An efficient Matlab code for two-dimensional heat transfer analysis applying the finite-volume theory

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Abstract. In engineering, professionals must understand the properties and performance of specific materials before exposing them to the conditions of a particular application. In this context, engineers and scientists have been seeking materials with unique properties that can withstand extreme adversities while responding with high performance to the imposed challenges. Heat transfer studies are essential and have been extensively explored due to their relevance in everyday life, such as thermal insulation systems and electronic device applications. The finite-volume theory was first introduced in 2003 using Cartesian coordinates. It establishes continuity and boundary conditions through the faces of the discretized analysis domain in a surface-averaging sense. The theory also satisfies flux balance equations in the subvolumes in a volume-averaged sense. Temperature fields are approximated by second-degree polynomials expressed in the local coordinates of the subvolumes. This work demonstrates the application of finite-volume theory to a two-dimensional thermal problem, using the FVT2DTHERMAL software to model heat transfer efficiently. The code closely matches analytical results, shows computational efficiency, and allows for adjustments in boundary conditions.

Keywords: heat transfer analysis, computational simulations, finite-volume theory.

1 Introduction

With technological advancements, heat transfer problems have gained prominence in engineering, as industrial processes often involve equipment that exchanges heat with the environment. Additionally, manufacturing industries, such as automotive and aerospace, require this knowledge to determine the best materials for developing components. Incropera *et al.* [1] state that one of the primary objectives of heat conduction analysis is to determine the temperature field within a medium due to the conditions imposed on its boundaries. In other words, the goal is to understand the temperature distribution, which represents how temperature varies with position within the medium.

Aiming to understand how temperature behaves throughout a material, one can resort to laboratory experiments, analytical solutions, or numerical models. Although they provide precise data, laboratory experiments have disadvantages, such as cost and the inability to replicate the conditions the material will face in practice. Analytical solutions, on the other hand, require time for resolution, and, in many cases, the problems are governed by complex differential equations and boundary conditions, which makes the analytical solution impractical. In this context, engineers turn to numerical modeling, which closely approximates the expected results with considerable accuracy. Veronese *et al.* [2] state that numerical approximation techniques have gained ground over

experimentation and analytical methods due to the increasing need for quick accurate solutions. Experimentation is almost always time-consuming and costly, and the expenses for acquiring and calibrating equipment are enormous for each new situation, while classical analytical methods have limitations.

The finite-volume theory originates from higher-order theory for functionally graded materials. Bansal and Pindera [3] presented the reformulation of the Higher-Order Theory for functionally graded materials, simplifying domain discretization and implementing a new approach for local and global matrices assemblings. In 2007, Cavalcante *et al.* [4, 5] verified the efficiency of this new technique by analyzing its results in thermal and mechanical problems. However, despite the advantages presented by numerical methods, it is necessary to use computational resources that drastically reduce processing time, especially in models with many subvolumes that require a significant amount of time to present results, as illustrated by Araújo *et al.* [6].

This work employs the MATLAB® platform to present a tool based on the finite-volume theory for analyzing the temperature field in a two-dimensional domain. The process spans domain discretization to post-processing. The technique includes a modified symmetric thermal conductivity matrix and MATLAB® advanced resources, significantly reducing computational cost compared to previous conventional algorithms. Details about the implementation can be found here.

2 The Finite-Volume Theory (FVT)

This technique employs the volume average of the different fields that define the material/solid behavior and imposes boundary and continuity conditions between adjacent subvolumes in a surface-averaged sense. Moreover, the heat conduction equation is satisfied in an averaged sense in the subvolumes of the discretized analysis domain, and the subvolume's temperature field is modeled by second-order polynomials defined in local coordinates [7-9].

2.1 *Discretization and displacement field representation*

The Figure 1 shows a rectangular solid subdivided in $N_s = N_\beta N_\gamma$ rectangular subdomains called subvolumes. N_β and N_γ indicate the number of subdivisions corresponding to the intervals $0 \leq x_1 \leq L$ and $0 \leq x_2 \leq H$, respectively. Each subvolume can be denoted by a single index s ($1 \leq s \leq N_s$) or by a pair of indexes $\beta = 1, \dots, N_\beta$ and $\gamma = 1, \dots, N_\gamma$, where s can be evaluated from β and γ. The subvolume (β,γ) occupies the position β in the horizontal direction and the position γ in the vertical direction, or $s = \gamma + (\beta - 1)N_\gamma$ for the discretized domain of analysis. Here, we adopt the finite-volume theory for rectangular analysis domains discretized in rectangular subvolumes, as shown in fig. 1, and an incomplete quadratic temperature field representation in the subvolume s,

$$T^{(s)}\left(x_1^{(s)}, x_2^{(s)}\right) = T_{(00)}^{(s)} + x_1^{(s)} T_{(10)}^{(s)} + x_2^{(s)} T_{(01)}^{(s)} + \frac{1}{2}\left(3\left(x_1^{(s)}\right)^2 - \frac{l_s^2}{4}\right) T_{(20)}^{(s)} + \frac{1}{2}\left(3\left(x_2^{(s)}\right)^2 - \frac{h_s^2}{4}\right) T_{(02)}^{(s)} \tag{1}$$

where $T_{(mn)}^{(s)}$ are unknown coefficients.

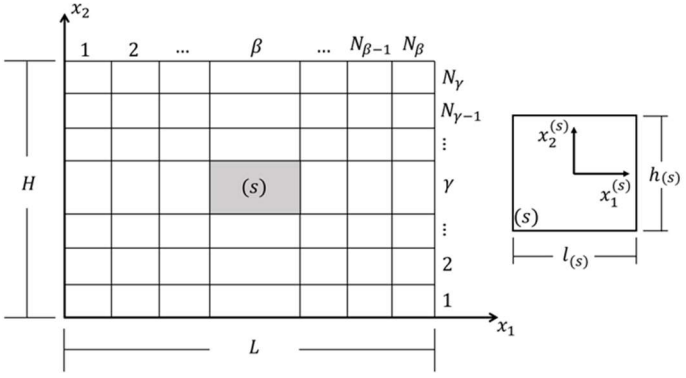

Figura 1: Discretized analysis domain in rectangular subvolumes and local coordinate system of a generic subvolume.

2.2 *Local conductivity matrix*

The surface-averaged temperatures at the subvolume faces can be evaluated by the following expressions:

$$\bar{T}^{(s,p)} = \frac{1}{l_s}\int_{-l_s/2}^{+l_s/2} T^{(s)}\left(x_1^{(s)}, \mp h_s/2\right) dx_1^{(s)}, \text{ for } p = 1,3 \tag{2}$$

$$\bar{T}^{(s,p)} = \frac{1}{h_s}\int_{-h_s/2}^{+h_s/2} T^{(s)}\left(\pm l_s/2\,, x_2^{(s)}\right) dx_2^{(s)}, \text{ for } p = 2,4 \tag{3}$$

where $\bar{T}^{(s,p)}$ are the surface-averaged temperatures of a generic subvolume s.

Employing the Fourier's law, the heat flux density vector of a generic isotropic subvolume s can be expressed as shown:

$$\boldsymbol{q}^{(s)}\left(x_1^{(s)}, x_2^{(s)}\right) = -\begin{bmatrix}\kappa_s & 0\\ 0 & \kappa_s\end{bmatrix}\begin{bmatrix}\partial T^{(s)}/\partial x_1^{(s)}\\ \partial T^{(s)}/\partial x_2^{(s)}\end{bmatrix} \tag{4}$$

where κ_s is the thermal conductivity and $\partial T^{(s)}/\partial x_i^{(s)}$ is the temperature gradient for a generic isotropic subvolume s.

The normal surface-averaged heat flux densities occurring on the faces of a generic subvolume s can be evaluated as follows:

$$\bar{q}^{(s,p)} = \mp\frac{1}{l_s}\int_{-l_s/2}^{+l_s/2} q_2\left(x_1^{(s)}, \mp h_s/2\right) dx_1^{(s)}, \text{ for } p = 1,3 \tag{5}$$

$$\bar{q}^{(s,p)} = \pm\frac{1}{h_s}\int_{-h_s/2}^{+h_s/2} q_1\left(\pm l_s/2\,, x_2^{(s)}\right) dx_2^{(s)}, \text{ for } p = 2,4 \tag{6}$$

The local system of equations of a generic subvolume can be expressed by:

$$\overline{\boldsymbol{q}}^{(s)} = \boldsymbol{\kappa}_{(4\times4)}^{(s)}\overline{\boldsymbol{T}}^{(s)} \tag{7}$$

where $\overline{\boldsymbol{q}}^{(s)} = \left[\bar{q}^{(s,1)}, \bar{q}^{(s,2)}, \bar{q}^{(s,3)}, \bar{q}^{(s,4)}\right]^T$ is the normal surface-averaged heat flux density vector, $\overline{\boldsymbol{T}}^{(s)} = \left[\bar{T}^{(s,1)}, \bar{T}^{(s,2)}, \bar{T}^{(s,3)}, \bar{T}^{(s,4)}\right]^T$ is the surface-averaged temperature vector and $\boldsymbol{\kappa}_{(4\times4)}^{(s)}$ is the local conductivity matrix of a generic subvolume s. The matrix $\boldsymbol{\kappa}_{(4\times4)}^{(s)}$ is non-symmetric, fact that imposes a computational cost on the solution of the global system of equations employing the MATLAB® solver when compared to an approach based on the finite element method. Moreover, the surface-averaged temperatures are not associated with the normal heat fluxes occurring on the faces of the subvolume. This suggests viewing $\boldsymbol{\kappa}_{(4\times4)}^{(s)}$ as a pseudo thermal conductivity matrix. A new local system of equations can be defined relating surface-averaged temperatures with the normal heat fluxes occurring on the faces of the subvolume s, as follows:

$$\boldsymbol{Q}^{(s)} = \overline{\boldsymbol{\kappa}}_{(4\times4)}^{(s)}\overline{\boldsymbol{T}}^{(s)} \tag{8}$$

$$\boldsymbol{Q}^{(s)} = \boldsymbol{L}^{(s)}\overline{\boldsymbol{q}}^{(s)} \tag{9}$$

$$\boldsymbol{L}^{(s)} = \begin{bmatrix} L_1^{(s)} & 0 & 0 & 0\\ 0 & L_2^{(s)} & 0 & 0\\ 0 & 0 & L_3^{(s)} & 0\\ 0 & 0 & 0 & L_4^{(s)}\end{bmatrix} \tag{10}$$

where $\overline{\boldsymbol{\kappa}}_{(4\times4)}^{(s)} = \boldsymbol{L}^{(s)}\boldsymbol{\kappa}_{(4\times4)}^{(s)}$ is the modified local thermal conductivity matrix, which is symmetric, resulting in a symmetric global thermal conductivity matrix, and $L_1^{(s)} = l_s$, $L_2^{(s)} = h_s$, $L_3^{(s)} = l_s$ and $L_4^{(s)} = h_s$ are the faces' lengths of the subvolume s.

2.3 *Global conductivity matrix assemblage*

The global thermal conductivity matrix is assembled considering the individual contribution of each subvolume of the discretized domain. If the solid is subdivided in $N_s = N_\beta N_\gamma$ subvolumes, the number of faces can be evaluate as $N_f = N_\beta(N_\gamma + 1) + (N_\beta + 1)N_\gamma$, which defines the size of the global system of equations. Based on the compatibility conditions for common faces, the expression that defines the global system of equations can be written as

$$\overline{\boldsymbol{q}}_{(N_f \times 1)} = \boldsymbol{\kappa}_{(N_f \times N_f)} \overline{\boldsymbol{T}}_{(N_f \times 1)} \tag{11}$$

where $\overline{\mathbf{T}}_{(N_f \times 1)}$ and $\overline{\mathbf{q}}_{(N_f \times 1)}$ are the global surface-averaged temperature vector and the global normal surface-averaged heat flux density vector, respectively, and the global conductivity matrix can be obtained by the following equation:

$$\boldsymbol{\kappa}_{(N_f \times N_f)} = \sum_{s=1}^{N_s} \left[\left(\boldsymbol{L}^{(s)}_{(4 \times N_f)} \right)^T \boldsymbol{\kappa}^{(s)}_{(4 \times 4)} \boldsymbol{L}^{(s)}_{(4 \times N_f)} \right] \tag{12}$$

where $\boldsymbol{L}^{(s)}_{(4 \times N_f)}$ and $\left(\boldsymbol{L}^{(s)}_{(4 \times N_f)} \right)^T$ are the temperature and normal heat flux compatibility matrices of the discretized analysis domain, respectively.

The modified global system of equations can be written as:

$$\boldsymbol{Q}_{(N_f \times 1)} = \overline{\boldsymbol{\kappa}}_{(N_f \times N_f)} \overline{\boldsymbol{T}}_{(N_f \times 1)} \tag{13}$$

where $\overline{\boldsymbol{\kappa}}_{(N_f \times N_f)} = \sum_{s=1}^{N_s} \left(\boldsymbol{L}^{(s)}_{(4 \times N_f)} \right)^T \overline{\boldsymbol{\kappa}}^{(s)}_{(4x4)} \boldsymbol{L}^{(s)}_{(4 \times N_f)}$ is the modified global conductivity matrix, obtained by summing the individual contribution of each subvolume of the discretized domain, where $\boldsymbol{Q}_{(N_f \times 1)}$ is the global normal heat flux vector.

3 Numerical Investigation

The previous section presented the two-dimensional formulation of the finite-volume theory for the thermal problem. This section will introduce the classic problem to be investigated through an implementation in MATLAB® (FVT2DTHERMAL). The numerical investigation will focus on examining the convergence of the temperature field by comparing it with results obtained from an analytical solution.

The convergence of the temperature field with mesh refinement was verified using a cross-sectional area in the x_1 and x_2 plane. The solid is homogeneous, with a thermal conductivity of 25 W/(m°C) (see Figure 2a), and is subjected to a uniform temperature of 100 °C on the bottom, top, and left edges. In contrast, the right edge is maintained at 200 °C. As illustrated in Figure 2b, the solid was discretized into a 50x50 mesh of subvolumes. The analytical solution for the temperature field was obtained by solving the Laplace equation using the standard Fourier series approach, as described by Zhong [10].

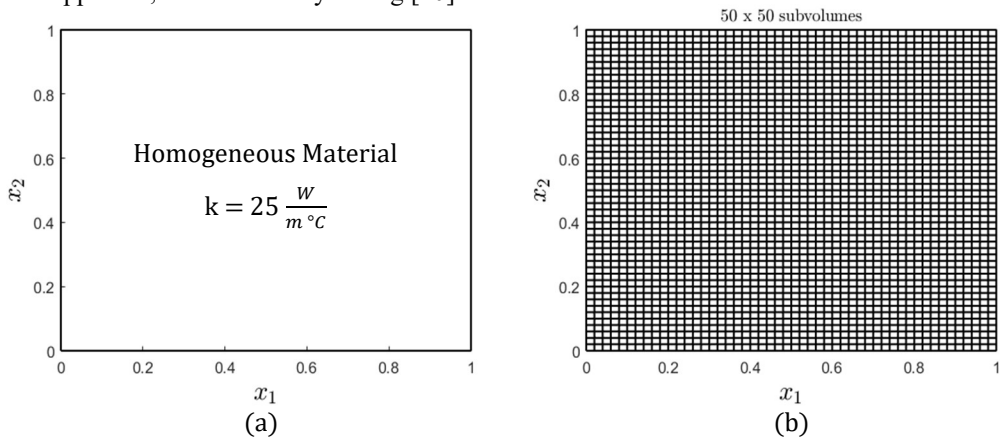

Figure 2: Problem definition: (a) investigated geometry and (b) discretization.

The global temperature field in the plate was investigated with the boundary conditions described in Figure 2a. It was observed that meshes with small discretization did not provide satisfactory results; however, the discretization illustrated in Figure 2b resulted in a convergent solution with the analytical solution and a smooth temperature field. Figure 3a shows the global temperature field of the plate using the finite-volume theory, while Figure 3b presents the temperature field obtained through the analytical solution, which allows for verifying the quality of the obtained numerical result. Figures 4a-d illustrate the comparison between the analytical solution and the numerical results of the temperature field along cuts in the x_1 and x_2 directions. The temperature field along these cuts made in the plate matches perfectly with the analytical solution. Based on these results, it can be concluded that the heat transfer problem can be satisfactorily modeled and predicted using the finite-volume theory.

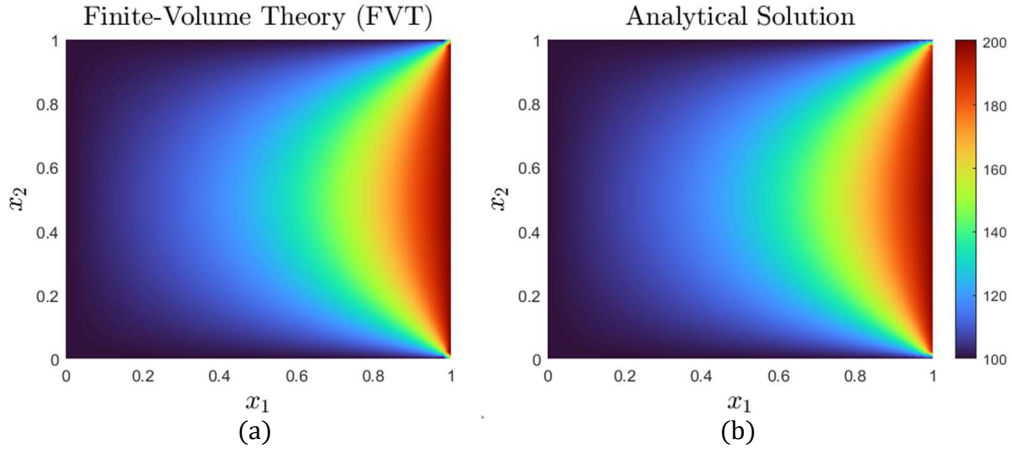

Figure 3: Temperature field: (a) FVT with 50x50 mesh of subvolumes and (b) analytical solution.

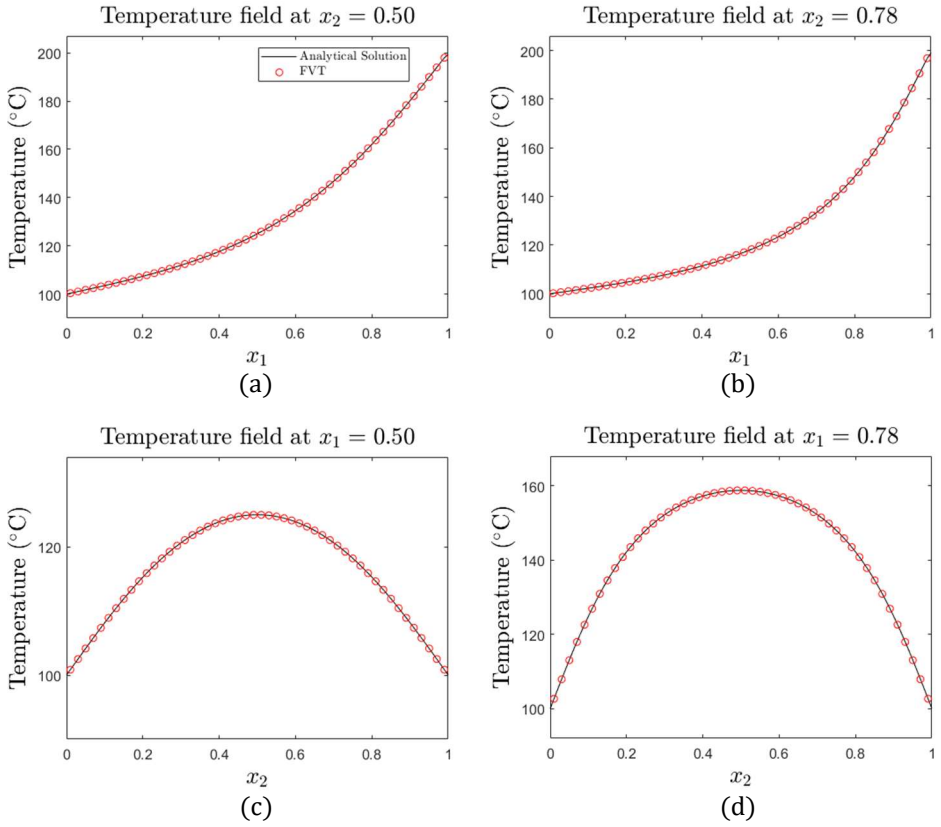

Figure 4: Comparison of the surface-averaged temperatures on the faces of the subvolumes with the results obtained from the analytical solution: (a) $x_2 = 0.50$, (b) $x_2 = 0.78$, (c) $x_1 = 0.50$ and (d) $x_1 = 0.78$.

4 Computational gain analysis using MATLAB® features

The finite-volume theory has proven to be numerically efficient in obtaining the temperature field in two-dimensional analyses. The algorithm developed in this work used the MATLAB® language and was initially composed only of basic programming commands. However, the goal is to provide an optimized code as an educational tool for the scientific community and students. In this regard, significant computational gains were observed in MATLAB® when advanced language features, such as vectorization and matrix sparsity, were incorporated into the algorithm, resulting in the code-named FVT2DTHERMAL.

Initially, it was observed that the global thermal conductivity matrix was predominantly composed of null elements. The massive presence of zeros resulted in redundant operations that compromised the code's computational efficiency. The matrix sparsity technique was adopted to address this issue. The sparse command in MATLAB® allows for the compact definition of matrices, storing only non-zero elements and their respective positions. This approach optimizes memory usage and improves computational performance, especially for large matrices with a high proportion of zero elements.

The FVT2DTHERMAL utilizes vectorization as a computational resource for code optimization. Instead of iterating over each node individually, the unique coordinates of the nodes are identified in a vectorized manner, allowing all points to be processed simultaneously and avoiding redundant calculations. The *unique* and *accumarray* functions efficiently accumulate the temperatures associated with each unique coordinate and calculate the average temperatures at the node. Vectorization reduces the need for explicit loops and simultaneously performs operations on entire vectors and matrices. This method simplifies the code and significantly improves computational performance, especially for large data sets, making the process faster and more efficient. Below is a code snippet that utilizes this resource:

```
[coords_u, ~, dx] = unique(Tno(:, 1:2), 'rows', 'stable');
temperatures = accumarray(dx, Tno(:, 3), [], @sum);
count = accumarray(dx, 1);
m_temperatures = temperatures ./ count;
Tno_M = [coords_u, m_temperatures];
```

The computational environment, in terms of programming language and machine, is defined as MATLAB® R2022b (64-bit) for Windows 11, with an Intel(R) Core(TM) 10th Gen i5-10210U 1.60 GHz processor and 16.0 GB of DDR4 RAM. For the discretization illustrated in Figure 2b, the educational code FVT2DTHERMAL was evaluated regarding the time spent in the preprocessing, processing, and post-processing phases. The execution time for each of these phases is detailed in Table 1.

Table 1. Computational performance in terms of execution times.

Phase	Execution time (s)
Preprocessing	0.14
Processing	0.72
Post-processing	40.40
Total time	41.26

5 Conclusions

This study developed and implemented an efficient MATLAB code for two-dimensional heat transfer analysis using finite-volume theory. The methodology involved discretizing the analysis domain, applying boundary conditions, and incorporating computational optimizations such as matrix sparsity and vectorization. The FVT2DTHERMAL code was tested against a classic heat transfer problem, and the results were compared with an analytical solution. The comparison revealed excellent agreement between the numerical and analytical results, confirming the proposed method's accuracy and the computational approach's robustness.

Implementing advanced MATLAB features, particularly matrix sparsity and vectorization, significantly reduced the computational cost. This is especially notable in the processing of large-scale systems of equations, where the optimizations minimize redundant calculations and improve overall efficiency. As a result, FVT2DTHERMAL

proves to be an effective educational tool for learning heat transfer analysis and a practical and reliable option for professional applications. Its capability to manage complex boundary conditions and mesh refinement with minimal computational cost underscores its potential as a versatile tool for engineers seeking efficient solutions to thermal problems.

Code availability. The authors are responsible for developing the code used in this work. The code can be accessed via the link: https://github.com/diogotiago04/FVT2D-THERMAL. The repository contains the instructions to replicate the two-dimensional heat transfer analysis described in this work.

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