Finite difference modeling of cylindrical panels under Pasternak contact constraints

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Abstract. This study examines the radial displacements of cylindrical panels in contact with the soil (or rock), approximated here by the Pasternak elastic model. The contact problem is characterized as bilateral due to the permanent contact situation between the structure and the medium. A simply supported is defined and then subjected to a concentrated load, with the contact system differential equation derivatives approximated by using the finite difference method (FDM). The cylindrical panel analysis is based on Sanders' theory, where quadratic terms are omitted under the assumption that they are negligible for small displacement conditions. This simplification allows for a more streamlined approach while maintaining accuracy in scenarios where displacements are minimal. Therefore, a Fortran-based computational tool is developed for the analysis of this particular bilateral contact problem. The results focus on the panel radial is on the panel radial displacements for different meshes, considering variations in nodal points in different directions. The FDM was efficient and suitable for this contact problem between the structure and medium, providing a straightforward numerical implementation of the cylindrical shell and elastic base model theories.

Keywords: bilateral contact problem, finite difference method, Pasternak foundation, cylindrical panel, modified sanders theory

1 Introduction

The study of contact problems between structures and deformable foundations is crucial in civil engineering, especially in soil-structure interaction scenarios. Structural elements frequently interact with the ground, as they are typically constructed on soil or rock formations. Understanding these interactions is vital for ensuring the stability and safety of structures, as the foundation behavior directly influences the overall performance and integrity of the structure. This analysis helps in predicting and mitigating potential issues that could arise from the ground-structure contact. In this context, analyzing structure-foundation systems focuses on the response of soil or rock to applied loads at the contact region. This involves incorporating the foundation's reaction into the differential equations governing problems related to beams, plates, or shells. By integrating these reactions, engineers can more accurately predict the behavior of structural elements under load, ensuring that both the structure and its foundation perform optimally under various conditions [1].

The simplest mathematical model to simulate the behavior of soil or rock typically uses only one parameter, usually representing its stiffness. This behavior is modeled by considering the medium as an elastic, deformable base that does not respond to tensile forces. The Winkler discrete spring model, distributed in the interaction region between the structure and the base, is widely used in engineering projects and scientific research. In this model, the base deflection is assumed to occur exclusively at the point of contact with the structure. On the other hand, when considering soil as a continuous medium, two-parameter models provide a more accurate representation of the elastic base's behavior. These models, like Pasternak's, account for the interaction between the springs, offering a more comprehensive and accurate analysis analysis. The primary effect of using these models is to increase the overall rigidity of the system, resulting in a more realistic portrayal of the soil-structure interaction [2].

The study of contact between shells/panels and an elastic base focuses on understanding the interaction between these elements. When the parts remain permanently fixed without losing contact, the problem is characterized as bilateral contact problem (BCP) (Figure 1). This scenario ensures continuous contact, affecting the analysis and behavior of the system under various loading conditions.

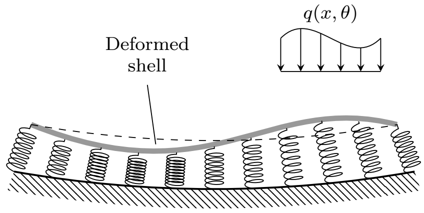

Figure 1. Bilateral contact problem

Cylindrical shells and panels are vital components in various engineering structures, including pipelines, tanks, cooling towers, silos, containment structures, chimneys, and roofs. In tunnel structures, wells, or buried pipelines, these shells interact with the surrounding soil or rock. Depending on the situation, the shells may either support the surrounding medium, be supported by it, or work in tandem with the soil, providing mutual support and bearing capacity. This interaction plays a critical role in the stability and functionality of the overall structural system.

The soil-shell contact problem solution is typically achieved through numerical methods, with the finite element method (FEM) being the most common. Other methods like the boundary element method (BEM), discrete element method (DEM), and finite difference method (FDM) are also used. This study aims to enhance a Fortran-based computational tool for the linear analysis of cylindrical shells, originally developed by Silveira [3]. The focus is on bilateral contact constraints using FDM due to its effectiveness in solving differential equations with contact restrictions.

In this article, the general mathematical framework is outlined for solving the bilateral contact problems between cylindrical panels and elastic bases using the finite difference method. Cylindrical panels are used as a practical application of the general shell equations.

2 Theoretical formulation

2.1 Theory of slender cylindrical shells

The analytical methods used to characterize the behavior of a thin shell under applied loads rely on assumptions concerning the shell's characteristic dimensions, the extent of deflection, and the torsion perpendicular to the reference surface at each point. To facilitate this, simplified theories make certain assumptions that streamline the analysis of shell-related problems in the study of deformations on the reference surface. The formulation used is [4]:

i. The normal stresses acting in the shell radial direction are negligible compared to the other stresses;

ii. Deformations due to transverse shear are neglected; the fibers normal to the undeformed reference surface remain normal and do not change in length during the deformation process;

iii. The shell is considered slender;

iv. The shell material is homogeneous and exhibits a linear elastic behavior; and

v. The forces acting on the system are conservative.

Sanders (1963) [5] developed expressions for small deformations and moderate to large rotations to enhance the accuracy of measurements for membrane and bending deformations. These equations fully account for geometric nonlinearity in the deformation-displacement relationships and are applicable to all shell types. When quadratic terms are negligible under small displacement conditions, the kinematic relations can be derived as follows:

$$\varepsilon_x = \partial_x u, \quad \varepsilon_\theta = \frac{1}{R}\partial_\theta v + \frac{1}{R}w, \quad \gamma_{x\theta} = \frac{1}{R}(\partial_\theta u + R\partial_x v) - \frac{1}{R}(\partial_x)v, \tag{1}$$

where ε_x and ε_θ are the normal strains and $\gamma_{x\theta}$ the shear strain. Displacements in the x and θ directions are denoted by u an v, respectively, and R is the radius of curvature of the slender shell. Displacement perpendicular to the surface and oriented in a radial direction is denoted by w. The linear approximations of curvature changes, concerning the adopted theory, are:

$$\chi_x = -\partial_{xx} w, \quad \chi_\theta = \frac{1}{R^2}(\partial_\theta v - \partial_{\theta\theta} w), \quad \chi_{x\theta} = -\frac{1}{4R^2}\partial_\theta u + \frac{3}{4R}\partial_x v - \frac{1}{R}\partial_{x\theta} w, \tag{2}$$

where χ_x and χ_θ are the changes of the main lines and $\chi_{x\theta}$ is the twist of the reference surface.

The analysis of static equilibrium conditions involves evaluating the effects of a set of forces on the shell element under investigation and ensuring equilibrium throughout the structure. The equilibrium differential equations for this infinitesimal element are derived by summing the forces and moments in the undeformed cylindrical shell element. The response of the cylindrical shell to specific loads, with bilateral contact constraints imposed by elastic bases, is considered only in the radial direction w. Given the assumptions of small displacements, elastic material properties, and specific load types, it is assumed that the equations governing the membrane effects of the shell have minimal impact on the system's response and are therefore neglected. For the reference surface of the cylindrical shells and panels, the equilibrium equation can be written as follows:

$$\frac{C}{R^2} w + D\,\nabla^4 w = q(x,\theta) + r_b(x,\theta), \tag{3}$$

where ∇^4 is the bilaplacian operator in cylindrical coordinates, q represents the acting lateral pressure and r_b characterizes the reaction of the elastic base. C and D are the membrane and bending stiffnesses, respectively, defined as:

$$C = \frac{Et}{1-\nu^2}, \quad D = \frac{Et^3}{12(1-\nu^2)}. \tag{4}$$

where E is the modulus of elasticity and ν is the Poisson's ratio.

2.2 Elastic foundations

The mathematical model used to represent the foundation varies depending on the specific application, with determining the contact pressure between components being a key challenge in analyzing the interaction between the ground and the structure. Common methods for investigating the structural system include numerical modeling approaches, particularly those that treat the foundation as an elastic medium. Among these, models that use discrete springs are prevalent. In such models, the behavior of the elastic foundation is represented using (i) a single parameter, where the soil-structure interaction is modeled with independent discrete springs whose stiffness is linked to the properties of the elastic foundation material, and (ii) two parameters, which incorporate interactions between the springs.

The simplest mechanical model was proposed by Winkler in 1867 [6]. In this representation, the pressure exerted by the base at a given point is directly proportional to deflection on site, i.e.:

$$r_b(x,\theta) = k_B w_b(x,\theta) \tag{5}$$

where k_B is the foundation modulus and $w_b(x,\theta)$ is the deflection. As soil may exhibit considerable interaction action among its elements, the Winkler springs have inherited deficiency in simulating such behavior. Pasternak model assumes the existence of shear interactions between the spring elements [7]. Mathematically:

$$r_b(x,\theta) = k_B w_b(x,\theta) - k_M \nabla^2 w_b(x,\theta) \tag{6}$$

where the second term on the right-hand side is the effect of the shear interactions of the vertical elements, ∇^2 is the laplacian operator in cylindrical coordinates and k_M represents the shear stiffness parameter of the layer. The two stiffness parameters in the Pasternak model, k_B and k_M, depend on the physical and mechanical properties of the elastic foundation and the system's geometry, being influenced by the constitution of the soil (or elastic material).

2.3 Approximation of the equilibrium differential equation

The soil-shell equilibrium equation (3) is solved here by the finite difference method. This method promotes the discretization of space by a mesh of discrete points, with the unknows variable w and its their derivatives being replaced by approximations at the points of the grid through difference quotients. Characteristically, for each nodal point, an algebric equation is obtained.

The process of discretization of space takes into account the organization of the stitched points, thus defining the mesh of finite differences. Depending on the arrangement imposed, the mesh can be composed of structures of different shapes, the most common squares, rectangles and triangles, always depending on each specific problem and methodology used. The cylindrical panel plan is shown in Figure 2 with a rectangular mesh consisting of sides of length Δx and $R\Delta\theta$.

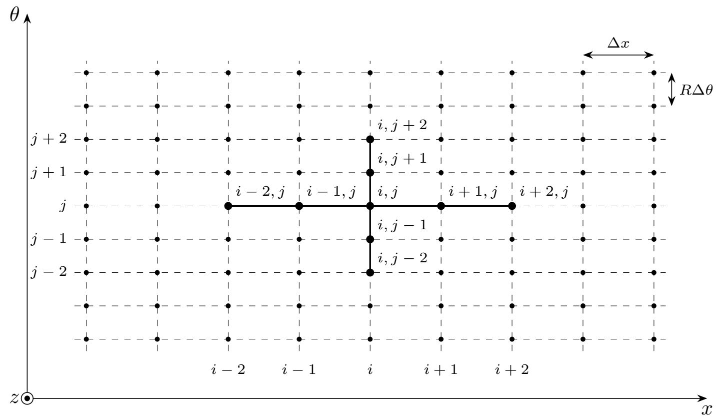

Figure 2. Planning of cylindrical shell for use in the FDM

The algebraic equation for the pivotal point (i, j), after applying the finite difference approximations in Equation (3), is expressed as follows (t is the thickness of the cylindrical panel):

$$
\begin{aligned}
&\left[\frac{t^2}{12(\Delta x)^4}\right](w_{i-2,j} + w_{i+2,j}) + \left[\frac{t^2}{12R^4(\Delta\theta)^4}\right](w_{i,j-2} + w_{i,j+2}) + \\
&\left[\frac{t^2}{6R^2(\Delta x)^2(\Delta\theta)^2}\right](w_{i-1,j-1} + w_{i+1,j-1} + w_{i-1,j+1} + w_{i+1,j+1}) + \\
&\left[\frac{t^2}{3R^4(\Delta\theta)^4} + \frac{t^2}{3R^2(\Delta x)^2(\Delta\theta)^2}\right][-w_{i,j-1} - w_{i,j+1}) + \\
&\left[\frac{t^2}{3(\Delta x)^4} + \frac{t^2}{3R^2(\Delta x)^2(\Delta\theta)^2}\right][-w_{i-1,j} - w_{i+1,j}] + \\
&\left[\frac{t^2}{2(\Delta x)^4} + \frac{2t^2}{3R^2(\Delta x)^2(\Delta\theta)^2} + \frac{t^2}{2R^4(\Delta\theta)^4} + \frac{1}{R^2}\right][w_{i,j}] = \frac{q(x,\theta)}{C} + \\
&k_B\,(w_{i,j}) - k_M\left[\frac{1}{(\Delta x)^2}(w_{i+1,j} + w_{i-j,j} - 2w_{i,j}) + \frac{1}{R^2(\Delta\theta)^2}(w_{i,j+1} + w_{i,j-1} - 2w_{i,j})\right].
\end{aligned}
\tag{7}
$$

3 Numerical Aplications

3.1 Solution strategy

The numerical strategy used in this work for the approximate solution of bilateral contact problems have as main characteristics:

i. The use of the FDM, which replaces the original domain of the bodies (structure and elastic base) and their respective contours with a mesh. As a consequence, the algebraic equation system that governs the PCB is reached;
ii. After this system discretization, the solution of the problem can be directly achieved;
iii. The computer program used for numerical simulation was made in the Fortran 90 language.

The algorithm adopted in this work for solving the BCP follows the following steps:
1. Data reading: geometric, material and system load properties (shell-base);
2. The finite difference mesh of the system is defined;
3. The external force vector is assembled;
4. The shell stiffness matrix is assembled;
5. The elastic base stiffness matrix is assembled;
6. The structural system stiffness matrix of the structural system is calculated;
7. The boundary conditions are introduced;
8. The algebric equation system is solved;
9. The radial displacement w and the internal forces are calculated;
10. Print the results.

3.2 BCP: panel with concentrated load

Figure 3 shows the cylindrical panel with length $L = 12$ m and radius $R = 24$ m, with all four sides simply supported under a load of $q = 10^6 N$ applied at its central point. The radial displacements w are determined at the point where the load is applied. The panel analyzed is made of isotropic material, with modulus of elasticity 2.05×10^{11} Pa and Poisson's ratio 0.3. The following thicknesses were adopted and analysed: 0.096 m, 0.192 m, 0.288 m, 0.384 m and 0.480 m. Therefore, the thickness (t) / radius (R) ratio were evaluated and are: 0.4, 0.8, 1.2, 1.6 and 2.0.

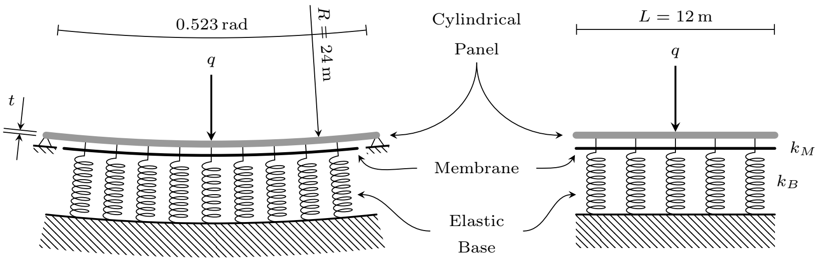

Figure 3. Cylindrical panel based on elastic base - bilateral contact

For the Pasternak-type base, the first parameter k_B assumed the values 10^6 N/m^3, 10^7 N/m^3 and 10^8 N/m^3. The second parameter k_M assumed the values 10^6 N/m, 10^7 N/m and 10^8 N/m. The k_B/k_M ratios analyzed are shown in Table 1. Additionally, the same panels on an elastic foundation with one parameter ($k_M = 0$), Winkler model, and without no base ($k_B = 0$ and $k_M = 0$) were considered. For the application of the FDM and for each direction (x and θ), the following number of nodal points was adopted: 11, 21, 31, 41, 51, totaling 25 meshes.

Table 1. k_B/k_M ratios analyzed

k_B(N/m^3)	10^6	10^6	10^6	10^7	10^7	10^7	10^8	10^8	10^8
k_M(N/m)	10^6	10^7	10^8	10^6	10^7	10^8	10^6	10^7	10^8
k_B/k_M(m^{-2})	1	0.1	0.01	10	1	0.1	100	10	1

Table 2 compares, for the different thicknesses and values of k_B and k_M, and for the different types of contact, the average values w at the panels center point radial displacement, with the standard deviations σ associated. All values are given in meters. The effective gain in rigidity provided by the use of Pasternak-type bases can be noted as the panels become less slender, mainly for the higher values of k_B and k_M.

Considering the meshes with the same variation of nodal points in the two directions already considered (11,21,31,41,51), the displacements at the point of application of the load were determined. Figures 4a and 4b illustrate the behavior of radial displacements, as presented in Table 1, for $k_B = 10^6$ N/m^3 and $k_B = 10^8$ N/m^3, respectively. Similarly, Figures 5a and 5b show the radial displacement behavior for $k_M = 10^6$ N/m and $k_M = 10^8$ N/m. The values $k_B = 10^7$ N/m^3 and $k_M = 10^7$ N/m were omitted, as their behavior was similar to that observed for $k_B = 10^6$ N/m^3 and $k_M = 10^6$ N/m, for the corresponding standard deviations. In both cases, there is a similar behavior for the gain in stiffness. The effective gain in rigidity provided by the use of Pasternak-type base can be noted as the panels become less slender. Convergence is observed for different contact situations as the slenderness of the cylindrical panel decreases.

Table 2. Mean values and associated standard deviations for different thicknesses

		$k_M(\mathrm{N/m})$							
		0		10^6		10^7		10^8	
$t/R(\%)$	$k_B(\mathrm{N/m^3})$	$\overline{w}$	σ	$\overline{w}$	σ	$\overline{w}$	σ	$\overline{w}$	σ
0.4	0	0.0567	0.0019						
	10^6	0.0560	0.0019	0.0550	0.0018	0.0480	0.0026	0.0247	0.0026
	10^7	0.0505	0.0016	0.0497	0.0016	0.0438	0.0023	0.0234	0.0025
	10^8	0.0296	0.0011	0.0293	0.0012	0.0271	0.0016	0.0171	0.0019
0.8	0	0.0069	0.0003						
	10^6	0.0069	0.0003	0.0069	0.0003	0.0066	0.0002	0.0050	0.0004
	10^7	0.0065	0.0002	0.0065	0.0002	0.0063	0.0002	0.0048	0.0003
	10^8	0.0046	0.0002	0.0046	0.0002	0.0045	0.0002	0.0037	0.0002
1.2	0	0.0020	0.0001						
	10^6	0.0020	0.0001	0.0020	0.0001	0.0020	0.0001	0.0017	0.0001
	10^7	0.0019	0.0001	0.0019	0.0001	0.0019	0.0001	0.0017	0.0001
	10^8	0.0015	0.0001	0.0015	0.0001	0.0015	0.0001	0.0013	0.0001
1.6	0	0.0009	0.0000						
	10^6	0.0009	0.0000	0.0009	0.0000	0.0008	0.0000	0.0008	0.0000
	10^7	0.0008	0.0000	0.0008	0.0000	0.0008	0.0000	0.0008	0.0000
	10^8	0.0007	0.0000	0.0007	0.0000	0.0007	0.0000	0.0006	0.0000
2.0	0	0.0004	0.0000						
	10^6	0.0004	0.0000	0.0004	0.0000	0.0004	0.0000	0.0004	0.0000
	10^7	0.0004	0.0000	0.0004	0.0000	0.0004	0.0000	0.0004	0.0000
	10^8	0.0004	0.0000	0.0004	0.0000	0.0004	0.0000	0.0003	0.0000

As the panel thickness increases, the stiffness gain offered by the two-parameter base model effectively reduces the disturbance caused by boundary conditions. As expected, for high values of the thickness/radius ratio, there is a large increase in stiffness, characterized by a decrease in radial displacement.

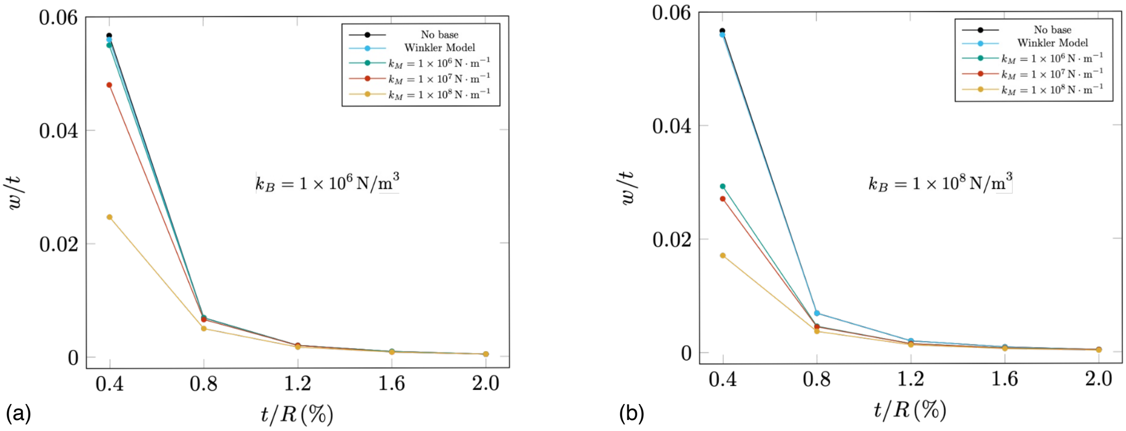

Figure 4. Radial displacements w evaluated at the load application point for different k_B values

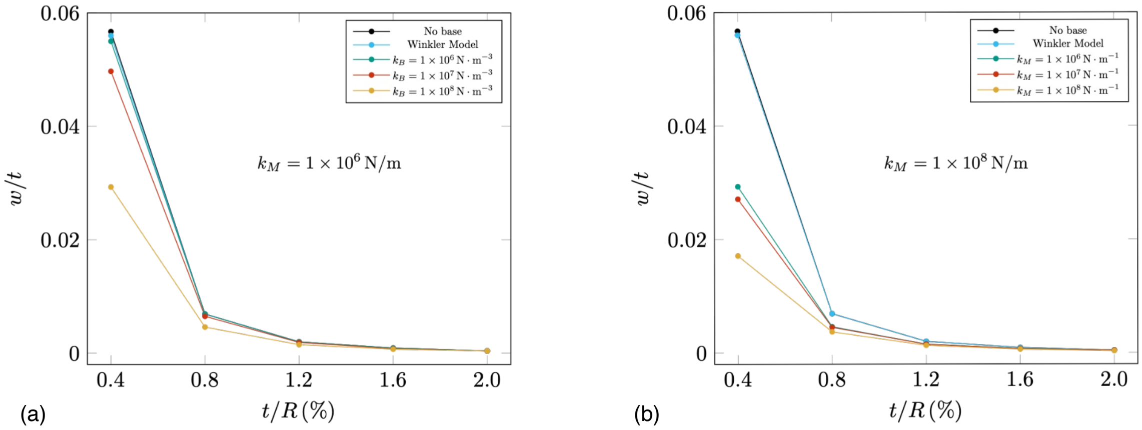

Figure 5. Radial displacements w evaluated at the load application point for different k_M values

4 Conclusions

A computational tool for the study and analysis of problems involving cylindrical panels with bilateral contact restrictions imposed by elastic bases was developed. This numerical study considered the FDM for discretization of the use of the FDM as a tool for discretion of the continuum, transforming the differential equations of both the shell and soil models into algebraic equations. The FDM presented efficiency in discretization of the medium-structure contact problem, with ease implementation of both cylindrical shell theory and also elastic foundation models. Therefore, it can be considered a good alternative to other numerical methods to solve this king of interation problem, mainly for this particular soil-structure interative problem. The evaluation of the Winkler and Pasternak models highlighted the differences in base behavior when subject to varying stiffness parameters and thickness. This analysis provided insight into how changes in stiffness influence the interaction between the structure and its foundation, with each model offering distinct responses to these variations.

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References

[1] F. C. S. Machado. *Análise Computacional via MDF de Cascas Cilíndricas com Restrições Bilaterais e Unilaterais de Contato*. Tese (doutorado), Programa de Pós Graduação em Engenharia Civil, Universidade Federal de Ouro Preto, 2013.
[2] W. T. Straughan. *Analysis of plates on elastic foundations*. PhD thesis, Texas Tech University, 1990.
[3] R. A. M. Silveira. *Análise de Elementos Estruturais Esbeltos com Restrições Unilaterais de Contato*. Tese (doutorado), Departamento de Engenharia Civil, PUC-RJ, 1995.
[4] P. Gould. *Analysis of Shells and Plates*. Springer Science & Business Media, 1998.
[5] J. L. Sanders Jr. Nonlinear theories for thin shells. *Quarterly of Applied Mathematics*, vol. 21, n. 1, pp. 21–36, 1963.
[6] E. Winkler. *Die Lehre von der Elasticitaet und Festigkeit: mit besonderer Rü cksicht auf ihre Anwendung in der Technik, für polytechnische Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architecten, etc*. H. Dominicus, 1867.
[7] P. Pasternak. On a new method of analysis of an elastic foundation by means of two foundation constants. *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu i Arkhitekture*, 1954.