Stress Recovery Techniques for the Modified Local Green's Function Method

Ramon Macedo Corrêa[1], Marcos Arndt[1,2], Roberto Dalledone Machado[1,2]

[1]*Graduate Program in Civil Engineering, Federal University of Paraná*
81531-990, Curitiba, Paraná, Brazil
ramon.correa@ufpr.br, arndt@ufpr.br, rdm@ufpr.br
[2]*Graduate Program in Numerical Methods in Engineering, Federal University of Paraná*
81531-990, Curitiba, Paraná, Brazil

Abstract. The Modified Local Green's Function Method (MLGFM) is an integral method hybrid of the Finite Element Method (FEM) and the Boundary Element Method (BEM). The method uses the FEM to create discrete projections of the Green's functions that will be used as fundamental solutions in BEM formulation. The MLGFM has the advantage of presenting high convergence for the displacements in the domain, inherited from the FEM, and for the normal stress on the boundary, inherited from the BEM. Despite these advantages, the recovered stress in the domain has not been explored in previous works. As the MLGFM is a hybrid of the FEM and BEM, techniques used in these methods will be explored in this paper to study the advantages and disadvantages of each one. The techniques that will be used here are the Least Squares procedure, the Zienkiewicz and Zhu (ZZ) recovery, both widely used in FEM, and using the integral form derived from the fundamental solution used in the BEM. The techniques will be analyzed in terms of errors and computational cost of each one.

Keywords: Green's Functions, Finite Element, Stress recovery

1 Introduction

The Modified Local Green's Function Method (MLGFM) was proposed, initially, in the work of Barcellos and Silva [1], being described as a hybrid method of the Finite Element Method (FEM) [2] and the Boundary Element Method (BEM) [3, 4]. The MLGFM uses the FEM technique to obtain discrete projections of the Green's Functions and use them as Fundamental Solutions in BEM.

The main advantage of the MLGFM is that its presents good convergence for both displacements in all the domain and the tractions in all the boundary [5]. The mathematical background of the method was presently with details in Barbieri et al. [5].

In almost all engineering applications the recovery of the stress field in all the domain is mandatory to a correct design of the structural element. In all the previous works about MLGFM, the Least Squares procedure was used to recover the stress field when necessary. But due the method characteristics, other procedures usually applicable in FEM and BEM are also applicable in the MLGFM. Recently, Corrêa et al. [6] developed a new approach to the MLGFM, this paper is based in this formulation.

The goal of this work is to compare some stress recovery techniques used in FEM and BEM and to analyze the advantages and disadvantages of each one. In this paper, the Least Squares procedure [7], the Zienkiewicz and Zhu (ZZ) recovery [8, 9], both widely used in FEM, and the integral form used in BEM [4] are applied.

2 Modified Local Green's Function Method formulation

As mentioned before, the formulation present here is based on a new approach proposed by the authors in Corrêa et al. [6]. Given a general elastic problem, the following boundary value problem can be found [2]:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{b} \quad \text{in} \quad \Omega, \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{b}$ are the body forces, and ∇ is the gradient operator, with the following boundary conditions:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{in} \quad \Gamma_{\mathrm{D}}, \tag{2}$$

and

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{in} \quad \Gamma_{\mathrm{N}}, \tag{3}$$

where $\mathbf{n}$ is the normal vector to the boundary, $\mathbf{t}$ are the boundary reactions, and Γ_{D} and Γ_{N} are the Dirichlet and Neumann boundaries, respectively.

Applying the weighted residual method over the domain using $\mathbf{w}$ as a weight function, and applying the Green's theorem, the following equation can be obtained:

$$\int_{\Omega} \nabla^s \mathbf{w}^T \, \mathbf{C} \, \nabla^s \mathbf{u} \, d\Omega = \int_{\Omega} \mathbf{w}^T \, \mathbf{b} \, d\Omega + \int_{\Gamma} \mathbf{w}^T \, \mathbf{t} \, d\Gamma, \tag{4}$$

where $\mathbf{C}$ is the constitutive relationship tensor and $\nabla^s(\cdot)$ is the symmetrical part of tensor $\nabla(\cdot)$.

Now, using the domain, $\boldsymbol{\Psi}(\mathbf{x})$, and boundary, $\boldsymbol{\Phi}(\mathbf{x})$, shape functions, the domain and boundary fields are described as:

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\Psi}(\mathbf{x}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}(\mathbf{x}) \end{bmatrix} \mathbf{u}_{\mathrm{D}}, \tag{5}$$

$$\tilde{\mathbf{b}}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\Psi}(\mathbf{x}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}(\mathbf{x}) \end{bmatrix} \mathbf{b}_{\mathrm{D}}, \tag{6}$$

and

$$\tilde{\mathbf{t}}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\Phi}(\mathbf{x}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}(\mathbf{x}) \end{bmatrix} \mathbf{t}_{\mathrm{B}}, \tag{7}$$

where, for 2D case, $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u}_{\mathrm{D}}$, $\mathbf{t}_{\mathrm{B}}$ and $\mathbf{b}_{\mathrm{D}}$ are the vector with nodal values of $\mathbf{u}$, $\mathbf{t}$ and $\mathbf{b}$, respectively.

The domain shape functions $\boldsymbol{\Psi}(\mathbf{x})$ are the Finite Element Method shape functions, and the boundary shape functions $\boldsymbol{\Phi}(\mathbf{x})$ need to respect the trace property, in other words:

$$\boldsymbol{\Phi}(\mathbf{x}) = \lim_{\mathbf{x} \to \Gamma} \boldsymbol{\Psi}(\mathbf{x}), \tag{8}$$

The main difference between the MLGFM and the FEM is that the first one does not require that the weight functions are null in the Dirichlet boundary, as the FEM. Since the weight functions are not null, it is necessary to define an approximation for the tractions in all boundary by eq. (7). Using this approximation, and the ones in eq. (5) and (6), and using the same basis functions as weight functions $\mathbf{w}$, the eq. (4) can be rewritten as:

$$\mathbf{K} \, \mathbf{u}_{\mathrm{D}} = \mathbf{D} \, \mathbf{t}_{\mathrm{B}} + \mathbf{A} \, \mathbf{b}_{\mathrm{D}}, \tag{9}$$

where $\mathbf{K}$ is the same stiffness matrix as the FEM, $\mathbf{D}$ is the matrix of the inner dot product on the boundary shape functions, and $\mathbf{A}$ is the matrix of the inner dot product on the domain shape functions.

The Green's functions are not required to be obtained in this approach, they are implicit in the formulation and can be defined as [6]:

$$\mathbf{K} \, \mathbf{G}^{\mathrm{DQ}} = \mathbf{A} \quad \text{or} \quad \mathbf{G}^{\mathrm{DQ}} = \mathbf{K}^{-1} \, \mathbf{A}; \tag{10a}$$

$$\mathbf{K} \, \mathbf{G}^{\mathrm{BQ}} = \mathbf{D} \quad \text{or} \quad \mathbf{G}^{\mathrm{BQ}} = \mathbf{K}^{-1} \, \mathbf{D}; \tag{10b}$$

where $\mathbf{G}^{\mathrm{DQ}}$ and $\mathbf{G}^{\mathrm{BQ}}$ are the nodal values of the Green's functions projections when the source is on the domain and on the boundary, respectively. More details about these projections can be found in Barbieri et al. [5].

The system in eq. (9) can be solved by imposing the boundary conditions in a similar way to the BEM, switching columns of $\mathbf{K}$ and $\mathbf{B}$ related to the prescribed boundary conditions as:

$$\left[-\mathbf{D} \quad | \quad \mathbf{K}\right] \left\{ \begin{array}{c} \mathbf{t}_{\mathrm{B}} \\ \mathbf{u}_{\mathrm{D}} \end{array} \right\} = \left[-\bar{\mathbf{K}} \quad | \quad \bar{\mathbf{D}}\right] \left\{ \begin{array}{c} \bar{\mathbf{u}}_{\mathrm{D}} \\ \bar{\mathbf{t}}_{\mathrm{B}} \end{array} \right\} + \mathbf{A}\,\mathbf{b}_{\mathrm{D}}, \tag{11}$$

where $\bar{\mathbf{u}}_{\mathrm{D}}$ and $\bar{\mathbf{t}}_{\mathrm{B}}$ are the prescribed displacements and tractions in the boundary.

3 Least Squares technique

Proposed in Zienkiewicz and Zhu [9] this technique is widely used in FEM context, the idea is to represent each component of the stress tensor $\boldsymbol{\sigma}$, let's call a generic component of $\boldsymbol{\sigma}$ as $S(\mathbf{x})$, in the form [10]:

$$S(\mathbf{x}) = \mathbf{p}(\mathbf{x})^T\,\mathbf{S}, \tag{12}$$

where $\mathbf{p}(\mathbf{x})$ is a vector of monomials and $\mathbf{S}$ is a vector to be determined in order to minimize error related to $S(\mathbf{x})$. $\mathbf{S}$ is obtained by the minimization of the square error, as:

$$E(\mathbf{S}) = \frac{1}{2} \sum_{j \in \mathbf{A}^k} \left[S(\mathbf{x_j}) - \mathbf{p}(\mathbf{x_j})^T\,\mathbf{S}\right]^2. \tag{13}$$

If the number of the terms of the basis $\mathbf{p}(\mathbf{x})$ are equal or smaller than the number of points in the cloud k, the system can be solved as:

$$\mathbf{E}\,\mathbf{S} = \mathbf{F} \tag{14}$$

where:

$$\mathbf{E} = \sum_{j \in \mathbf{A}^k} \mathbf{p}(\mathbf{x_j})\,\mathbf{p}(\mathbf{x_j})^T \tag{15}$$

and

$$\mathbf{F} = \sum_{j \in \mathbf{A}^k} \mathbf{p}(\mathbf{x_j})\,S(\mathbf{x_j}). \tag{16}$$

This process is repeated for each node k in the mesh.

4 Zienkiewicz and Zhu (ZZ) recovery technique

The second stress recovery technique is also proposed in Zienkiewicz and Zhu [9] and consists in a L_2 projection in an element patch. The goal is to represent each component of the stress tensor $\boldsymbol{\sigma}$, let´s say $S(\mathbf{x})$, as:

$$S^*(\mathbf{x}) = \boldsymbol{\Psi}(\mathbf{x})^T\,\mathbf{S}^*, \tag{17}$$

where $\boldsymbol{\Psi}(\mathbf{x})$ are the same basis used in the FEM approximation and $\mathbf{S}$ are the set of nodal values of $S^*(\mathbf{x})$, the approximation of $S(\mathbf{x})$.

In order to minimize the error of the approximation given by:

$$E_{L_2} = \int_\Omega \left(S(\mathbf{x}) - S^*(\mathbf{x})\right)^2\,d\Omega = \int_\Omega \left(S(\mathbf{x}) - \boldsymbol{\Psi}(\mathbf{x})^T\,\mathbf{S}^*\right)^2\,d\Omega, \tag{18}$$

the minimization condition of eq. (18) leads to:

$$\int_{\Omega} \mathbf{\Psi}(\mathbf{x})^T \, \mathbf{\Psi}(\mathbf{x}) \, d\Omega \, \mathbf{S}^* = \int_{\Omega} \mathbf{\Psi}(\mathbf{x})^T \, \mathbf{S}(\mathbf{x}) \, d\Omega, \tag{19}$$

that results in a system in the form:

$$\mathbf{E} \, \mathbf{S}^* = \mathbf{F}. \tag{20}$$

5 BEM Integral form

Once the values of $\mathbf{u}$ and $\mathbf{t}$ are known on the boundary, after the solution of eq. (9), the stress tensor, for each source point ξ, can be obtained by the integral form[3, 4]:

$$\sigma_{ij}(\xi) = \int_{\Gamma} D_{kij}(\xi, \mathbf{x}) \, \Phi_k(\mathbf{x}) \, d\Gamma \, t_k - \int_{\Gamma} S_{kij}(\xi, \mathbf{x}) \, \Phi_k(\mathbf{x}) \, d\Gamma \, u_k + \int_{\Omega} D_{kij}(\xi, \mathbf{x}) \, \Psi_k(\mathbf{x}) \, d\Omega \, b_k \tag{21}$$

where:

$$D_{kij}(\xi, \mathbf{x}) = \frac{1}{r^{\alpha}} \left\{ (1-2\nu) \left\{ \delta_{ki} \, r_{,j} + \delta_{kj} \, r_{,i} - \delta_{ij} \, r_{,k} \right\} + \beta \, r_{,i} \, r_{,j} \, r_{,k} \right\} \frac{1}{4 \, \alpha \, \pi \, (1-\nu)} \tag{22}$$

and

$$\begin{aligned} S_{kij}(\xi, \mathbf{x}) = & \frac{2}{r^{\beta}} \Bigg\{ \beta \frac{\mathbf{r}}{\partial \mathbf{n}} \left[(1-2\nu) \delta_{ij} \, r_{,k} + \nu \left(\delta_{ik} \, r_{,j} + \delta_{jk} \, r_{,i} \right) - \gamma \, r_{,i} \, r_{,j} \, r_{,k} \right] \\ & + \beta \, \nu \left(n_i \, r_{,j} \, r_{,k} + n_j \, r_{,i} \, r_{,k} \right) \\ & + (1-2\nu) \left(\beta \, n_k \, r_{,i} \, r_{,j} + n_j \, \delta_{ik} + n_i \, \delta_{jk} \right) - (1-4\nu) \, n_k \, \delta_{ij} \Bigg\} \frac{1}{4\alpha \, \pi (1-\nu)}, \end{aligned} \tag{23}$$

with $\alpha = 1, \beta = 2$ and $\gamma = 4$ for 2D problems, and $\alpha = 2, \beta = 3$ and $\gamma = 5$ for 3D problems, δ_{ij} is the Kronecker delta, $\mathbf{n}$ is the normal vector to the boundary, μ is the Lamé's constants, ν is the Poisson ratio, r is the distance between the source and the field point and:

$$r_{,i} = \frac{\partial r}{\partial x_i}. \tag{24}$$

6 Numerical Example

The numerical example analyzed here is an L-shaped plane elastic domain with thickness t and having a singularity point. The model scheme is shown in Fig. 1 and the load is obtained from the analytical solution of stress [11]:

$$\sigma_x = K_1 \, \lambda_1 \, r^{\lambda_1 - 1} \left\{ \left[2 - Q_1 \, (\lambda_1 + 1) \right] \cos \left(\lambda_1 - 1 \right) \theta - \left(\lambda_1 - 1 \right) \cos \left(\lambda_1 - 3 \right) \theta \right\}, \tag{25}$$

$$\sigma_y = K_1 \, \lambda_1 \, r^{\lambda_1 - 1} \left\{ \left[2 + Q_1 \, (\lambda_1 + 1) \right] \cos \left(\lambda_1 - 1 \right) \theta + \left(\lambda_1 - 1 \right) \cos \left(\lambda_1 - 3 \right) \theta \right\}, \tag{26}$$

$$\tau_{xy} = K_1 \, \lambda_1 \, r^{\lambda_1 - 1} \left[\left(\lambda_1 - 1 \right) \sin \left(\lambda_1 - 3 \right) \theta + Q_1 \left(\lambda_1 + 1 \right) \sin \left(\lambda_1 - 1 \right) \theta \right], \tag{27}$$

where r and θ are the polar coordinates shown in Fig. 1, K_1 is a generalized stress-intensity factor (here assumed as $K_1 = 1$), $\lambda_1 = 0.544483737$ and $Q_1 = 0.543075579$.

The Dirichlet boundary conditions are imposed by the analytical solution of displacements [11]:

$$u_x = \frac{K_1}{2\,G} r^{\lambda_1} \left\{ \left[\kappa - Q_1 \left(\lambda_1 + 1 \right) \right] \cos \lambda_1 \, \theta - \lambda_1 \cos \left(\lambda_1 - 2 \right) \theta \right\}, \tag{28}$$

$$u_y = \frac{K_1}{2\,G} r^{\lambda_1} \left\{ \left[\kappa + Q_1 \left(\lambda_1 + 1 \right) \right] \sin \lambda_1 \, \theta - \lambda_1 \sin \left(\lambda_1 - 2 \right) \theta \right\}, \tag{29}$$

where G is the shear modulus and κ depends on Poisson's ratio ν only. For plane strain, the following relation is valid: $\kappa = 3 - 4\nu$ [11]. Here is assumed $\nu = 0.3$.

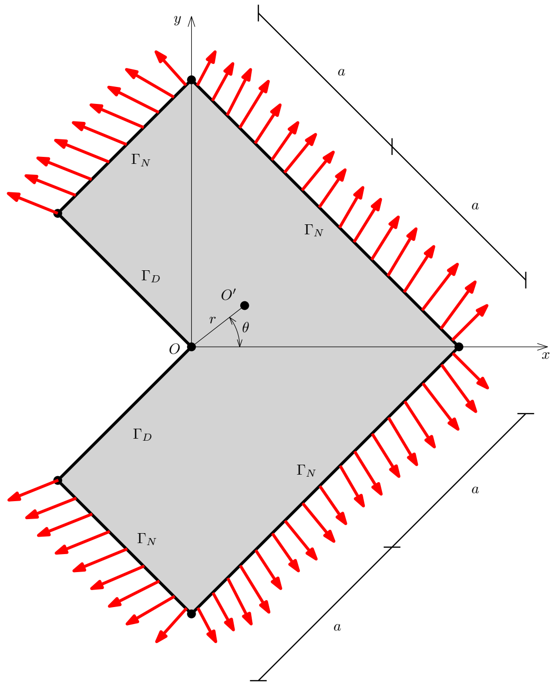

Figure 1. L-shaped domain scheme, $a = 10$.

For the MLGFM model, Lagrange quadratic functions were used for the boundary meshes and bi-quadratic Lagrange functions for the domain meshes. For error analysis, an approximation of the L_2-norm error was used, for each stress component, defined as:

$$\epsilon_{er} = \frac{1}{|S|_{\text{max}}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left(S_i^{(e)} - S_i^{(n)} \right)^2}, \tag{30}$$

where N is the number of nodes, $|S|_{\text{max}}$ is the maximum value on N samples nodes, $S_i^{(e)}$ is the exact value on node i and $S_i^{(n)}$ is the approximate value on node i. The error for σ_x, σ_y and τ_{xy} components using the three recovery techniques are presented in Fig. 2, Fig. 3 and Fig. 4. Due to the hypersingularity of the integral form for boundary points ($r \to 0$), defined in eq. (21), only the stresses in internal points are analyzed in this paper.

As can be noticed in Figs. 2, 3 and 4 the errors using the Least Squares and the ZZ recovery are small and very similar to each other, but the errors using the BEM Integral form are considerable smaller than the others. In terms of convergence rates, the three techniques showed similar rates for all stress components. This can be explained due the fact that the integral form presents less numerical degradation than the other stress recovery techniques.

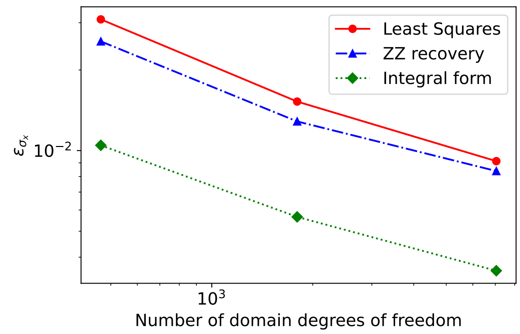

Figure 2. Error for σ_x stress component.

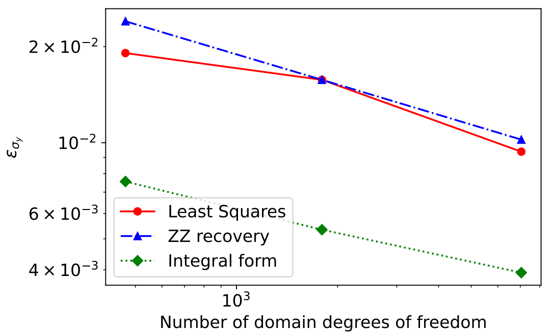

Figure 3. Error for σ_y stress component.

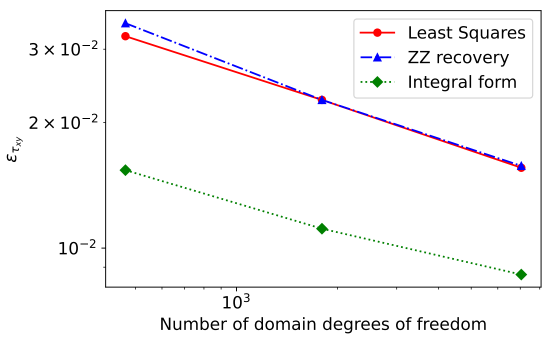

Figure 4. Error for τ_{xy} stress component.

7 Conclusions

In this paper three procedures to recover the stress field in MLGFM are analyzed, two of them (Least Squares and ZZ recovery) proposed by Zienkiewicz and Zhu [9] and widely used in methods based on the FEM, and one where the stresses are obtained directly from the BEM integral form. Although Least Squares and ZZ recovery present small errors with values very similar to each other, the integral form presents smaller errors compared to these other two techniques. These results are expected due the integral form doesn't present the numerical degradation of the result that the methods used in FEM present.

For future works, the idea is to explore some techniques to avoid the problem of the hypersingularity of the integral form in boundary points and explore other examples and other types of problems.

Acknowledgements. The authors gratefully acknowledge CAPES (grant number: 88887.825830/2023-00) and CNPq (grant numbers: 301802/2022-0 and 316985/2021-0) for their financial support. The authors also acknowledge the PPGEC (Graduated Program in Civil Engineering) and the UFPR (Federal University of Paraná) for their institutional support.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] C. S. Barcellos and L. H. M. Silva. Elastic membrane solution by a modified local greens function method. In C. A. Brebbia and C. S. Venturine, eds, *Proceedings of the international conference on boundary element technology*, Southampton. Computational Mechanics Publications, 1987.
[2] K. Bathe. *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, New Jersey, 1996.
[3] C. A. Brebbia, J. C. F. Telles, and L. C. Wrobel. *Boundary Element Techniques: Theory and Applications in Engineering*. Springer-Verlag Berlin Heidelberg, 1 edition, 1984.
[4] C. A. Brebbia and J. Dominguez. *Boundary Elements: an introductory course*. WIT Press/Computational Mechanics Publications, 1992.
[5] R. Barbieri, P. A. R. Muñoz, and R. D. Machado. Modified local Green's function method (MLGFM) part I. mathematical background and formulation. *Engineering Analysis with Boundary Elements*, vol. 22, n. 2, pp. 141–151, 1998.
[6] R. M. Corrêa, M. Arndt, and R. D. Machado. A new approach for the Modified Local Green's Function Method applied to solid mechanics problems. *Engineering Analysis with Boundary Elements*, vol. 155, pp. 907–919, 2023.
[7] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Butterworth-Heinemann, 6 edition, 2005.
[8] O. C. Zienkiewicz and J. Z. Zhu. A simple error estimator and adaptive procedure for practical engineerng analysis. *International Journal for Numerical Methods in Engineering*, vol. 24, n. 2, pp. 337–357, 1987.
[9] O. C. Zienkiewicz and J. Z. Zhu. The superconvergent patch recovery anda posteriori error estimates. part 1: The recovery technique. *International Journal for Numerical Methods in Engineering*, vol. 33, n. 7, pp. 1331–1364, 1992.
[10] P. T. Mendonça and E. A. Fancello. *O Método dos Elementos Finitos aplicado à Mecânica dos Sólidos*. Orsa Maggiore, 1th edition, 2019.
[11] B. A. Szabó. Mesh design for the p-version of the finite element method. *Computer Methods in Applied Mechanics and Engineering*, vol. 55, n. 1-2, pp. 181–197, 1986.