Numerical Modelling of Crack Propagation Using BemCracker2D Program and Comparison of Different Criteria

G. Gomes[1], A. M. Delgado Neto[1], I. A. A. Lustosa[2], D. C. de Moura[1]

[1]*Dept. of Civil Engineering. University of Brasília*
Asa Norte, 70000-900, Brasília/DF, Brazil
ggomes@unb.br, alvaro.martins.bok@gmail.com, danilo.cmoura@hotmail.com
[2]*Instituto Federal do Piauí, Parnaíba/PI - 64211-145*
eng.iuri@gmail.com

Abstract. The study of crack propagation modeling and its industrial applications often focuses on finding numerical and computational techniques that simulate geometry and loading conditions as realistically as possible while minimizing computational effort. Traditionally, the Finite Element Method (FEM) and its extended or enriched forms (XFEM/GFEM) have been widely used. However, these methods still face challenges with remeshing during incremental analyses. In this paper, we address the problem of modeling moving discontinuities, such as crack propagation, using the Dual Boundary Elements Method (DBEM) and the BemCracker2D program. We employ three different criteria for crack propagation: Maximum Principal Stress, Maximum Energy Release Rate, and Strain Energy Density. Our methodology involves analyzing a PMMA beam, calculating the Stress Intensity Factors (SIFs) using the J-Integral, and computing the propagation directions for each increment based on the three employed criteria. To ensure the continuity of equations for each criterion, our approach requires correcting the crack path direction for the nth increment. We compare our results with a numerical study performed using FEM, which has been previously validated with an experimental study of the same beam. Our findings demonstrate the efficiency of the adopted methodology, the accuracy of the results, and the effectiveness of the BemCracker2D code.

Keywords: crack growth path, DBEM, stress intensity factors; *BemCracker2D*.

1 Introduction

Fracture mechanics is a relatively new science, having been established less than 40 years ago, with its primary focus on preventing brittle fractures. Despite its recent formalization, the concern for preventing such types of rupture is longstanding [1]. With the advancement of science, there has been a considerable increase in studies on stresses and crack propagations, as well as the directions in which they occur.

One of the main challenges in the study of fracture mechanics is the complexity of solving engineering problems, resulting in a limited number of available solutions. This issue has been mitigated by the constant increase in computer processing power, along with the development of numerical methods for function approximation, such as the Finite Element Method (FEM) and the Boundary Element Method (BEM) [2].

Among numerical methods, FEM has gained significant popularity in recent decades. However, the traditional use of finite elements to solve fracture mechanics problems involving singularities, such as cracks, has several disadvantages, including the need for remeshing to accommodate crack propagation and reduced efficiency [3].

These difficulties have led to the development of new techniques that eliminate the need for remeshing. The Extended Finite Element Method (XFEM) allows for crack propagation modeling without generating new meshes for cracks with small curvature, though it loses precision for elements with high curvature [4]. Another technique is the Generalized Finite Element Method (GFEM), which uses an enriched approach to model 3D crack growth, introducing a crack representation with triangles [5].

Therefore, one of the reasons for using BEM in this work is the advantage of not requiring remeshing each time the crack extends. In BEM, displacement and body forces are interpolated variables, and from the boundary values, internal stresses are found. Standard BEM effectively resolves problems of elasticity in infinite or semi-infinite domains. However, when discontinuities such as cracks are present, the method becomes ineffective due to the distinct faces of cracks sharing the same coordinates [2]. To address this issue, Portela [6] developed the Dual Boundary Elements Method (DBEM) by applying displacement boundary integral equations on one crack face and traction boundary integral equations on the other. DBEM offers advantages such as simplified modeling of the cracking area, reduced processing time, and precise simulation of crack growth [6].

This paper aims to contribute to the development of BEM-based analysis programs, which are relatively scarce compared to FEM-based programs. BemCracker2D is the software used in this analysis. For fracture mechanics problems, it calculates tensile stresses using standard BEM and employs DBEM incrementally for crack growth analysis [7].

This work is organized as follows: Section 2, Theory, presents basic concepts of fracture mechanics, such as stress intensity factors, energy release rate, J-integral, and three criteria for calculating crack propagation: Maximum Principal Stress (MPS), Maximum Energy Release Rate (MERR), and Strain Energy Density (SED). It also covers concepts of BEM and DBEM. Section 3, Materials and Methods, introduces the BEMLAB2D GUI and BemCracker2D software, outlines the modeling strategy, and describes the PMMA beam specimen. Section 4 presents the modeling process. Section 5 compares the results, showing how the modeling was conducted and the results obtained. Section 6 provides the final considerations.

2 Theoretical Background

In this section, we present the essential theoretical concepts and formulations that support our study of crack propagation using the BemCracker2D program. We begin by exploring the basic principles of fracture mechanics, and we discuss the three criteria for crack propagation, as well as we provide an overview of the Boundary Element Method (BEM) and its extension to the Dual Boundary Element Method (DBEM), highlighting their relevance and application in the context of crack modeling.

2.1 Linear Elastic Fracture Mechanics

The Linear Elastic Fracture Mechanics (LEFM) has the purpose of studying how tensions behave and distribute themselves, as well as their greatness, in regions close to a crack. These tensions and their magnitude can be described by the Stress Intensity Factor (SIF) - K. Thus, having material parameters and K it is possible to make forecasts of propagations and failures of a crack when subjected to certain loading.

Stress Intensity Factors. The Stress Intensity Factors, (SIFs), K, is one of the most important parameters in linear elastic fracture mechanics and quantifies the stress field near a crack, providing fundamental information on how the crack starts and propagates [3]. This factor is associated with the opening mode of the crack and there are three modes: Mode I describes the loading or displacement that induces crack opening, separating its two ends within the plane; Mode II results from shear forces acting at the crack tip within the plane; and Mode III also involves shear forces at the crack tip, but these forces act outside the plane

In general, it can be calculated using the equation (1):

$$K = \sigma\sqrt{\pi a} * Y\left(\frac{a}{w}\right) \tag{1}$$

Energy Release Rate. It is well-established that all materials experience internal tensile forces that maintain their structural integrity. When these tensile forces are exceeded and no longer present, cracks form. At this juncture, the system may either maintain equilibrium or not. In the former case, the system's overall energy remains unchanged; in the latter, potential energy diminishes until the system reaches equilibrium or experiences total failure [8]. In essence, the formation or propagation of a crack occurs when the system's potential energy decreases or remains constant [9].

In 1956, Irwin [10] introduced an energy-based approach akin to Griffith's model but more suitable for engineering applications. He defined the Energy Release Rate (G), which quantifies the energy available per unit crack extension, as:

$$\mathcal{G} = -\frac{d\Pi}{dA} = \frac{\pi * \sigma^2 * a}{E} \tag{2}$$

J-Integral. The J-Integral, developed by Rice, consists of a line integral that has the same value for any path around the tip of the crack in a two-dimensional deformation field of an elastic or elastoplastic body [11] and that for any closed boundary, the value of J is equal zero, given by equation (3):

$$J = \int_{\Gamma} \left(w d_y - T_i \frac{\partial u_i}{\partial_x} d_s \right) \tag{3}$$

It is known that $J = \mathcal{G}$, therefore, it relates to K by means of equation (4):

$$J = \mathcal{G} = \frac{{K_I}^2}{E'} + \frac{{K_{II}}^2}{E'} + \frac{{K_{III}}^2}{2 * \mu} \tag{4}$$

Criteria for Calculating Crack Propagation:

I. **Maximum Principal Stress Criterion (MPS):** According to the maximum principal stress criterion, crack propagation occurs perpendicular to the plane where the circumferential stress is maximum, and where the second derivative of circumferential stress with respect to θ is negative, indicating zero shear stress. However, propagation only occurs if the circumferential stress exceeds the critical stress intensity factor K_{IC} [12]. The propagation angle is given by equation (5):

$$\theta = 2\tan^{-1}\left(\frac{1}{4}\frac{K_I}{K_{II}} \pm \frac{1}{4}\sqrt{\left(\frac{K_I}{K_{II}}\right)^2 + 8} \right) \tag{5}$$

II. **Maximum Energy Release Rate Criterion (MERR):** According to this criterion, crack propagation will occur in the direction θ where there is the greatest release of energy. Specifically, propagation happens when the maximum energy release rate exceeds the critical energy rate required for cracking [13]. The equation $g(\theta)$, which represents the energy release rate as a function of the angle θ relative to the crack tip, and which needs to be maximized, is given by equation (6):

$$\mathcal{G}(\boldsymbol{\theta}) = \frac{4}{E}\left(\frac{1}{3 + \cos^2\theta}\right)^2 \left(\frac{1 - \frac{\theta}{\pi}}{1 + \frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}} \left[(1 + 3\cos^2\theta){K_I}^2 + 8\sin\theta\cos\theta K_{II} + (9 - 5\cos^2\theta){K_{II}}^2\right] \tag{6}$$

III. **Strain Energy Density Criterion (SED):** This criterion, proposed by Sih [14], utilizes the strain energy density SSS to determine the direction of crack propagation. Crack propagation occurs when the strain energy density S(θ)S(\theta)S(θ) equals a critical strain energy density factor ScrS_{cr}Scr, and it proceeds in the direction where the strain energy density is minimized [14].

The expression $S(\theta)$ is given by equation (7):

$$S(\theta) = a_{11}{K_I}^2 + 2a_{12}K_I K_{II} + a_{22}{K_{II}}^2 \quad (7)$$

Where the coefficients a_{11}, a_{12} and a_{22} represent the relevant terms of the elasticity or stiffness matrix that depend on the direction θ where the strain energy density is being evaluated.

Boundary Element Method. The partial differential equation of a problem describes how the unknowns behave within the domain and on its boundary. The Boundary Element Method (BEM) transforms these equations into surface integrals that relate values only on the boundary, providing a numerical solution [15]. Its discretization uses elements only on the boundary, which integrate the equations for each point, making it effective for solving boundary integral equations. The variables considered are displacements and tractions. Because the method operates entirely on the boundary, the problem's dimensionality decreases by one [16]. This reduction is advantageous as it allows working with a much smaller system of equations and less data volume, thereby reducing computational effort compared to other methods [17].

Dual Boundary Element Method. While BEM performs well for linear elastic problems in continuous domains, discontinuities such as cracks - whether internal or on the boundary, create challenges. Cracks lack volume or area and introduce discontinuities in the strain field [6]. Consequently, standard BEM fails in crack problems due to singularities arising from coincident crack surfaces, where equations for points on one crack face are identical to those for points on the opposite face at the same coordinates [6].

To address this issue, the Dual Boundary Element Method (DBEM) applies displacement boundary integral equations on one crack face and traction boundary integral equations on the opposite face. This approach ensures that even though points on one side of the crack are coincident with those on the other side, the boundary equations differ, thereby avoiding singularities [6].

3 Material and Methods

Here, will be presented the softwares *BEMLAB2D* GUI and *BemCracker2D* that will be used to the process of automatic modelling and calculation. A specimen of previous papers [18] will be shown. Its geometry and proprieties will be presented. The modelling strategy will be shown. All these aspects compose the material and methods applied in this paper.

Software *BEMCRACKER2D:* The *BemCracker2D* (BC2D) is a software written in a C++ programming language using Object Oriented Programming concepts (OOP) for the purpose of analyzing two-dimensional elastostatic problems, using BEM [2] and it has 3 calculations modules:

I. Standard BEM (module I) – for analysis of elements without discontinuity, that is, without cracks and using BEM;
II. DBEM without propagation (module II) – for analysis of elements with discontinuity, however, without propagation and using the DBEM;
III. DBEM with propagation (module III) – for analysis of elements with discontinuity when there is propagation and using the DBEM.

The program does the processing as follows:

a. Stress analysis using BEM and DBEM;
b. Evaluation of Stress Intensity Factors (SIFs) using the J integral method;
c. Evaluation of the direction/correction of the propagation of the crack using the following methods: Maximum Principal Stress; Maximum Energy Release Rate; Strain Energy Density;

d. Fatigue Life evaluation using Paris law;

Software *BEMLAB2D* GUI: The *BEMLAB2D* GUI in Figure 1(a), is a graphical interface written in MATLAB with the purpose of pre and post-processing. Pre-processing constitutes modeling by means of points, lines, arcs and zones with posterior generation of boundary elements and application of traction and strain boundary conditions. Post-processing constitutes the generation of graphics by reading previously processed data [19].

The interface is GUI type – Graphical User Interface – and its most advantageous purposes are: to generate meshes for two-dimensional problems and to enable its visualization; Display graphics of elastostatic analysis processed by the *BemCracker2D* program [20].

Actions are defined by the user through buttons, mouse, and dialog. The dialogues facilitate the execution and understanding of the functions. The modules of the program are:

I. GEOMETRY (module I);
II. MESH (module II);
III. BOUNDARY CONDITIONS (module III);
IV. ELASTOSTATIC ANALYSIS (module IV);
V. GRAPHICAL RESULTS (module V) – Figure 1 (b).

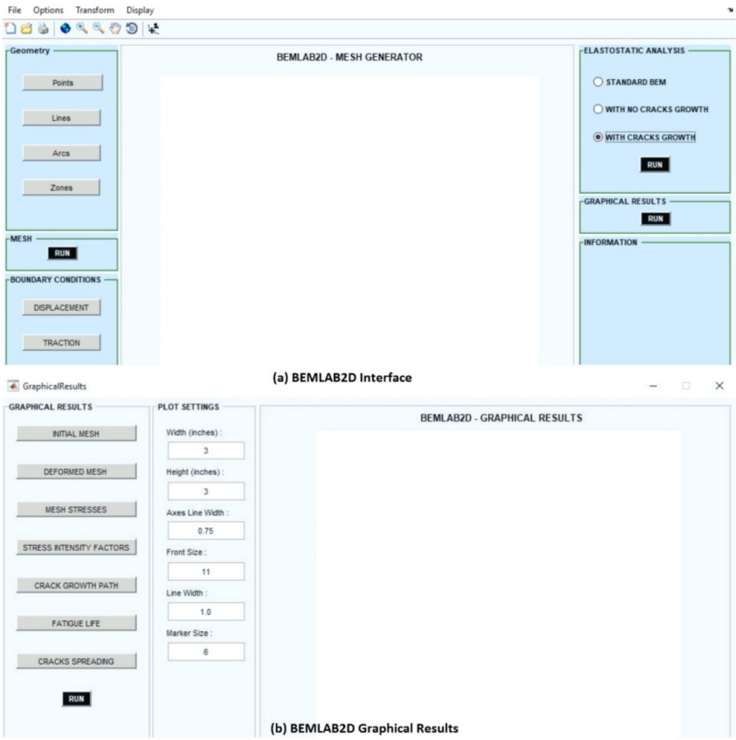

Figure 1 – (a) *BEMLAB2D* GUI interface. (b) Graphical Results.

Modelling Strategy: The process of modelling follows the steps shown in Figure 2. First process is called pre-processing and it's done in *BEMLAB2D* GUI. The stage consists of launching points for drawing lines and arcs. Then, the zones are defined with posterior generations of the BEM mesh. At the end, will be launched the boundary conditions of displacement and traction and defined the type of analysis will be. A file with geometric and material parameters will be written. Second process is called processing and it's done in *BemCracker2D*. Here, the file with geometric and material parameters is read and all the analysis will be done. Several files will be written containing information as SIFs, propagation angle, crack growth path, deformed mesh, stresses etc. Third and final process is called post-processing and it's shown in *BEMLAB2D* GUI. The program read all files written and plot several graphics as crack growth path, SIFs, deformed mesh etc.

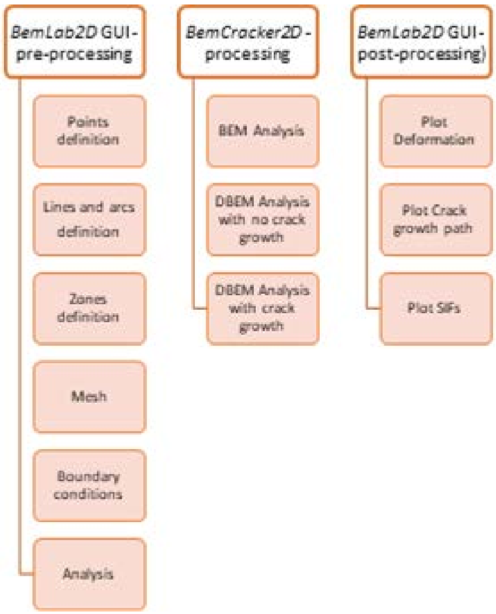

Figure 2 – Flow chart of modelling strategy.

PMMA Beam with an initial notch: The PMME Beam model is a bi-supported beam with central loading and a notch in the lower left corner, as showed in Figure 3. Boljanovic and Maksimovic [18], have made a numerical simulations with Finite Element Method to examine the propagation of the crack formed from the notch, as well as the Stress Intensity Factors in each propagation increment [18].

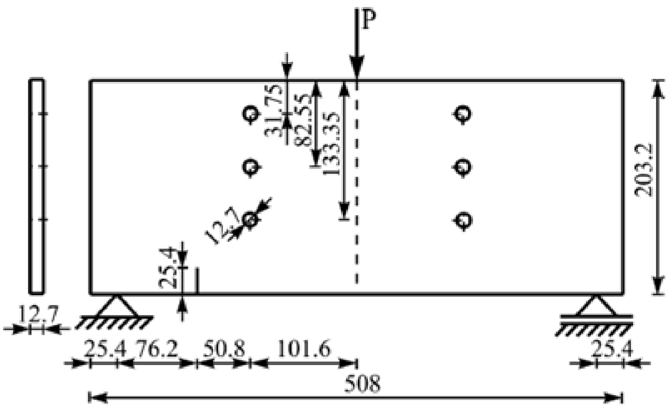

Figure 3 – Initial geometry of the PMMA Beam. Dimensions are in millimetre [18].

The propagation results were compared with an experimental study carried out by Ingraffea and Grigoriu [21]. The beam's material was polymethylmethacrylate. The loading was cyclical with a maximum load of $5.7kN$ and minimum of at least 10% of the maximum. The modulus of elasticity of the material is $E = 3103MPa$ and the Poisson coefficient is $\nu = 0.36$ [21]. The criterion used for the calculation of the propagation was the MPS. Nine increments were considered with length of $12.7mm$, however, the last increment was considered with length of $6.35mm$ because the tenth increment is already very close to the hole.

4 Modelling

This section presents the procedure of launching the PMMA beam, firstly using *BEMLAB2D* GUI for generate the boundary mesh, followed by *BemCracker2D* for DBEM analysis.

The PMMA Beam in *BEMLAB2D* GUI: The zones were released with the parameters of the polymethylmethacrylate material being six holes zones and one master zone involving the contour of the beam and the crack. The holes mesh was defined with six continuous quadratic elements in each hole. The crack was made with 3 discontinuous quadratic elements on each face and with a ratio of 0.5, 0.3, 0.2. The mesh boundary of the beam was defined with 50 continuous quadratic elements. The boundary conditions of displacement were two supports: one of second degree on the left side and another of first degree on the right side. The boundary condition of traction was a concentrated loading in the middle of the upper side of the beam. The Figure 4 shows details of the modeling.

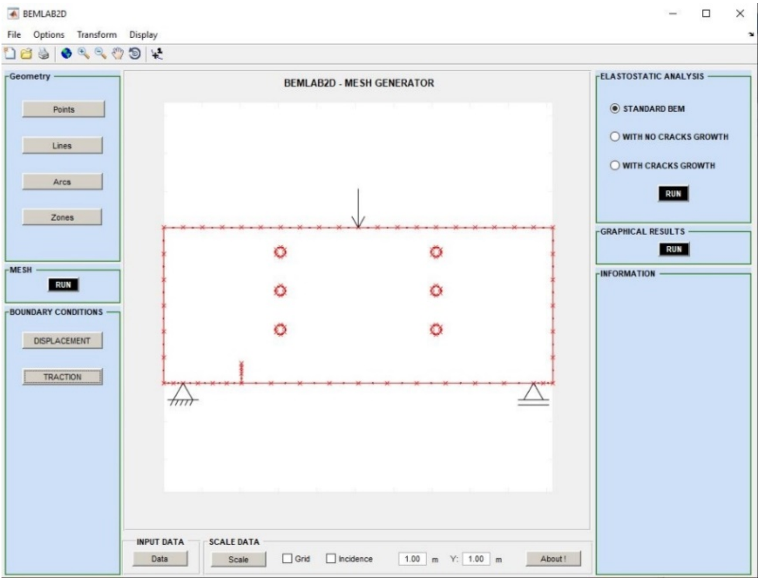

Figure 4 – Modelling of PMMA beam in *BEMLAB2D* GUI.

Finally, the elastostatic analysis with crack growth was selected – Figure 5. The 127 mm step was then defined for the crack growth.

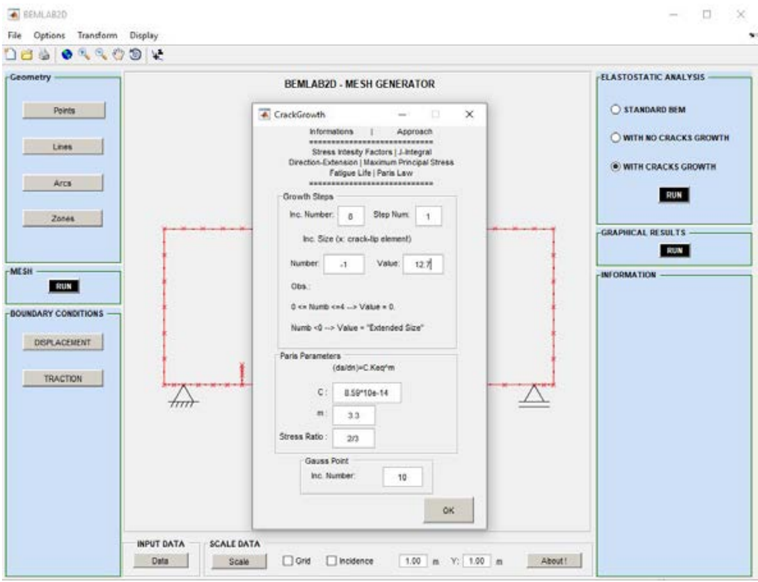

Figure 5 – Crack Growth Parameters in *BEMLAB2D* GUI.

5 Comparison of Results

The results of Boljanovic and Maksimovic's work [18] were compared with the experimental ones made by Ingraffea and Grigoriu [21]. The conclusion they had was that the numerical simulation of crack propagation calculated by the MPS can accurately predict the values found empirically. Thus, the numerical values found by Boljanovic and Maksomovic will be compared with the processing data made by the *BemCracker2D* from the modeling in the *BEMLAB2D* GUI.

Stress Intensity Factors: During propagation, the values of K_I e K_{II} were calculated using three criteria – MPS, MERR and SED – and compared with the values exposed by Boljanovic and Maksimovic [18] – Table 1.

Table 1 – SIFs with *BemCracker2D* and Boljanovic and Maksimovic's paper. Unit in $MPa.\sqrt{m}$.

	BC2D MPS		BC2D MERR		BC2D SED		Boljanovic and Maksimovic	
Step	KI	KII	KI	KII	KI	KII	KI	KII
1	0.6903	0.0800	0.6903	0.0800	0.6903	0.0800	0.69	0.08
2	0.8807	0.0101	0.8814	0.0083	0.8807	0.0102	0.88	0.04
3	1.0833	0.0099	1.0844	0.0084	1.0834	0.0099	1.08	0.04
4	1.3212	0.0107	1.3227	0.0091	1.3212	0.0108	1.31	0.04
5	1.5887	0.0258	1.5907	0.0248	1.5888	0.0259	1.57	0.05
6	1.8704	0.0083	1.8717	0.0076	1.8705	0.0083	1.86	0.2
7	2.2582	-0.0207	2.2594	-0.0185	2.2583	-0.0208	2.18	�0.24
8	2.9135	-0.0120	2.9166	-0.0101	2.9137	-0.0120	2.84	�0.03
9	2.9313	-0.9116	3.2765	-0.3017	FALHA	FALHA	3.39	�0.01
10	FALHA	FALHA	FALHA	FALHA	FALHA	FALHA	FALHA	FALHA

It is noted that the three numerical criteria calculated by the *BemCracker2D* bring satisfactory results. The values of SIFs have precision around two decimal places. The first increment is the crack tip, so the SIFs values are the same. The others have a slight divergence, since the coordinates of the subsequent propagation points are slightly different. The ninth increment calculated by SED has very divergent values because the crack is already very close to the hole. In this case, there has been a failure of the element.

It is also perceived that the K_I tends to increase its value, while the K_{II} tends to remain constant – Figure 6. This shows the behavior of the crack of starting to propagate in mixed mode – mode I and mode II – and, after some increments, continue its propagation only in mode I, since the value of K_I becomes much higher than K_{II}.

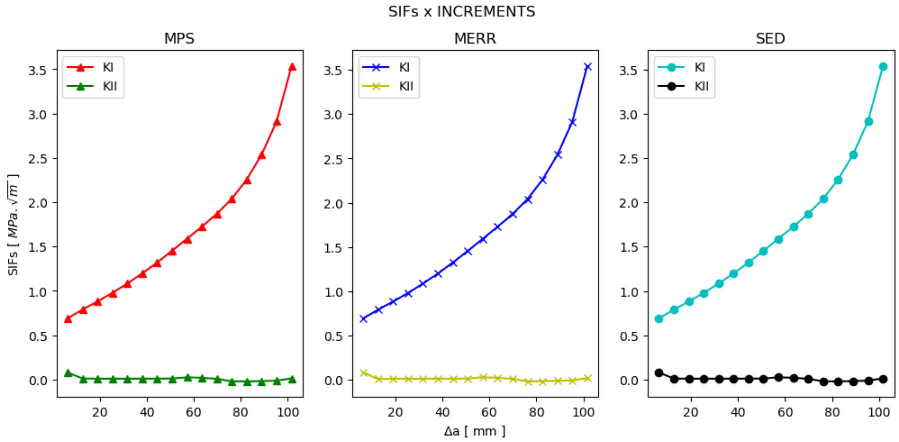

Figure 6 – Graphics of SIFs vs. Increments of the three criteria calculated through *BemCracker2D*.

When comparing the three curves of SIFs between each other – Figure 7, it is possible to perceive that there is almost no variation from one curve to another, since the propagation also occurs in a very similar way. The comparison made with the values calculated by Boljanovic and Maksimovic [18] shows a minimum difference and the same tendency of the curves.

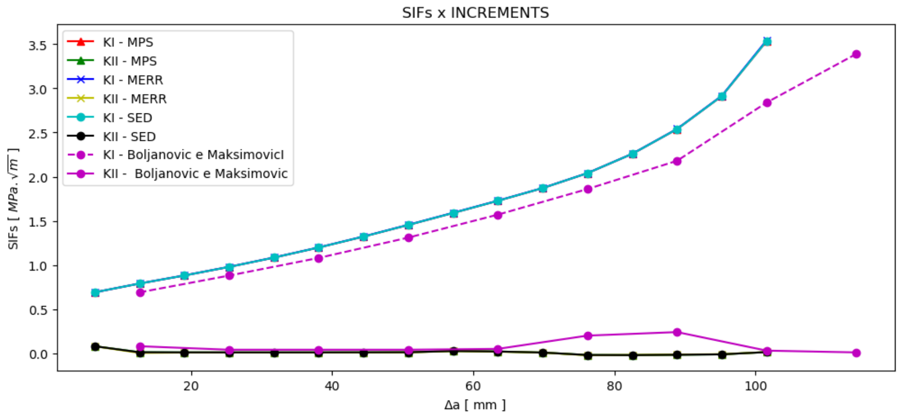

Figure 7 – SIFs vs. Increments with *BemCracker2D* and Boljanovic and Maksinovic's paper.

Mesh: The Figure 8 (a) shows the mesh made of finite elements by Boljanonic and Maksimovic. Figure 8 (b) is the mesh made of boundary elements in *BEMLAB2D* GUI.

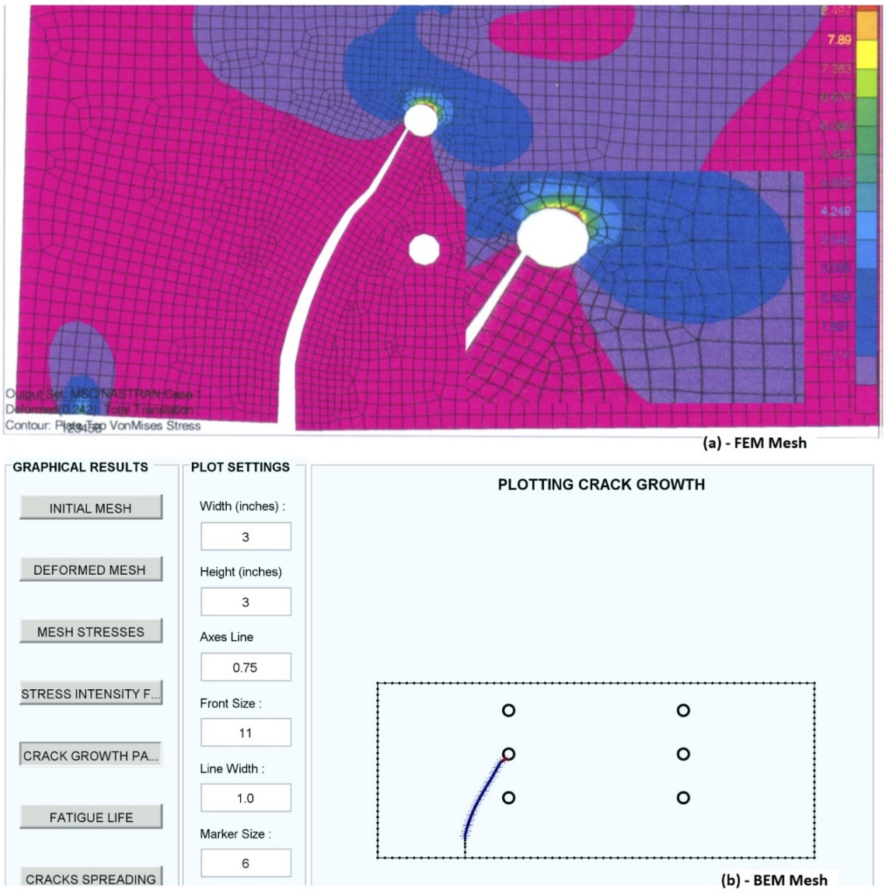

Figure 8 – (a) Finite element analysis [18]; (b) Boundary Elements mesh in *BEMLAB2D* GUI.

Crack Propagation Path: The angles of each increment calculated for propagation by means of the various criteria are in the Table 2. It is perceived that there is a slight variation in the initial values that attenuates over the remainder of the propagation. The Figure 9 shows the propagation curve for the three criteria of *BemCracker2D* program. Next figure compares the curve of the three criteria with Boljanovic and Maksimovic's propagation – Figure 10.

Table 2 – Comparison between angles of crack propagation. Unit in degree (º).

Step	BC2D MPS	BC2D MERR	BC2D SED	Boljanovic and Maksimovic
1	0.000	0.000	0.000	0.000
2	-12.890	-14.158	-13.011	-12.900
3	-2.639	-2.593	-2.652	-5.100
4	-2.136	-2.120	-2.134	-4.800
5	-1.934	-1.937	-1.936	-3.900
6	-1.862	-1.977	-1.867	-3.480
7	-0.509	-0.516	-0.510	-11.820
8	1.052	1.037	1.054	12.420
9	0.472	0.441	0.469	1.240
10	FALHA	FALHA	FALHA	FALHA

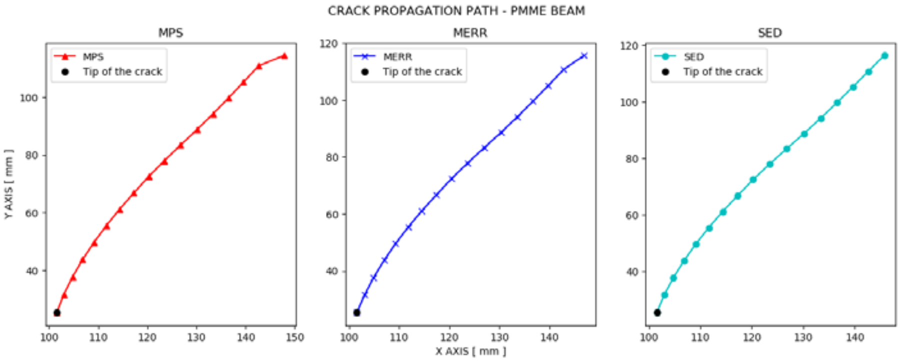

Figure 9 – Graphics of crack propagation of the three criterions calculated through *BemCracker2D*.

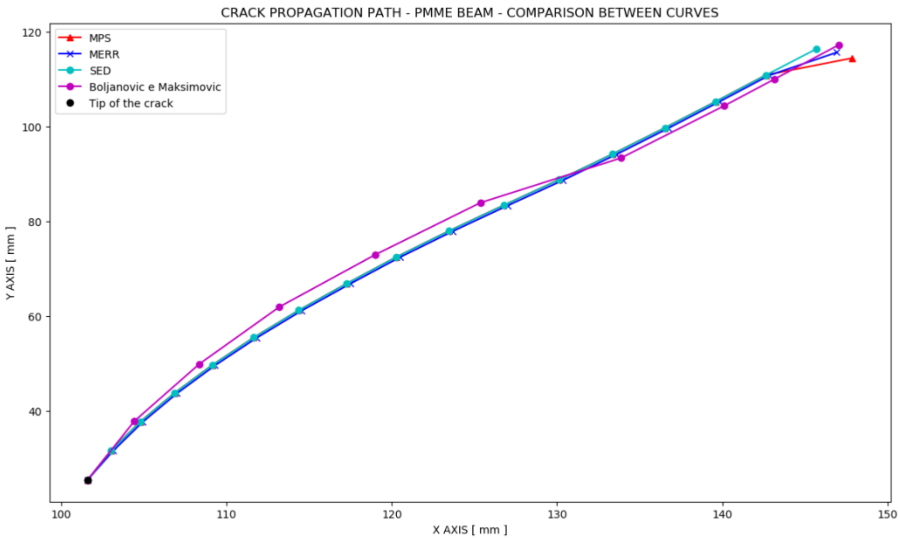

Figure 10 – Comparison of curve of the three BC2D's criterions with Boljanovic and Maksimovic's propagation.

The Table 3 shows in Cartesian coordinates which was the propagation path. It is perceived that the values are close.

Table 3 – Coordinates of crack growth path with BC2D and Boljanovic and Maksinoniv's paper. Unit in mm.

	BC2D MPS		BC2D MERR		BC2D SED		Boljanovic and Maksimovic	
Step	X	Y	X	Y	X	Y	X	Y
1	101.6000	25.4000	101.6000	25.4000	101.6000	25.4000	101.6	25.4
2	104.7880	37.6879	104.9440	37.6494	104.7997	37.6853	104.42	37.77
3	109.1338	49.6184	109.3158	49.5704	109.1451	49.6159	108.36	49.86
4	114.3646	61.1890	114.5659	61.1323	114.3752	61.1869	113.21	61.92
5	120.2725	72.4307	120.4877	72.3667	120.2826	72.4288	119	72.92
6	126.7607	83.3472	127.0027	83.2674	126.7709	83.3453	125.37	83.9
7	133.3702	94.1911	133.6089	94.1129	133.3805	94.1891	133.86	93.32
8	139.6008	105.2574	139.8153	105.1928	139.6109	105.2555	140.13	104.37
9	142.6332	110.8366	142.8404	110.7759	142.6434	110.8346	143.15	109.96
10	147.8689209	114.4295936	146.9359967	115.6285505	145.717629	116.3908452	147.06	117.22

***BEMLAB2D* GUI Propagation Path:** The graphic results for the crack propagation have been obtained by the *BEMLAB2D* GUI post-processor – Figure 11.

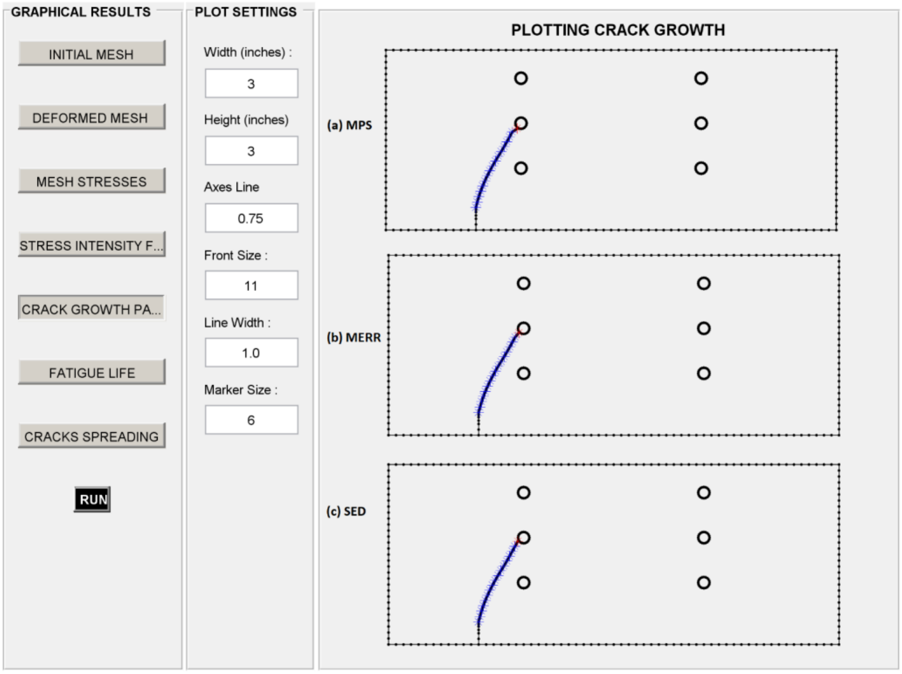

Figure 11 – Crack Growth Path criteria in *BEMLAB2D* GUI – (a) MPS; (b) MERR; (c) SED.

The Figure 12 shows the four curves in the PMMA specimen.

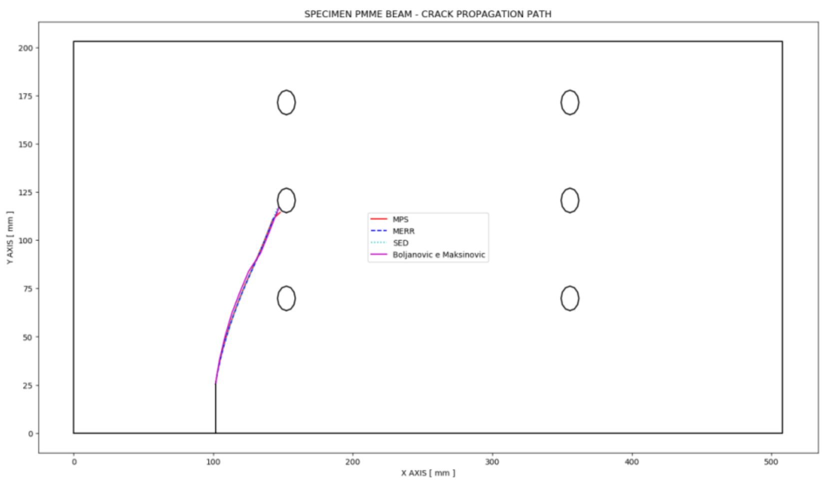

Figure 12 – The criteria curves in the PMMA specimen.

6 Final Remark

In this work, the programs *BemCracker2D* and *BEMLAB2D* GUI were used, respectively, to calculate and model the two-dimensional propagation of cracks, as well as to find values of SIFs, calculated by means of J-Integral. The analysis was based on the boundary elements method in which it was compared with other numerical results [18] of previous studies performed and validated experimentally.

The modeling procedure through the *BEMLAB2D* GUI – using GUI interface – makes the procedure more efficient. The geometry is released in a simple way and the mesh of BEM is done automatically.

The results found through the *BemCracker2D*, both the SIFs and the crack propagation, have inexpressible differences, validating their results. In this way, the program becomes a powerful tool in the two-dimensional elastostatic calculation.

The MPS also dispenses material parameters. With the stress values it's possible to calculate the SIFs and consequently the angle of propagation. However, in order to use MERR, it is necessary to have the modulus of elasticity of the material and in the SED, in addition, the Poisson's coefficient and the plane state to which it is subjected, whether of stress or strain.

Finally, the MPS becomes more practical and easily implemented. However, in terms of numerical values and propagation path, the three are practically equal.

Acknowledgements

The authors are grateful to the Brazilian Coordination for the Improvement of Higher Education (CAPES) for the supporting funds for this research. The authors also thank the Graduate Programme in Structural Engineering and Civil Construction in the Department of Civil and Environmental Engineering at the University of Brasilia.

References

[1] SANFORD, R. J. "Fundamentals of Frature Mechanics". [S.l.]: Prentice Hall, 2002.
[2] GOMES, G.; DELGADO NETO, Á. M.; WROBEL, L. C. "Modelagem e visualização de trincas 2D usando equação integral de contorno dual". XXXVII IBERIAN LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING, Brasília, v. 37, 2016.
[3] HE, W.-Y.; REN, W.-X.; YANG, Z.-J. "Computation of Plane Crack Stress Intensity Factors Using Trigonometric Wavelet Finite Element Method". Fatigue & Fracture of Engineering Materials & Structures, v. 35, n. 8, p. 732–741, 2011.
[4] MOËS, N.; DOLBOW, J.; BELYTSCHKO, T. "A finite element method for crack growth without remeshing". International Journal for Numerical Methods in Engineering, v. 46, p. 131-150, 1999.
[5] DUARTE, C. A. et al. "A generalized finite element method for the simulation of three dimensional dynamic crack propagation". Computer Methods in Applied Mechanics and Engineering, v. 190, p. 2227-2262, 2001.
[6] PORTELA, A.; ALIABADI, M. H.; ROOKE, D. P. "The Dual Bounday Element Method: Effective Implementation for Crack Problems". International Journal for Numerical Method in Engineering, v. 33, p. 1269-1287, 1992.
[7] OLIVEIRA, T. A. A.; GOMES, G.; EVANGELISTA J., F. "Multiscale aircraft fuselage fatigue analysis by the dual boundary element method". Engineering Analysis with Boundary Elements, v. 104, p. 107-119, 2019.
[8] GRIFFITH, A. A. "The Phenomena of Rupture and Flow in Solids". Philosophical Transactions of the Royal Society of London, v. 221, p. 163-198, 1921.
[9] ANDERSON, T. L. "Fracture Mechanics: Fundamentals and Application". 3rd. ed. [S.l.]: CPC Press, 2005.
[10] IRWIN, G. "Analysis of Stresses and Strains near the End of a Crack Traversing a Plate". Journal of Applied Mechanics, v. 24, p. 361-364, 1957.
[11] RICE, J. R. "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks". Journal of Applied Mechanics, v. 35, n. 2, p. 379, 1968.
[12] ERDOGAN, F.; SIH, G. C. "On the Crack Extension in Plates Under Plane Loading and Transverse Shear". Journal of Basic Engineering, v. 85, n. 4, p. 519, 1963.
[13] HUSSAIN, M. A.; PU, S. U.; UNDERWOOD, J. "Strain Energy Release Rate for a Crack under Combined Mode I and II". ASTM STP 560, p. 2-28, 1974.
[14] SIH, G. C. "Strain-energy-density factor applied to mixed mode crack problems". International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts, v. 11, n. 12, p. 305-321, 1974.
[15] BREBBIA, C. A.; DOMINGUEZ, J. "Boundary Elements: An Introductory Course". 2ª. ed. [S.l.]: WIT Press, 1992.
[16] GOMES, G. "Aplicação dos métodos de elementos de contorno e reciprocidade dual em problemas de plasticidade 2D orientada a objeto". Brasília - DF: Tese (Doutorado em Engenharia Civil) – Programa de Pós-graduação em Estrutura e Construção Civil, Universidade de Brasília, UnB, 2006. 168 p.
[17] SCUCIATO, R. F. "Estudo teórico e implementação computacional do método de elementos de contorno aplicado a problemas elastostáticos lineares bidimensionais". Paraná: Dissertação (Mestrado em Métodos Numéricos em Engenharia) –

Curso de Pós-graduação em Métodos Numéricos em Engenharia, Setor de Tecnologia, Universidade Federal do Paraná, UFPR, 2007. 191 p.
[18] BOLJANOVIC, S.; MAKSIMOVIC, S. "Analysis of the crack growth propagation process under mixed-mode". Engineering Fracture Mechanics, v. 78, p. 1565-1576, 2011.
[19] GOMES, G.; MIRANDA, A. C. O. "Analysis of crack growth problems using the object-oriented program bemcracker2D", v. 45, p. 67-85, 2018.
[20] DELGADO NETO, Á. M. "BEMLAB2D: Interface Gráfica de Modelagem, Visualização e Análise com Elementos de Contorno – Uma Aplicação em Problemas Elastostáticos". Brasília: Dissertação (Mestrado em Engenharia Civil) – Programa de Pós-graduação em Estrutura e Construção Civil, Universidade de Brasília, UnB, 2017. 112 p.
[21] INGRAFFEA, A. R.; GRIGORIU, M. "Probabilistic fracture mechanics: a validation of predictive capability". Department of Structural Engineering, Cornell University, 1990.