure are defined accordingly to the material studied. Therefore, the constitutive response of a particular point of the macro-continuum is defined by the homogenized response of the RVE that represents that point. Both the 2D-dimensional problem (macro-continuum problem, whose formulation was developed by Fernandes and Souza Neto in [1]) and the RVE problem (material microstructure, whose formulation was developed by Fernandes et al in [2]) are modelled by Boundary Element formulations, being the results compared to a formulation where the RVE is modelled considering the Finite Element Method, developed by Fernandes et al in [3].

To model structures at different scales is important to better represent the mechanical behaviour of heterogeneous materials, as we can see in the works Fernandes et al [2] and [3], Peric et al [4], Somer et al [5], Silva et al [6], Pituba et al [7] and Fernandes et al [8], where the mechanical behaviour of different kind of materials have been modelled using a multi-scale framework. In many situations of traditional one scale formulations of structures, sophisticate constitutive models must be used to obtain better results, while in multi-scale modelling good results are obtained using simple constitutive models at the micro-scale. Besides, the dissipative phenomena that occur at microstructure level can be better modelled if a multi-scale modelling is considered because in this case, we can treat the different dissipative phenomena of the microstructure separately as well as modelling the mechanical behaviour of the different phases of the microstructure separately. On the other hand, in a one scale analysis of a structure (macro-continuum), the material behaviour is governed by a constitutive model that considers the material to be homogeneous at macrostructure level.

Most of the materials have a heterogeneous microstructure at the microscale. In the present work, the heterogeneity at microscale is defined by different volume fraction and sizes of the inclusions and of the voids as well as by the different material properties of the microstructure phases. The numerical examples are developed in the context of Metal Matrix Composites (MMC) which are metallic materials reinforced by more rigid metal inclusions or in the context of porous metal materials.

2 Equilibrium Problem for the Macro-continuum

Let us consider a flat plate of thickness t, external boundary Γ and domain Ω which supports only loads acting in the x_1 and $x_2$directions over the plate middle surface and the case of small strains. The following values are defined on the plate surface: in-plane tractions ($\dot{p}_n$ and $\dot{p}_s$) and in-plane displacements ($\dot{u}_n$ and $\dot{u}_s$), with n and s referring to the boundary normal and tangential directions, respectively. As we deal with dissipative phenomena, all variables are expressed in their time derivatives, i.e., $(\dot{x}) = dx/dt$ and the total strain is split into its elastic ($\dot{\varepsilon}_{ij}^{e}$) and inelastic ($\dot{\varepsilon}_{ij}^{0}$) parts. In this case, the inelastic membrane force rate $\dot{N}_{ij}^{0}$, which in the present paper are represented by the plastic forces, is defined as:

$$\dot{N}_{ij}^{0} = \dot{N}_{ij}^{e} - \dot{N}_{ij} \qquad \text{i, j} = 1, 2 \tag{1}$$

where the forces $\dot{N}_{ij}$ are obtained from the stress vector that satisfies the constitutive model, which in the present paper, corresponds to the RVE homogenized response and the forces $\dot{N}_{ij}^{e}$ can be written in terms of the total strain $\dot{\varepsilon}_{ij}$ or in terms of the total displacements derivatives, by applying the Hooke's law.

From Betti's reciprocal theorem we can obtain the integral representation for the plate stretching problem (see more details in in the work developed by Fernandes and Souza Neto [1]), which after integrating by parts leads to the well-known representation of in-plane displacements:

$$K_i\,(q)\dot{u}_i\,(q) = -\int_{\Gamma}\left(\dot{u}_n p_{in}^{*} + \dot{u}_s p_{is}^{*}\right)d\Gamma + \int_{\Gamma}\left(u_{in}^{*}\dot{p}_n + u_{is}^{*}\dot{p}_s\right)d\Gamma + \int_{\Omega_b}\left(u_{in}^{*}\dot{b}_n + u_{is}^{*}\dot{b}_s\right)d\Omega$$
$$+\int_{\Omega}\varepsilon_{ijk}^{*}\dot{N}_{jk}^{0}\,d\Omega \qquad \text{k, i, j} = 1,2 \tag{2}$$

where q is the source point, the values of the fundamental problem are denoted by the superscript *; the subscript i is the direction of the fundamental load; Ω_b is the plate loaded area; $K_i(q)$ is the free term, being K(q)=1 and K(q)=1 /2 for internal and boundary points, respectively.

When dissipative phenomena are considered, the load must be divided into increments, being the equilibrium equation checked for each load increment. For that, we consider that a variable over the time step Δt is given by: $\Delta a = a_{n+1} - a_n = \Delta t \dot{a}_{n+1}$, being n the increment number. Thus, in what follows we will write the values in terms of their increments instead of their rates. To solve the problem, we need also to write the integral representation for the forces $\dot{N}_{ij}^{e}$, obtained from the integral representation for rotations and Hookes's law. And to obtain the integral representation for rotations, we must only differentiate eq. (2).

To obtain the algebraic equations, we adopt geometrically linear elements where the displacements and tractions are approximated by quadratic shape functions. Over the plate domain we adopt triangular cells where the inelastic force increments $\Delta \boldsymbol{N}^0$ are approximated by linear shape functions. After writing two in-plane displacements equations at each boundary node of the plate (macro-continuum), the set of equations is obtained, which after applying the boundary conditions to the plate, can be written as:

$$\Delta\boldsymbol{X} = \Delta\boldsymbol{L} + \boldsymbol{R}_N \Delta\boldsymbol{N}^0 \tag{3}$$

where $\Delta\boldsymbol{X}$ is the unknown vector on the plate boundary; the vector $\Delta\boldsymbol{L}$ represents the elastic solution for the boundary values; $\boldsymbol{R}_N = \boldsymbol{A}^{-1}\boldsymbol{E}$ express the corrections due to the inelastic force increment (ΔN^0), being the matrix $\boldsymbol{E}$ obtained by integrating the cells and the matrix $\boldsymbol{A}$ obtained after imposing the boundary conditions to the set of equations.

Three algebraic equations of elastic force increments $\Delta \boldsymbol{N}^{e(BEM)}$ are written at each cell node of the macro-continuum, resulting into (see more details in the work developed by Fernandes and Souza Neto [1]):

$$\Delta \boldsymbol{N}^{e(BEM)} = \Delta \boldsymbol{K}_N + \boldsymbol{S}_N \Delta \boldsymbol{N}^0 \tag{4}$$

where $\Delta\mathbf{K}_N$ is the elastic solution for the membrane forces of the macro-continuum, the matrix $\mathbf{S_N}$ expresses the corrections due to the inelastic force increment ΔN^0.

The equilibrium of the plate, for a load increment *n*, is achieved when the following equilibrium equation is verified:

$$\boldsymbol{R}_N\left(\Delta\boldsymbol{\varepsilon}_n\right) = 2\Delta\boldsymbol{K}_{N(n)} - \boldsymbol{C}_N\Delta\boldsymbol{\varepsilon}_n + \boldsymbol{S}_N\left(\boldsymbol{C}_N\Delta\boldsymbol{\varepsilon}_n - \Delta\boldsymbol{N}_n\right) - \Delta\boldsymbol{N}_n = 0 \tag{5}$$

where $\mathbf{C}_N$ is the matrix that contains the elastic matrices of all cell nodes; for a particular node k, $\Delta\boldsymbol{N}^{(k)}$ is computed from the homogenized stress vector of the RVE. The plate equilibrium is achieved when the strain vector ($\boldsymbol{\Delta\varepsilon}$) that satisfies eq. (5) is found. For that, an iterative procedure must be considered where additive corrections $\delta\Delta\boldsymbol{\varepsilon}_n^{i+1}$ is computed as follows:

$$\delta\Delta\boldsymbol{\varepsilon}_n^{i+1} = -\left[\frac{\partial \boldsymbol{R}_N\left(\Delta\boldsymbol{\varepsilon}_n^i\right)}{\partial\,\Delta\boldsymbol{\varepsilon}_n^i}\right]^{-1}\boldsymbol{R}_N\left(\Delta\boldsymbol{\varepsilon}_n^i\right) \tag{6}$$

3 Equilibrium Problem for the Micro-continuum (RVE)

The RVE (Representative Volume Element) represents the microstructure of the material, being the mechanical behaviour of a macro-continuum point governed by the homogenized response of the RVE. At micro-scale, the two-dimensional problem considering dissipative phenomena will also represent the problem to be studied. To represent the heterogeneity of the microstructure, the RVE is modelled as a plate divided into sub-regions, where solid and void parts can be defined. The solid part can be composed of different phases, for which different elastic properties and constitutive models can be defined. To obtain the integral equation of in-plane displacements rates for the microstructure, the Betti's theorem is applied to each sub-region. Then, the integral equation for the zoned plate can be obtained by summing the equations of all sub-regions. Assuming that the RVE is composed of N_{inc} inclusions and N_{voids} voids, whose boundaries are adopted such that do not coincide to the RVE external boundary, and where no loads b_i is defined, the following displacement representation can be written:

$$\begin{aligned}
C_{k1}\dot{u}_1^{\mu}(q) + C_{k2}\dot{u}_2^{\mu}(q) = &-\frac{\overline{\overline{E}}_{\mu(1)}\nu_{\mu(1)}}{\overline{\overline{E}}_{\mu}\nu_{\mu}}\int_{\Gamma_1}\left(\dot{u}_1^{\mu}(P)\,p_{k1}^{*(\mu)}(q,P) + \dot{u}_2^{\mu}(P)\,p_{k2}^{*(\mu)}(q,P)\right)d\Gamma + \\
&-\sum_{m=1}^{N_{inc}}\frac{\left(\overline{\overline{E}}_{\mu(m)}\nu_{\mu(m)} - \overline{\overline{E}}_{\mu(1)}\nu_{\mu(1)}\right)}{\overline{\overline{E}}_{\mu}\nu_{\mu}}\int_{\Gamma_{m1}}\left(\dot{u}_1^{\mu}(P)\,p_{k1}^{*(\mu)}(q,P) + \dot{u}_2^{\mu}(P)\,p_{k2}^{*(\mu)}(q,P)\right)d\Gamma_{m1} \\
&-\sum_{m=1}^{N_{voids}}\frac{\overline{\overline{E}}_{\mu(1)}\nu_{\mu(1)}}{\overline{\overline{E}}_{\mu}\nu_{\mu}}\int_{\Gamma_{1m}}\left(\dot{u}_1^{\mu}(P)\,p_{k1}^{*(\mu)}(q,P) + \dot{u}_2^{\mu}(P)\,p_{k2}^{*(\mu)}(q,P)\right)d\Gamma_{1m} \\
&+\int_{\Gamma_1}\left(u_{k1}^{*(\mu)}(q,P)\,\dot{p}_1^{\mu}(P) + u_{k2}^{*(\mu)}(q,P)\,\dot{p}_2^{\mu}(P)\right)d\Gamma + \sum_{m=1}^{N_s}\overline{\overline{E}}_{\mu(m)}\left(1 - \frac{\nu_{\mu(m)}}{\nu_{\mu}}\right)\int_{\Omega_m}\dot{u}_i^{\mu}(p)\,\varepsilon_{kij,j}^{*(\mu)}(q,p)\,d\Omega_m \\
&-\overline{\overline{E}}_{\mu(1)}\left(1 - \frac{\nu_{\mu(1)}}{\nu_{\mu}}\right)\int_{\Gamma_1}\left[\dot{u}_1^{\mu}(P)\,\varepsilon_{k1}^{*(\mu)}(q,P) + \dot{u}_2^{\mu}(P)\,\varepsilon_{k2}^{*(\mu)}(q,P)\right]d\Gamma + \sum_{m=1}^{N_S}\left[\int_{\Omega_m}\varepsilon_{kij}^{m*(\mu)}(q,p)\,\dot{N}_{ij}^{0(\mu)}(p)\,d\Omega\right] \\
&-\sum_{m=1}^{N_{inc}}\left[\overline{\overline{E}}_{\mu(m)}\left(1 - \frac{\nu_{\mu(m)}}{\nu_{\mu}}\right) - \overline{\overline{E}}_{\mu(1)}\left(1 - \frac{\nu_{\mu(1)}}{\nu_{\mu}}\right)\right]\int_{\Gamma_{m1}}\left[\dot{u}_1^{\mu}(P)\,\varepsilon_{k1}^{*(\mu)}(q,P) + \dot{u}_2^{\mu}(P)\,\varepsilon_{k2}^{*(\mu)}(q,P)\right]d\Gamma_{m1} \quad \text{k, i, j=1,2}
\end{aligned} \tag{7}$$

where q is the source point, k is the fundamental load direction; Ω_1 and Γ_1 represent, respectively, the domain and the external boundary of the matrix; Γ_{m1} is the interface between the matrix and inclusion m and Γ_{1m} the interface between the matrix and a void m; $\overline{\overline{E}}_{\mu(m)} = \dfrac{\overline{E}_{\mu(m)}}{\left(1-\nu_{\mu(m)}{}^2\right)}$, $\overline{E}_{\mu(m)} = E_m t$, the subscript "1" refers to the matrix, the values

ν_μ, E_μ, $\overline{E}_\mu$ and $\overline{\overline{E}}_\mu$ are referred to the sub-region where the source point q is placed; the free terms values (C_{k1} and C_{k2}) depend on the position of the source point q: internal, on the external boundary, on matrix/inclusion interface or on matrix/void interface (see the work developed by Fernandes et al [8] for their definition).

To obtain the algebraic equations, the external boundary and interfaces of the RVE are discretized into geometrically linear elements and the domain into triangular cells. The values are approximated by linear shape functions over the elements and linear shape functions are adopted to approximate the displacements in the cells. But the inelastic forces ($N_{11}^0, N_{12}^0, N_{22}^0$) are assumed constant over the cells in the RVE domain.

After writing two displacements equations for all RVE nodes (internals, on the interfaces and on the boundary), one can get the set of equations necessary to compute the unknowns. After imposing to the boundary nodes, the linear displacements field computed from the macro-strain vector ($\Delta\boldsymbol{\varepsilon}_n^i$), the RVE unknowns' vector ($\Delta\boldsymbol{X}_\mu$) can be obtained from the following expression:

$$\Delta\boldsymbol{X}_\mu = \Delta\boldsymbol{L}_\mu + \boldsymbol{R}_\mu \Delta\boldsymbol{N}_\mu^0 \tag{8}$$

where the ΔX_μ vector contains the boundary tractions and the displacements at interfaces and internal nodes i.e.: $\Delta\boldsymbol{X}_\mu = \begin{Bmatrix} \Delta\boldsymbol{P}_B^\mu \\ \Delta\boldsymbol{U}_{\text{int}}^\mu \end{Bmatrix}$; the vector $\Delta\boldsymbol{L}_\mu = \boldsymbol{A}_\mu^{-1}\Delta\boldsymbol{B}_\mu$ represents the RVE elastic solution for the boundary tractions and for the displacements at interfaces and internal nodes, being $\boldsymbol{A}_\mu = \begin{bmatrix} -\boldsymbol{G}_{BB}^\mu & \boldsymbol{H}_{Bi}^\mu \\ -\boldsymbol{G}_{iB}^\mu & \boldsymbol{H}_{ii}^\mu \end{bmatrix}$ and $\Delta\boldsymbol{B}_\mu = -\boldsymbol{H}_{BB}^\mu \Delta\boldsymbol{U}_B^\mu - \boldsymbol{H}_{iB}^\mu \Delta\boldsymbol{U}_B^\mu$; the matrix $\boldsymbol{R}$ is given by: $\boldsymbol{R}_\mu = \boldsymbol{A}_\mu^{-1}\boldsymbol{E}_\mu$, where $\boldsymbol{E}$ is obtained by integrating the cells. To complete the necessary set of equations, three equations of elastic membrane forces increment (ΔN_{11}^e ΔN_{12}^e and ΔN_{22}^e) are written at the center of each cell, using the following equation:

$$\Delta\boldsymbol{N}^e{}_\mu = \Delta\boldsymbol{K}_N^\mu + \boldsymbol{S}_\mu \Delta\boldsymbol{N}_\mu^0 \tag{9}$$

where the matrix $\boldsymbol{S}$ is defined as $\boldsymbol{S}_\mu = \boldsymbol{E}'_\mu - \boldsymbol{A}'_\mu \boldsymbol{R}_\mu$; the vector $\Delta\boldsymbol{K}_N^\mu$ is given by $\Delta\boldsymbol{K}_N^\mu = \Delta\boldsymbol{B}'_\mu - \boldsymbol{A}'_\mu \Delta\boldsymbol{L}_\mu$ and it represents the RVE elastic solution for the forces; $\boldsymbol{A}'_\mu = \begin{bmatrix} -\boldsymbol{G}'^\mu_B & \boldsymbol{H}'^\mu_i \end{bmatrix}$; $\Delta\boldsymbol{B}'_\mu = -\boldsymbol{H}'^\mu_B \Delta\boldsymbol{U}_B^\mu$. The microscopic displacement field $\boldsymbol{u}_\mu$ in the RVE is composed of two parts: the displacement fluctuation field $\Delta\tilde{\boldsymbol{u}}_\mu$ necessary to satisfy the RVE equilibrium equation and the displacement field $\Delta\boldsymbol{u}_\mu^\varepsilon(\boldsymbol{y})$ computed after imposing the macroscopic strain $\Delta\boldsymbol{\varepsilon}(\boldsymbol{x})$ to the RVE boundary:

$$\Delta\boldsymbol{u}_\mu(\boldsymbol{y}) = \Delta\boldsymbol{u}_\mu^\varepsilon(\boldsymbol{y}) + \Delta\tilde{\boldsymbol{u}}_\mu(\boldsymbol{y}) \tag{10}$$

After discretizing the RVE into N_{cel}^μ cells, the RVE equilibrium equation can be written as:

$$R_F = \sum_{e=1}^{N_{cel}^\mu} \boldsymbol{B}_e^T \boldsymbol{C}_{N(\mu)}^{ep(e)} \left(\Delta\boldsymbol{\varepsilon} + \boldsymbol{B}_e \Delta\tilde{\boldsymbol{u}}_\mu\right)_e A_e = 0 \tag{11}$$

where Ω_μ^h denotes the discretised RVE domain, $\boldsymbol{B}_e$ is the strain-displacement matrix, A_e is the cell area; $\boldsymbol{C}_{N(\mu)}^{ep(e)} = t\boldsymbol{C}_{(\mu)}^{ep(e)}$, being $\boldsymbol{C}_{(\mu)}^{ep(e)}$ the constitutive tensor for cell e, computed according to the constitutive model adopted for cell e. To find the displacement fluctuation field $\Delta\tilde{\boldsymbol{u}}_\mu^{i_{RVE}+1} = \Delta\tilde{\boldsymbol{u}}_\mu^{i_{RVE}} + \delta\tilde{\boldsymbol{u}}_\mu^{i_{RVE}+1}$ that satisfies eq. (11), the fluctuations corrections ($\delta\tilde{\boldsymbol{u}}_\mu^{i_{RVE}+1}$) to be imposed at iteration $i_{RVE}+1$ are computed from:

$$\boldsymbol{R}_{\boldsymbol{F}}^{\boldsymbol{i}_{RVE}} + \boldsymbol{K}^{\boldsymbol{i}_{RVE}} \delta\tilde{\boldsymbol{u}}_{\mu}^{\boldsymbol{i}_{RVE}+1} = 0 \tag{12}$$

where $\boldsymbol{K}$ is the consistent tangent operator obtained by derivation of Eq. (11), i.e.: $\boldsymbol{K}_{\mu}^{\boldsymbol{i}_{RVE}} = \frac{\partial \boldsymbol{R}_{\boldsymbol{F}}}{\partial \Delta\tilde{\boldsymbol{u}}} = \sum_{\boldsymbol{e}=1}^{N_{cel}^{\mu}} \boldsymbol{B}_{\boldsymbol{e}}^{\boldsymbol{T}} \boldsymbol{C}_{N\,\mu}^{ep(\boldsymbol{i}_{RVE})} \boldsymbol{B}_{\boldsymbol{e}}\ A_e$.

4 Constitutive Model: RVE homogenized response

The mechanical behaviour of a macro-continuum point is governed by the homogenised stress vector $\boldsymbol{\sigma(x)=\sigma}$ (eq. 13) and the homogenised constitutive tensor $\boldsymbol{C}^{ep}$ (eq. 14) of the RVE assigned for that point.

$$\boldsymbol{\sigma} = \frac{1}{2V_{\mu}} \left(\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\sigma}}^{\boldsymbol{T}} \right) \tag{13}$$

where $\bar{\boldsymbol{\sigma}} = \sum_{\boldsymbol{k}=1}^{\boldsymbol{Nb}} \boldsymbol{F}_{\boldsymbol{k}} \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{T}}$, being $\boldsymbol{F}_{\boldsymbol{k}(\boldsymbol{n})} = \frac{1}{2}\left(L_{e-1} + L_e \right) \boldsymbol{P}_{\boldsymbol{B}(\boldsymbol{n})}^{\mu}$, N_b the number of nodes used to discretize the RVE boundary, n the load increment (related to the macro-continuum problem); L_e the length of element e where point k is placed; $\boldsymbol{P}_{\boldsymbol{B}(\boldsymbol{n})}^{\mu} = \boldsymbol{P}_{\boldsymbol{B}(\boldsymbol{n}-1)}^{\mu} + \Delta\boldsymbol{P}_{\boldsymbol{B}(\boldsymbol{n})}^{\mu}$, the tractions increment ($\Delta\boldsymbol{P}_{\boldsymbol{B}(\boldsymbol{n})}^{\mu}$) are computed from eq. (8), after updating the boundary displacements increments $\Delta\boldsymbol{U}_{\boldsymbol{B}(\boldsymbol{n})}^{\mu}$, by adding the displacement fluctuation field.

$$\boldsymbol{C}^{\boldsymbol{ep}} = \boldsymbol{C}^{\boldsymbol{ep}(\boldsymbol{Taylor})} + \tilde{\boldsymbol{C}}^{\boldsymbol{ep}} \tag{14}$$

where $\boldsymbol{C}^{\boldsymbol{ep}(\boldsymbol{Taylor})}$ is the Taylor model tangent tensor computed by the volume average of the microscopic constitutive tangent tensor $\boldsymbol{C}_{\mu}^{\boldsymbol{ep}}$, i.e.: $\boldsymbol{C}^{\boldsymbol{ep}(\boldsymbol{Taylor})} = \sum_{\boldsymbol{p}=1}^{\boldsymbol{N_p}} \frac{V_p}{V_{\mu}} \boldsymbol{C}_{\mu}^{\boldsymbol{ep}}$; $\tilde{\boldsymbol{C}}^{\boldsymbol{ep}}$ represents the influence of the displacement fluctuation field into the homogenised tangent tensor, defined as: $\tilde{\boldsymbol{C}}^{\boldsymbol{ep}} = -\frac{1}{V_{\mu}} \boldsymbol{G}_{\boldsymbol{R}} \boldsymbol{K}_{\boldsymbol{R}}^{-1} \boldsymbol{G}_{\boldsymbol{R}}^{\boldsymbol{T}}$, where N_p is the number of phases defined in the RVE; the matrices $\boldsymbol{K}_{\boldsymbol{R}}$ and $\boldsymbol{G}_{\boldsymbol{R}}$ are, respectively, reduced forms of $\boldsymbol{K}$ and $\boldsymbol{G}$, being $\boldsymbol{G} = \sum_{\boldsymbol{e}=1}^{N_{cel}^{\mu}} \boldsymbol{C}_{\mu}^{\boldsymbol{ep}(\boldsymbol{e})} \boldsymbol{B}_{\boldsymbol{e}}\ V_e$, and they are computed according to the boundary conditions adopted in terms of displacements fluctuations.

5 Numerical Example

The macro-continuum problem is defined in Fig 1a, where it is assumed a plane stress state and the following kinematic constraints have been adopted for the nodes on the fixed side: us=0 and un=0, being n and s the normal and tangential directions to the boundary. The other sides are assumed free. A tolerance tol = 1.0 E-5 has been adopted for the convergence of both iterative procedures: the one related to the macro-continuum and the other one required for achieving the RVE equilibrium. The plate thickness has been assumed equal to t=1mm. We have adopted the mesh depicted in Fig. 1b which has 30 elements and 64 nodes along the boundary and 48 triangular cells in the domain.

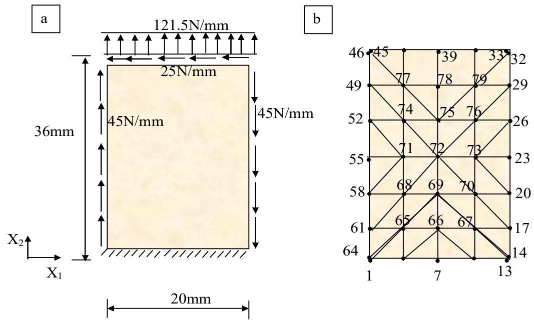

Figure 1- a) Plate definition, b) Plate Discretization.

The RVE depicted in Fig. 2a, which has only one centred inclusion, and the RVE defined in Fig 2b, with five inclusions randomly distributed, will be considered to analyse the influence on the plate mechanical behaviour of different distributions of inclusions in the RVE. The RVEs defined in Fig 2c and 2d will be considered to verify the influence on the plate mechanical behaviour of different distributions of voids in the RVE. Note that the RVEs have the same volume fraction (v_f) of inclusions or voids: v_f=37%. The inclusions have elastic behaviour with the following elastic properties: Young's modulus E=200GPa and Poisson's ratio ν=0.2, while the matrix is governed by the von Mises criterion assuming E=70GPa, ν=0.3, yield stress σ_y=243MPa and hardening modulus K=2.24GPa. The mesh related to the RVE1 with one inclusion has 576 cells, 321 nodes, 64 boundary elements and 36 interface elements (see Fig 2a), while the RVE2 with five inclusions is discretized into 488cells, 277 nodes, 64 boundary elements and 72 interface elements (see Fig 2b). The RVE3 (see Fig 2c) has been discretized into 356 cells, 228 nodes, 64 boundary elements and 36 interface elements while in the RVE4 (see Fig 2d) domain 546 cells, 357 nodes, 80 boundary elements and 96 interface elements have been defined.

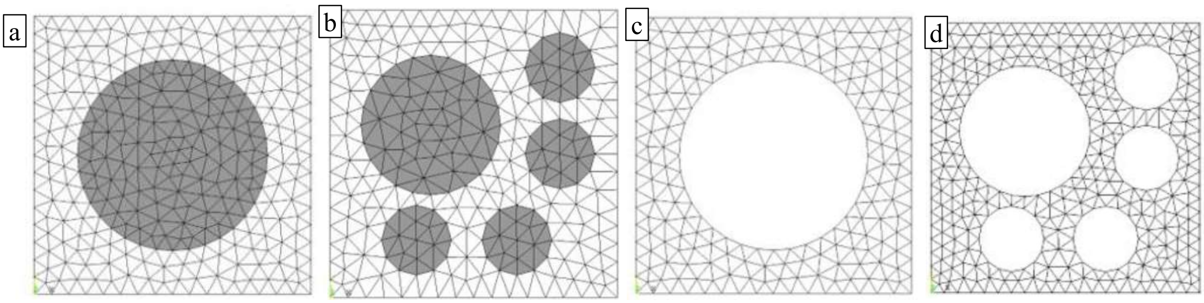

Figure 2–a) Mesh for RVE1 with one inclusion b) Mesh for the RVE2 with five inclusions c) Mesxh for the RVE3 with one void d) Mesh for the RVE4 with five voids

In Figs. 3a and 3b are depicted the displacement u_n at node 39. When RVES 1 and 2 are used (see Fig.3a), we can observe that the results compare very well to the BEM/FEM model, developed by Fernandes et al in [3]. Besides, we can conclude that adopting different distributions of inclusions in the RVE domain do not significantly affect the plate mechanical behaviour. When RVE 3 and 4 are considered (see Fig. 3b), we observe that the different distribution of voids has important influence on the plate mechanical behaviour, because the plate modelled with the RVE3, with only one void, presents bigger rigidity and strength. We can also note that the limit load achieved with the proposed model is considerably bigger than the one related to the BEM/FEM model.

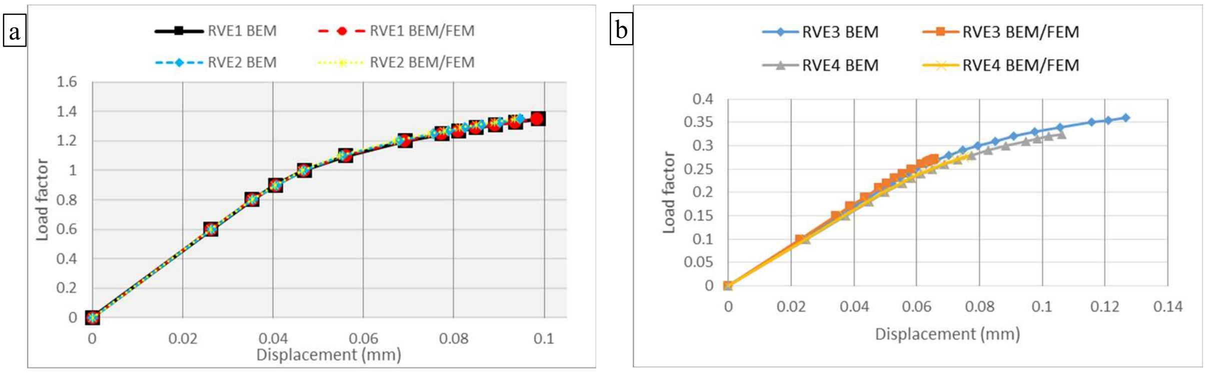

Figure 3 – Displacement u_n at node 39 along the load incremental process of the plate a) using RVEs 1 and 2; b) using RVEs 3 and 4.

6 Conclusions

In the present paper, a BEM formulation to model the stretching plate problem is coupled to a BEM formulation to model the constitutive response of heterogeneous materials, to perform a full coupled multi-scale analysis of plates considering dissipative phenomena. We have shown that different distribution of inclusions in the RVE domain do not affect the rigidity or the strength of the plate, but when five voids are defined in the RVE domain instead of only one void, both the strength and the rigidity of the plate are considerably reduced. The proposed modelling shows the BEM as a good alternative to perform multi-scale analysis, because it presented good qualitative results, the results compared well to the coupled BEM/FEM model, the model presented stability during the iterative procedures (for macro and micro scales) and it achieved the solution with a computation effort much smaller when compared to the BEM/FEM model.

Acknowledgements. The authors wish to thank CNPq (National Council for Scientific and Technological Development) and FAPEG (Goiás Research Foundation) for financial support.

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