# A co-rotational model for nonlinear analysis of slender section steel frames accounting for local buckling via lumped damage mechanics

Deborah C. Nardi[1], Sérgio G. F. Cordeiro[2], David L. N. F. Amorim[3,4]

[1] *School of Engineering of São Carlos, Structural Engineering Department, University of São Paulo*
*Av. Trabalhador São Carlense 400, 13566-590, São Carlos-SP, Brazil*
*deborahnardi@usp.br*
[2] *Civil Engineering Division, Aeronautics Institute of Technology*
*Praça Marechal Eduardo Gomes 50, 12228-900, São José dos Campos-SP, Brazil*
*cordeiro@ita.br*
[3] *Disaster Research Institute, Federal University of Sergipe*
*Av. Marcelo Deda Chagas s/n, 49107-230, São Cristóvão-SE, Brazil*
[4] *Graduate Program of Civil Engineering, Federal University of Alagoas*
*Av. Lourival Melo Mota s/n, 57072-970, São Maceió-AL, Brazil*
*davidnf@academico.ufs.br*

**Abstract.** To reduce engineering costs, the search for more efficient materials and design concepts leads to slender structures. Consequently, the need for geometrically nonlinear analysis which accounts for local inelastic instabilities becomes more evident, specially when dealing with structural components with cross-sections composed of slender plates and shells. The present work proposes an efficient geometrical nonlinear beam finite element model, developed for the analysis of slender section steel frames, accounting for local buckling. The finite element is locally formulated as the traditional linear Euler-Bernoulli beam element. A co-rotational description of motion is then employed to account for large displacements and rotations. The local buckling phenomenon is taken into account by a lumped damage model, which concentrates the effects at damage-plastic hinges. A predictor-corrector algorithm is employed at element level, whereas the Newton-Raphson method solves the global nonlinear equilibrium equations. To assess the accuracy of the proposed model, the numerical results obtained in this study are compared against available responses. The preliminary numerical results are reasonable, which corroborates that the proposed model might be used in further investigations.

**Keywords:** Co-rotational description of motion, Lumped damage model, Local buckling

## 1 Introduction

The nonlinearities that occur in the analysis of frame structures are mainly triggered by two sources: geometrical and material. With regard to the nonlinear geometric analysis, it can be efficiently handled by the co-rotational formulation [1–5]. The fundamental idea of such a formulation is to decompose the large motion of the element into rigid body and pure deformation parts, through the use of a local system which continuously rotates and translates with the element. The deformation is captured at the level of the local reference system, whereas the geometric nonlinearity induced by the large rigid-body motion is incorporated in the transformation matrices relating local and global displacements. The main interest is that the pure deformation part can be assumed as small and can be represented by a linear or a low order theory. Avoiding the nonlinear relationship between the strain tensor and the displacement gradient makes the co-rotational approach very attractive to deal with geometrical nonlinearity. With respect to the nonlinear material analysis of beams and frames, it can be distinguishable placed into two branches: the continuum inelasticity theories [4, 6] and the lumped inelasticity theories [7–9]. Herein, the latter one is adopted due to its efficiency in engineering practices. The so-called Lumped Damage Mechanics (LDM) is a theory based on concepts from classic plastic hinges and continuum damage mechanics. The local buckling phenomenon can be taken into account in LDM models by an internal damage variable, which is incorporated into plastic hinges. The works from [10–13] are examples where the LDM was successfully applied for the analysis of steel components considering local buckling. None of the works, however, considered a geometric nonlinear description of motion.

In the present work, a co-rotational finite element model is developed to analyze planar slender section steel frames, accounting for geometric nonlinearity and local buckling. The constrained nonlinear system of equation of the discrete constitutive problem at the element level is solved with the Newton-Raphson method, whereas a classical displacement-control procedure solves the global nonlinear equilibrium equations. An example with reference solution is presented to demonstrate the robustness of the proposed formulation.

## 2 Co-rotational finite element lumped damage model

A co-rotational model is adopted to describe the motion of the element from an initial configuration $C_0$ to the current configuration $C_n$. The rigid body motion is identified by an intermediate configuration $C_{0n}$ such that the motion between the intermediate configuration and the current configuration involves all the deformation of the element, under small strains assumption. Thus, the equilibrium equations of the element refereed to $C_{0n}$ can be obtained based on the Euler-Bernoulli beam linear theory [6]. The co-rotational model relates the total displacements of the element, i.e., from $C_0$ to $C_n$, with the displacements involved in the motion from $C_{0n}$ to $C_n$.

### 2.1 Equilibrium refereed to $C_{0n}$

The co-rotational displacement fields of the Euler-Bernoulli linear element with inelastic hinges can be rewritten in matrix notation as.

$$\left\{ \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right\} = \mathbf{\Phi}^e(\bar{x})\overline{\mathbf{u}}^e + \mathbf{\Phi}^p(\bar{x})\overline{\mathbf{u}}^p \quad \overline{\mathbf{u}}^e = \lfloor 0 \quad \bar{\theta}_1^e \quad \bar{u}_2^e \quad \bar{\theta}_2^e \rfloor^T \quad \overline{\mathbf{u}}^p = \lfloor \bar{u}_1^p \quad \bar{\theta}_1^p \quad \bar{u}_2^p \quad \bar{\theta}_2^p \rfloor^T \tag{1}$$

where $\overline{\mathbf{u}}^e$ is the elastic nodal displacement vector, $\overline{\mathbf{u}}^p$ is the plastic nodal displacement vector of the plastic hinges, $\mathbf{\Phi}^e(\bar{x})$ is the matrix of approximation functions related to the elastic nodal values and $\mathbf{\Phi}^p(\bar{x})$ the matrix of approximation functions related to the inelastic nodal values. The total co-rotational displacement vector $\overline{\mathbf{u}}$ can be written in terms of $\overline{\mathbf{u}}^e$ and $\overline{\mathbf{u}}^p$ as: $\overline{\mathbf{u}} = \overline{\mathbf{u}}^e + \overline{\mathbf{M}}\overline{\mathbf{u}}^p$, where $\overline{\mathbf{M}}$ is a matrix of zeros and ones. More details of the co-rotational model and the approximation functions $\mathbf{\Phi}^e(\bar{x})$ and $\mathbf{\Phi}^p(\bar{x})$ can be found in [14]. From the approximations, the generalized strain fields $\varepsilon_0 = d\bar{u}/d\bar{x}$ and $\kappa = d^2\bar{v}/d\bar{x}^2$ can be represented as

$$\left\{ \begin{array}{c} \varepsilon_0 \\ \kappa \end{array} \right\} = \left\{ \begin{array}{c} \varepsilon_0^e \\ \kappa^e \end{array} \right\} + \left\{ \begin{array}{c} \varepsilon_0^p \\ \kappa \end{array} \right\} = \overline{\mathbf{B}}^e(\bar{x})\overline{\mathbf{u}}^e + \overline{\mathbf{B}}^p(\bar{x})\overline{\mathbf{u}}^p \tag{2}$$

in which the matrices $\overline{\mathbf{B}}^e(\bar{x})$ and $\overline{\mathbf{B}}^p(\bar{x})$, derived from $\mathbf{\Phi}^e(\bar{x})$ and $\mathbf{\Phi}^p(\bar{x})$, can also be found in [14]. Using the Principle of Virtual Displacements, it is possible to establish the equilibrium equations of the element referred to $C_{0n}$

$$-\int_0^{L_0} \left\{ \begin{array}{c} \delta\varepsilon_0 \\ \delta\kappa \end{array} \right\}^T \left\{ \begin{array}{c} N \\ M \end{array} \right\} d\bar{x} + \delta\overline{\mathbf{u}}^T\overline{\mathbf{r}} + \int_0^{L_0} \left\{ \begin{array}{c} \delta\bar{u} \\ \delta\bar{v} \end{array} \right\}^T \left\{ \begin{array}{c} q_{\bar{x}} \\ q_{\bar{y}} \end{array} \right\} d\bar{x} = 0 \tag{3}$$

where $\overline{\mathbf{r}} = \lfloor F_{\bar{x}1} \quad M_{\bar{z}1} \quad F_{\bar{x}2} \quad M_{\bar{z}2} \rfloor^T$ are the co-rotational reactions forces, $N = EA\varepsilon_0^e, M = EI\kappa^e$ are the (elastic) axial force and bending moment, $\overline{\mathbf{r}}$ is the nodal reactions, $q_{\bar{x}}$ and $q_{\bar{v}}$ are the distributed forces on element. The substitution of the approximations (1) for the virtual displacements into the last integral in the L.H.S. of (3) allows to reduce this term to $\delta\overline{\mathbf{u}}^T\overline{\mathbf{p}}$, in which $\overline{\mathbf{p}} = \lfloor p_{\bar{u}_1} \quad p_{\bar{\theta}_1} \quad p_{\bar{u}_2} \quad p_{\bar{\theta}_2} \rfloor^T$ are consistent equivalent nodal forces. Replacing the approximations (2) for the virtual strains into the first integral in the L.H.S of (3), one obtains

$$-\int_0^{L_0} \left\{ \begin{array}{c} \delta\varepsilon_0 \\ \delta\kappa \end{array} \right\}^T \left\{ \begin{array}{c} N \\ M \end{array} \right\} d\bar{x} = -\delta\overline{\mathbf{u}}^{eT}\overline{\mathbf{f}} - \delta\overline{\mathbf{u}}^{pT}\overline{\mathbf{f}}^p \tag{4}$$

where

$$\overline{\mathbf{f}} = \left\{ \begin{array}{c} N_1 \\ M_1 \\ N_2 \\ M_2 \end{array} \right\} = \int_0^{L_0} \overline{\mathbf{B}}^{eT} \left[ \begin{array}{cc} EA & 0 \\ 0 & EI \end{array} \right] \overline{\mathbf{B}}^e d\bar{x}\overline{\mathbf{u}}^e = \frac{E}{L_0} \left[ \begin{array}{cccc} A & 0 & -A & 0 \\ & 4I & 0 & 2I \\ & & A & 0 \\ sym & & & 4I \end{array} \right] \left\{ \begin{array}{c} \bar{u}_1 \\ \bar{\theta}_1 - \bar{\theta}_1^p \\ \bar{u}_2 - \bar{u}^p \\ \bar{\theta}_2 - \bar{\theta}_2^p \end{array} \right\} \tag{5}$$

and $\overline{\mathbf{f}}^p = \lfloor N_1 \quad -M_2 \quad N_2 \quad M_1 \rfloor^T$, with $\bar{u}^p = \bar{u}_1^p + \bar{u}_2^p$ (see details in [14]). Once, by construction, $\bar{u}_1$ is null in the co-rotated frame, the displacements $\overline{\mathbf{u}}^e, \overline{\mathbf{u}}^p, \overline{\mathbf{u}}$ can be redefined as

$$\overline{\mathbf{u}}^e \leftarrow \lfloor \bar{\theta}_1^e \quad \bar{u}_2^e \quad \bar{\theta}_2^e \rfloor^T \quad \overline{\mathbf{u}}^p \leftarrow \lfloor \bar{\theta}_1^p \quad \bar{u}^p \quad \bar{\theta}_2^p \rfloor^T \quad \overline{\mathbf{u}} \leftarrow \lfloor \bar{\theta}_1 \quad \bar{u}_2 \quad \bar{\theta}_2 \rfloor^T = \overline{\mathbf{u}}^e + \overline{\mathbf{u}}^p. \tag{6}$$

the same applied to the vectors $\overline{\mathbf{r}}$, $\overline{\mathbf{p}}$, $\overline{\mathbf{f}}^p$:

$$\overline{\mathbf{r}} \leftarrow \lfloor M_{\bar{z}1} \quad F_{\bar{x}2} \quad M_{\bar{z}2} \rfloor^T \quad \overline{\mathbf{p}} \leftarrow \lfloor p_{\bar{\theta}_1} \quad p_{\bar{u}_2} \quad p_{\bar{\theta}_2} \rfloor^T \quad \overline{\mathbf{f}}^p \leftarrow \lfloor -M_2 \quad N_2 \quad M_1 \rfloor^T, \tag{7}$$

and to $\overline{\mathbf{f}}$:

$$\overline{\mathbf{f}} \leftarrow \begin{Bmatrix} M_1 \\ N_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L_0} & 0 & \frac{2EI}{L_0} \\ & \frac{EA}{L_0} & 0 \\ sym & & \frac{4EI}{L_0} \end{bmatrix} \begin{Bmatrix} \bar{\theta}_1 - \bar{\theta}_1^p \\ \bar{u}_2 - \bar{u}^p \\ \bar{\theta}_2 - \bar{\theta}_2^p \end{Bmatrix} = \overline{\mathbf{k}}^e \{\overline{\mathbf{u}} - \overline{\mathbf{u}}^p\} \tag{8}$$

Following the work by [15], rewriting Eq. (8) in terms of flexibility allows for the introduction of inelastic rotations due to damage, using concepts analogous to strain equivalence from classical damage mechanics. Then, rewriting it again in terms of stiffness results

$$\overline{\mathbf{f}} \leftarrow \begin{Bmatrix} M_1 \\ N_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{12(1-d_1)EI}{(3-d_1d_2+d_1+d_2)L_0} & 0 & -\frac{6EI(1-d_1)(1-d_2)}{(3-d_1d_2+d_1+d_2)L_0} \\ & \frac{EA}{L_0} & 0 \\ sym & & \frac{12(1-d_2)EI}{(3-d_1d_2+d_1+d_2)L_0} \end{bmatrix} \begin{Bmatrix} \bar{\theta}_1 - \bar{\theta}_1^p \\ \bar{u}_2 - \bar{u}^p \\ \bar{\theta}_2 - \bar{\theta}_2^p \end{Bmatrix} = \overline{\mathbf{k}} \{\overline{\mathbf{u}} - \overline{\mathbf{u}}^p\} \tag{9}$$

in which $d_i$ are damage variables for the hinges $i = 1, 2$. The stiffness matrix $\overline{\mathbf{k}}$ now considers the influence of the damage variables, and reduces to $\overline{\mathbf{k}}^e$ when $d_1 = d_2 = 0$. Replacing (6),(7) and (9) into (3) yields: $\delta\overline{\mathbf{u}}^{eT}\left\{-\overline{\mathbf{f}} + \overline{\mathbf{r}} + \overline{\mathbf{p}}^e\right\} + \delta\overline{\mathbf{u}}^{pT}\left\{-\overline{\mathbf{f}}^p + \overline{\mathbf{r}} + \overline{\mathbf{p}}^p\right\} = 0$. Since the virtual displacements $\delta\overline{\mathbf{u}}$ are arbitrary, it is possible to choose a virtual displacement field completely elastic. Supposing that $\delta\overline{\mathbf{u}}^p = \mathbf{0}$ then

$$-\overline{\mathbf{f}} + \overline{\mathbf{r}} + \overline{\mathbf{p}}^e = \mathbf{0} \quad \Rightarrow \quad \overline{\mathbf{k}}\{\overline{\mathbf{u}} - \overline{\mathbf{u}}^p\} = \overline{\mathbf{r}} + \overline{\mathbf{p}}^e \tag{10}$$

which represents the equilibrium of the element refereed to $C_{0n}$. Notice that the system (10) has more unknown than equations, requiring thus additional equations provided by the evolution laws of the internal variables.

### 2.2 Equilibrium refereed to $C_0$

The equilibrium equations of the element referred to $C_0$ can be obtained returning the rigid body motion that occurs from $C_0$ to $C_{0n}$ to the element, i.e., identifying the relationship between $\overline{\mathbf{u}} = \lfloor \bar{\theta}_1 \quad \bar{u}_2 \quad \bar{\theta}_2 \rfloor^T$ and $\mathbf{u} = \lfloor u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2 \rfloor^T$. Such relationship can be established in a differential form as (see [16] for details)

$$\delta\overline{\mathbf{u}} = \frac{\partial\overline{\mathbf{u}}}{\partial\mathbf{u}}\delta\mathbf{u} = \mathbf{T}\delta\mathbf{u} \qquad \mathbf{T}(\mathbf{u}) = \begin{bmatrix} \frac{\partial\bar{\theta}_1}{\partial u_1} & \frac{\partial\bar{\theta}_1}{\partial v_1} & \cdots & \frac{\partial\bar{\theta}_1}{\partial\theta_2} \\ \frac{\partial\bar{\theta}_2}{\partial u_1} & \frac{\partial\bar{\theta}_2}{\partial v_1} & \cdots & \frac{\partial\bar{\theta}_2}{\partial\theta_2} \\ \frac{\partial\bar{u}_2}{\partial u_1} & \frac{\partial\bar{u}_2}{\partial v_1} & \cdots & \frac{\partial\bar{u}_2}{\partial\theta_2}. \end{bmatrix} \tag{11}$$

Since the virtual work realized by nodal reactions $\overline{\mathbf{r}}$, internal nodal forces $\overline{\mathbf{f}}$ and equivalent nodal forces $\overline{\mathbf{p}}$ are invariant with respect to changes in the coordinate system, it is possible to obtain the equilibrium equations of the element referred to $C_0$ by pre-multiplying (10) by $\mathbf{T}^T$

$$\mathbf{T}^T\left\{-\overline{\mathbf{f}} + \overline{\mathbf{r}} + \overline{\mathbf{p}}\right\} = 0 \quad \Rightarrow \quad \Psi(\mathbf{u}) = -\mathbf{f} + \mathbf{r} + \mathbf{p} = \mathbf{0}. \tag{12}$$

The previous relation forms a system of nonlinear equations, which can be solved by the Newton-Raphson procedure. To calculate $\Psi(\mathbf{u})$ and $\partial\Psi/\partial\mathbf{u}$ it is necessary to obtain $\overline{\mathbf{f}}$ (and $\overline{\mathbf{u}}^p$ ) from $\mathbf{u}$. The computation of $\overline{\mathbf{f}}$ from $\mathbf{u}$ is described in section 2.4.

### 2.3 Evolution laws

**Plasticity evolution laws**

The evolution laws of the plastic displacements $\bar{u}_1^p, \bar{u}_2^p$ and plastic rotations $\bar{\theta}_1^p, \bar{\theta}_2^p$ of the inelastic hinges can be written as a function of the nodal internal forces $N_1, M_1$ and $N_2, M_2$ [17]. Let the respective yield functions for the hinges $i = 1, 2$ be represented by $f_i = f_i\left(N_i, M_i\right) \leq 0$. The evolution laws of the plastic displacements $\bar{u}_i^p$ and plastic rotations $\bar{\theta}_i^p$ are given by the associative normality rule: $\Delta\bar{u}_i^p = \Delta\lambda_i\ \partial f_i/\partial N_i$ and $\Delta\bar{\theta}_i^p = \Delta\lambda_i\ \partial f_i/\partial M_i$. Recording that $\bar{u}^p = \bar{u}_1^p + \bar{u}_2^p$, one writes

$$\Delta\bar{u}^p = \Delta\bar{u}_1^p + \Delta\bar{u}_2^p = \Delta\lambda_1\frac{\partial f_1}{\partial N_1} + \Delta\lambda_2\frac{\partial f_2}{\partial N_2}, \tag{13}$$

where $\lambda_i$ is the plastic multiplier of the plastic hinges $i$ . The evolution laws of $\lambda_i$ are

$$\Delta\lambda_i = 0 \quad if \quad f_i\left(N_i, M_i\right) < 0; \qquad f_i\left(N_i, M_i\right) = 0 \quad if \quad \Delta\lambda_i > 0. \tag{14}$$

Analytical expressions for the yielding functions $f_i$ depends on the cross-section geometry and on the constitutive model of the material. Suitable empirical expressions for symmetric cross-sections and elastic-plastic materials are given by [14]. Replacing the bending moments $M_i$ with the effective bending moments $\bar{M}_i = M_i/(1 - d_i)$ in the yield function from [14] allows to obtain the yielding functions for damage-plastic material [7], which are adopted herein

$$f_{\mathrm{i}}\left(N_{\mathrm{i}}, M_{\mathrm{i}}\right) = \left[\left(\frac{N_{\mathrm{i}}}{N_y}\right)^{\mathrm{e}} + \left(\frac{M_{\mathrm{i}}}{M_y\left(1 - d_i\right)}\right)^{\mathrm{e}}\right]^{1/e} - 1 \leq 0, \tag{15}$$

where $M_y$ is the yield moment of the cross-section without axial forces, $N_y$ produces the total plasticization of the element when there is no bending moments and the constant $e$ is a cross-section dependent parameter.

**Damage evolution laws**

The evolution laws for the damage variables $d_1$ and $d_2$ were proposed by [7] to address local buckling problems and are given by: $d_i = k_m\left\langle\left|\bar{\theta}_i^p\right| - p_{cr}\right\rangle_+$, where $p_{cr}$ is the critical plastic rotation that initiates the local buckling, $k_m$ is the slope of the damage evolution line and $\langle\rangle_+$ indicates that only positive values are taken. Under the assumption of monotonic loading, in which $\left|\bar{\theta}_i^p\right|$ is a monotonically increasing variable, the damage evolution laws can also be presented as

$$f_{di} = c_{di} l_i = 0, \tag{16}$$

in which $l_i = d_i - k_m\left(\left|\bar{\theta}_i^p\right| - p_{cr}\right)$, and $c_{di}$ are constant related to the damage state of a particular hinge $i = 1, 2$, which assumes the values $c_{pi} = 0$ or $c_{pi} = 1$ depending on the conditions $\left|\bar{\theta}_i^p\right| - p_{cr} < 0$ or $\left|\bar{\theta}_i^p\right| - p_{cr} \geq 0$, respectively.

### 2.4 Discrete constitutive equations

The constitutive equations of the element with hinges, i.e., the elasticity equations (9) and the evolution laws, allows the computations of the internal force vector $\bar{\mathbf{f}}$ from a given displacement vector $\mathbf{u}$. To accomplish this task, the evolution laws must be presented in a discrete form and combined with (9). Let $\bar{\mathbf{u}}_0^p$ be the vector of plastic displacements jumps of a previous known solution $\bar{\mathbf{u}}_0$. Let $c_{p_i}$ and $c_{d_i}$ be constants related to the plasticity and damage states, respectively, of a particular hinge $i$ ($i = 1, 2$). The constants $c_{p_i}$ assumes the values $c_{p_i} = 0$ or $c_{p_i} = 1$ depending on $f_i^* < 0$ or $f_i^* 0$, in which $f_i^*$ is the yield function of the hinge $i$ and evaluated for the elastic prediction $\bar{\mathbf{f}}^* = \bar{\mathbf{k}}^e\left(\bar{\mathbf{u}} - \bar{\mathbf{u}}_0^p\right)$. Assuming the plastic displacements $\bar{\mathbf{u}}^p$ as small, one decomposes $\bar{\mathbf{u}}^p$ in the additive form

$$\bar{\mathbf{u}}^p = \bar{\mathbf{u}}_0^p + \Delta\bar{\mathbf{u}}^p, \tag{17}$$

where, from the plastic displacements and rotations evolution laws

$$\Delta\bar{\mathbf{u}}^p = \begin{bmatrix} \Delta\bar{\theta}_1^p & \Delta\bar{u}^p & \Delta\bar{\theta}_2^p \end{bmatrix}^T = \left\lfloor c_{\mathrm{p}1}\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} \quad c_{\mathrm{p}1}\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial N_1} + c_{\mathrm{p}2}\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial N_2} \quad c_{\mathrm{p}2}\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2}\right\rfloor^T. \tag{18}$$

Based on the equations (9) and (17), one writes

$$\overline{\mathbf{f}} + \overline{\mathbf{k}}\Delta\overline{\mathbf{u}}^p = \overline{\mathbf{k}}\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\}. \tag{19}$$

From the definition of $\overline{\mathbf{k}} = \overline{\mathbf{k}}(d_1, d_2)$, one writes

$$\overline{\mathbf{k}}\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\} = \overline{\mathbf{k}}^d\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\} + \overline{\mathbf{k}}^e\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\}, \tag{20}$$

where $\overline{\mathbf{k}}^e$ is the elastic stiffness matrix defined in (8) and $\overline{\mathbf{k}}^d = \overline{\mathbf{k}}^d(d_1, d_2)$ is a damage correction stiffness matrix, defined as

$$\overline{\mathbf{k}}^d = \begin{bmatrix} \frac{12(d_1-1)EI}{(d_1d_2-d_1-d_2-3)L_0} - \frac{12EI}{3L_0} & 0 & -\frac{6EI(d_1-1)(d_2-1)}{(d_1d_2-d_1-d_2-3)L_0} - \frac{6EI}{3L_0} \\ & 0 & 0 \\ sym & & \frac{12(d_2-1)EI}{(d_1d_2-d_1-d_2-3)L_0} - \frac{12EI}{3L_0} \end{bmatrix} \tag{21}$$

Thus, (19) can be rewritten as

$$\overline{\mathbf{f}} + \overline{\mathbf{k}}\Delta\overline{\mathbf{u}}^p - \overline{\mathbf{k}}^d\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\} = \overline{\mathbf{f}}^*, \tag{22}$$

where $\overline{\mathbf{f}}^* = \overline{\mathbf{k}}^e\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\}$ is the elastic prediction, which assumes $\overline{\mathbf{u}}_0^p = \mathbf{0}$ and $\mathbf{d} = \lfloor d_1, d_2 \rfloor^T = \mathbf{0}$. A nonlinear system of equations arises from (22), resulting

$$\mathbf{g}\left(\mathbf{x}\right) = \overline{\mathbf{f}} + \overline{\mathbf{k}}\Delta\overline{\mathbf{u}}^p - \overline{\mathbf{k}}^d\left\{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0^p\right\} - \overline{\mathbf{f}}^* = \mathbf{0}, \tag{23}$$

in which $\mathbf{x} = \left\lfloor \overline{\mathbf{f}}, \Delta\lambda_1, \Delta\lambda_2, d_1, d_2 \right\rfloor^T$. Notice that the system (23) has three equations and seven unknowns. In order to make (23) solvable, the plasticity constraining equations: $c_{p1}f_1 = 0$, $c_{p2}f_2 = 0$, and the damage constraining equations: $c_{d1}l_1 = 0$, $c_{d2}l_2 = 0$ are added into it:

$$\mathbf{g}\left(\mathbf{x}\right) \leftarrow \left\{ \begin{array}{c} \mathbf{g}\left(\mathbf{x}\right) \\ c_{p_1}f_1\left(N_1, M_1, d_1\right) \\ c_{p_2}f_2\left(N_2, M_2, d_2\right) \\ c_{d_1}l_1\left(\Delta\lambda_1, d_1\right) \\ c_{d_2}l_2\left(\Delta\lambda_2, d_2\right) \end{array} \right\} - \left\{ \begin{array}{c} \overline{\mathbf{f}}^* \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}. \tag{24}$$

Once assumed values for the constants $c_{pi}$ and $c_{di}$, the above non-linear system of equations can be solved numerically by the Newton-Raphson method. The converged solution obtained for the provisional values of $c_{pi}$ and $c_{di}$ must satisfy the constrains to be consistent with the damage-plasticity constitutive model. If the constraints are not satisfied, a new prevision must be made for the constants $c_{pi}$ and $c_{di}$, and the local problem (24) must be solved again for the new constants.

## 3 Numerical results

The effectiveness of the proposed co-rotational finite element was verified with an example of a bi-clamped beam subjected to an asymmetrical concentrated load, as illustrated in Figure 1. The validation was performed by the results obtained with the developed formulation with the elastic-plastic solutions from Alhasawi et al. [8] and Tasinaffo et al. [14]. The inelastic hinges activated during the analysis are also presented in Fig. 1.

The example was analyzed in [8, 14] with an elastic perfectly plastic hinges models. It is worth to emphasize that [8, 14] adopted the yield functions defined in 15, but with $d_i = 0$ as damage was not considered in their analysis. The dimensions of the problem are $L = 720$ cm, $a = L/3$ and $b = 2L/3$. The beam cross-section is of type HEB 220 , with a plastic modulus of $z_y = 827.19\ \mathrm{cm}^3$. The damage-plastic material has a Young modulus $E = 210\mathrm{GPa}$ and a yield stress $\sigma_y = 355\mathrm{MPa}$. From $z_y$ and $\sigma_y$ it is possible to obtain $M_y = 29365.25\mathrm{kN.cm}$ and $N_y = 3230.50\mathrm{kN}$. The critical plastic rotation and the the slope of the damage evolution, adopted only in the present work analysis, are set as $p_{cr} = 0.15\mathrm{rad}$ and $k_m = 4.0$. The example is analyzed with 2 elements. The tolerance for the convergence of the global Newton-Raphson procedures is equal to $10^{-3}$ for both dimensionless displacement increments and forces, while the tolerance for the convergence of the local Newton-Raphson procedures is equal to $10^{-3}$ for dimensionless internal forces and $10^{-6}$ for damage-plasticity laws restrictions. The load $P = 1200\mathrm{kN}$ is applied in 20 load steps. Figure 2 shows the load-displacement results at the point of application of the concentrated load, as well as the damage evolution for hinges 1 and 2.

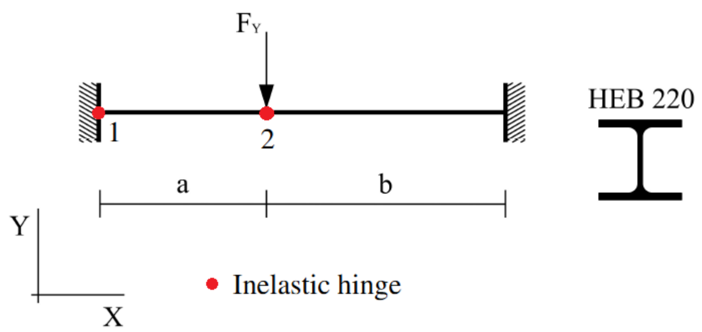
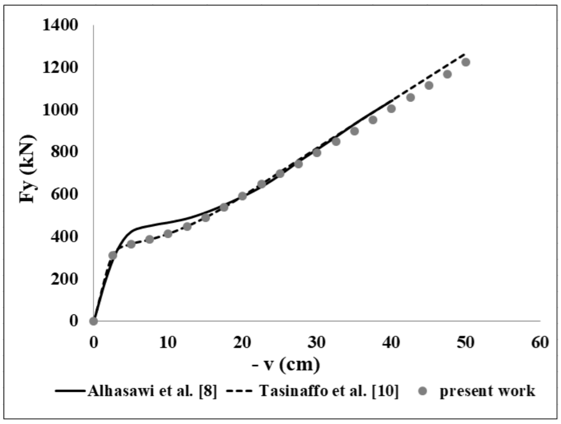
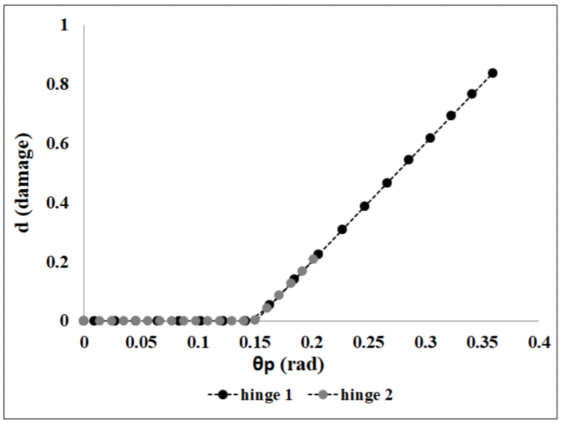



Figure 1. Bi-clamped beam subjected to an asymmetrical concentrated load.




a) load-displacement results




b) damage evolution results

Figure 2. Load-dsiplacement and damage evolution results.

Notice that the damage-plasticity load-displacement results starts to differ from the elastoplastic response from [8, 14] when the damage variable of the hinge 1 is activated, resulting a less stiff response. The numerically computed damage evolution also agrees with the imposed ones.

## 4 Final remarks

In this paper, a co-rotational model was developed to analyze planar slender section steel frames, accounting for geometric nonlinearity and local buckling. The local buckling phenomenon is taken into account by a lumped damage model, which was consistently incorporated into the co-rotational model. Both the constrained nonlinear system of equation of the discrete constitutive problem at element level and the nonlinear global equilibrium equations are solved by full Newton-Raphson procedures. A bi-clamped beam example with an available elastic-plastic reference solution was presented to validate the developed formulation. The results were consistent with the expected, as discussed in the numerical results section.

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