Numerical modeling of localization bands with mesh-independence using lumped damage mechanics

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Abstract. The Lumped Damage Mechanics for Continuous Media has successfully described the nonlinear physical behavior of engineering problems. The approach utilizes basic concepts from fracture and classic damage mechanics. In this regard, material degradation in two-dimensional media is computed from the evolution of localization bands concentrated on the finite elements' faces. The growth of these bands occurs based on the Griffith criterion. This work adopts a nonlinear evolution law with exponential softening to describe quasi-brittle failure behavior accurately. Numerical solutions show the mesh independence of the proposed approach and the reproduction of the size effect phenomenon. Finally, it was possible to reproduce experimental behavior in terms of equilibrium trajectory and cracking pattern of the failure region.

Keywords: lumped damage mechanics, mesh-independence, localization bands.

1 Introduction

Concrete crack propagation is still an important issue in structural mechanics. Plasticity models were often used and are even improved nowadays [1]. Significant developments in smeared crack models occurred in the past century [2-3]. Such advances were paramount for developing continuum damage models, especially considering fracture energy regularization techniques [4-5], which are often used to circumvent mesh dependence due to the softening phase, i.e., *strain localization* [6-7]. Other regularization models, based on gradient and nonlocal techniques [8-11], have successfully avoided strain localization.

Alternatively, different theories were combined to bypass the mathematical issues of strain localization. Regarding continuum damage models, Wolenski et al. [12] developed a technique based on the extended/generalized finite element method that properly assesses the strain localization. Recently, Pereira Junior et al. [13] designed an inverse analysis using the artificial bee colony algorithm to solve strain localization using continuum damage mechanics.

On the other hand, numerical models based on fracture mechanics are an interesting alternative to analyzing concrete cracking. Linear elastic fracture mechanics is characterized as not presenting the fracture process zone. There is a vast amount of literature on that theme for finite and boundary element methods (for a brief review, see [14-15] and the references therein). Cohesive fracture mechanics can model ductile and brittle behaviors [16-17], with several variations describing the fracture process zone behavior [18-20]. Usually based on damage and fracture mechanics concepts, phase-field models are a research hotspot with important developments for nonlinear solid mechanics [21-24].

Despite the importance of all aforementioned theories, this paper proposes to assess the cracking process in two-dimensional media by using the lumped damage mechanics (LDM). LDM was first proposed by Flórez-López [25] for reinforced concrete frames and then developed for several frame structures (see, e.g., [26-28] and the references therein for a brief review). Later, LDM was extended to plane stress/strain applications [29-31]. Recently, LDM was successfully used to describe reinforced concrete slabs' behavior and cracking pattern [32].

2 Lumped damage modeling for two-dimensional media

Consider the classic quadrilateral finite element with nodes on its vertexes (Fig. 1a), in which the nodal displacements matrix is given by $\{\mathbf{q}\}$. Then, the strain field of the finite element $\{\boldsymbol{\varepsilon}(x,y)\}$ can be expressed as:

$$\{\boldsymbol{\varepsilon}(x,y)\} = [\mathbf{B}(x,y)]\{\mathbf{q}\}\ , \tag{1}$$

where $[\mathbf{B}(x,y)]$ is the classic compatibility matrix. In view of the generalized Hooke's law for plane stress or plane strain, the stress field of the finite element $\{\boldsymbol{\sigma}(x,y)\}$ is written by:

$$\{\boldsymbol{\sigma}(x,y)\} = [\mathbf{H}]\{\boldsymbol{\varepsilon}(x,y)\} = [\mathbf{H}][\mathbf{B}(x,y)]\{\mathbf{q}\}\ , \tag{2}$$

being $[\mathbf{H}]$ the matrix of elastic coefficients. Then, the stiffness matrix of the finite element $[\mathbf{k}]$ is given by:

$$[\mathbf{k}] = \iiint_V [\mathbf{B}(x,y)]^T[\mathbf{H}][\mathbf{B}(x,y)]dV\ . \tag{3}$$

Regarding this element's displacement approach, its deformed shape depends on five constants. Therefore, in this formulation, the classic element is replaced by an equivalent truss named here as numerical extensometers or *numexes* (Fig. 1b). Note that the *numexes* elongations can also describe the deformed shape of the quadrilateral finite element. These elongations can be gathered in the matrix of generalized deformations, given by:

$$\{\boldsymbol{\delta}\}^T = \left\{\delta_{ij}, \delta_{ik}, \delta_{il}, \delta_{jk}, \delta_{kl}\right\}, \tag{4}$$

where δ_{ij} is the elongation of the *numex ij*, and so on.

Note that the *numexes*' elongations can be expressed by the nodal displacements, i.e.

$$\{\boldsymbol{\delta}\} = [\mathbf{b}]\{\mathbf{q}\}\,, \tag{5}$$

being $[\mathbf{b}]$ the kinematic transformation matrix of the finite element, expressed in its reference space (Fig. 1c) by:

$$[\mathbf{b}] = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}. \tag{6}$$

Now, the elastic relations of the finite element (1-3) are rewritten in terms of the *numexes*' elongations:

$$\{\boldsymbol{\varepsilon}(x,y)\} = [\mathbf{T}(x,y)]\{\boldsymbol{\delta}\}\,, \tag{7}$$

$$\{\boldsymbol{\sigma}(x,y)\} = [\mathbf{H}][\mathbf{T}(x,y)]\{\boldsymbol{\delta}\}\,, \tag{8}$$

$$[\mathbf{k}] = [\mathbf{b}]^T\left(\iiint_V [\mathbf{T}(x,y)]^T[\mathbf{H}][\mathbf{T}(x,y)]dV\right)[\mathbf{b}]\,, \tag{9}$$

where $[\mathbf{B}(x,y)] = [\mathbf{T}(x,y)][\mathbf{b}]$ and $[\mathbf{T}(x,y)]$ is the transformation matrix, given as follows for the reference space:

$$[\mathbf{T}] = \frac{1}{4}\begin{bmatrix} 1-s & 0 & 0 & 0 & 1+s \\ 0 & 0 & 1-t & 1+t & 0 \\ -1-t & 2\sqrt{2} & -1-s & s-1 & t-1 \end{bmatrix}. \tag{10}$$

Regarding the elongations equivalence hypothesis [30], the total elongations of the *numexes* are:

$$\{\boldsymbol{\delta}\} = \{\boldsymbol{\delta}^{\mathbf{e}}\} + \{\boldsymbol{\delta}^{\mathbf{d}}\}\,, \tag{11}$$

being $\{\boldsymbol{\delta}^{\mathbf{e}}\}$ its elastic part and $\{\boldsymbol{\delta}^{\mathbf{d}}\}$ the damaged one, expressed in the reference space by “jumps” on the displacement field (Fig. 1d):

$$\{\boldsymbol{\delta}^{\mathbf{d}}\} = \begin{Bmatrix} \delta^d_{ij} \\ \delta^d_{ik} \\ \delta^d_{il} \\ \delta^d_{jk} \\ \delta^d_{kl} \end{Bmatrix} = \begin{Bmatrix} e^{ij}_i - e^{ik}_i\sqrt{2}/2 + e^{il}_j\sqrt{2}/2 \\ e^{il}_i\sqrt{2}/2 + e^{ij}_i\sqrt{2}/2 \\ e^{ij}_i + e^{ik}_i\sqrt{2}/2 + e^{il}_l\sqrt{2}/2 \\ e^{ij}_j - e^{jl}_j\sqrt{2}/2 + e^{ik}_k\sqrt{2}/2 \\ e^{il}_l - e^{jl}_l\sqrt{2}/2 + e^{ik}_k\sqrt{2}/2 \end{Bmatrix}, \tag{12}$$

where e_i^{ij} is the displacement “jump” on the node i at side ij, and so on (Fig. 1d).

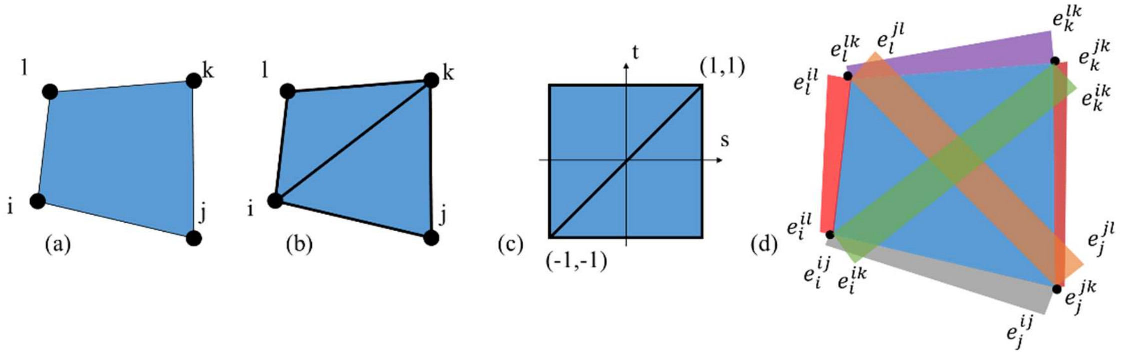

Figure 1. (a) Quadrilateral finite element, (b) numerical extensometers, (c) finite element in the reference space, and (d) localization bands

Then, the elastic relations (8) must be rewritten as:

$$\{\boldsymbol{\sigma}(x,y)\} = [\mathbf{H}][\mathbf{T}(x,y)]\{\boldsymbol{\delta} - \boldsymbol{\delta}^{\mathbf{d}}\}\,. \tag{13}$$

Each “jump” grows by a damage evolution law and follows an exponential decay as an attempt to reproduce experimental behavior, i.e.

$$\begin{cases} (\sigma^{ij}_j \dot{e}^{ij}_j - e^{ij}_j \dot{\sigma}^{ij}_j) = 0 & \text{if } h < 0 \\ h = 0 & \text{if } (\sigma^{ij}_j \dot{e}^{ij}_j - e^{ij}_j \dot{\sigma}^{ij}_j) > 0 \end{cases} \quad \text{with } h = \sigma^{ij}_i - \sigma_{cr}\exp(q e^{ij}_i) \le 0\ , \tag{14}$$

where σ_{cr} is the tensile strength and q is the material internal parameter. Such parameter q can be expressed in terms of the tensile strength and the fracture energy (G_f), considering the following development:

$$G_f = \int_0^{\infty} \sigma(e)de = \int_0^{\infty} \sigma_{cr} \exp(q e_i^{ij}) de = -\frac{\sigma_{cr}}{q} \Rightarrow q = -\frac{\sigma_{cr}}{G_f}. \tag{15}$$

3 Numerical applications

3.1 The double-hexagon example ("*rubber duck*")

Picón et al. [30] previously presented this example's geometry, which is depicted in Fig. 2. The specimen is clamped at segment AB, and an imposed displacement inclined at 45º is applied in segment FG. The material's properties are also presented in Fig. 2.

To analyze solution convergence, this example was simulated with three different meshes containing 230, 672, and 829 elements (Fig. 2). Note that the collapse mechanism is similar in the analyzed meshes, especially for the more refined ones (672 and 829 elements). Still, considering the more refined meshes, note that both force-displacement curves are quite fitted, which indicates mesh independence. Similar lumped damage models for two-dimensional media [29-31] and bending plates [32] also show mesh independence.

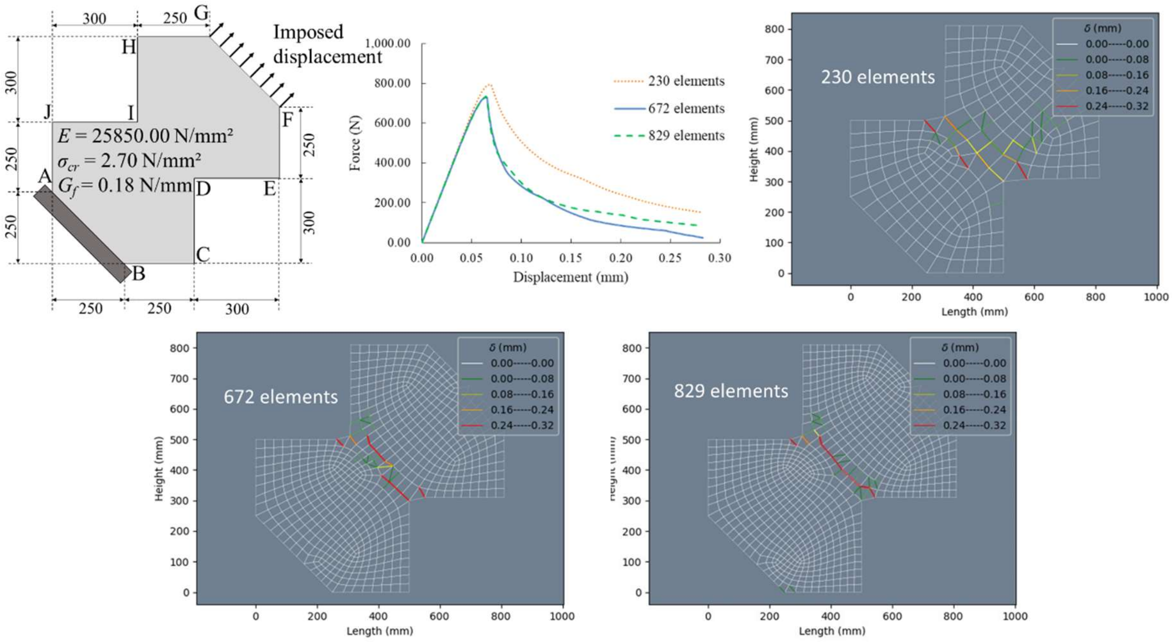

Figure 2. The double-hexagon test: geometry (dimensions in mm), force-displacement curves, and collapse mechanisms of the analyzed meshes

3.2 Size effect

This example addresses the capability to reproduce the size effect phenomenon. In all analyses, a cantilever beam with 300 elements is modeled where the height (h) varies from 50mm to 1000mm. This beam presents an aspect ratio (length/height) equal to 3.0 (Fig. 3) and unitary thickness (1.0mm). The material properties adopted are: E = 26GPa, ν = 0.25, σ_{cr} = 3MPa, and G_f = 0.32N/mm. Note that the softening behavior of the analyzed examples indicates the size effect phenomenon. For all analyses, the ultimate load (P_u) was measured and then the collapse stress (σ_t) was calculated by:

$$\sigma_t = \frac{P_u(3h)h}{2I} = \frac{P_u(3h)h}{2}\frac{12}{h^3} = \frac{18P_u}{h}. \tag{16}$$

As expected, the relative collapse stress (σ_t/σ_{cr}) decreases as the dimensions increase (Fig. 3).

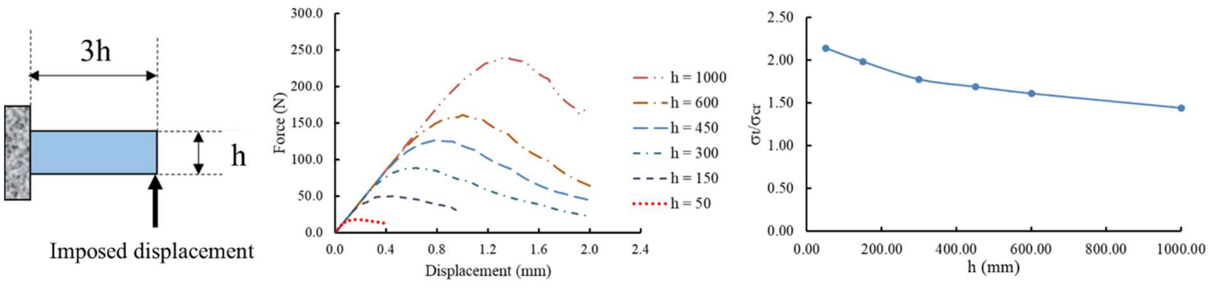

Figure 3. Size effect: geometry, force-displacement curves, and size effect graph

3.3 Three-point bending test

The third example is a set of three fiber-reinforced concrete beams, experimentally tested by Köksal et al. [33] using a three-point bending test setup. The beams had lengths and spans equal to 900 mm and 700 mm, respectively, with square cross-sections of 150 mm x 150 mm and a notch of 25 mm depth. The beams were batched with different fiber ratios, equal to 0.33%, 0.67%, and 1%, respectively, for beams #1, #2, and #3. The elasticity modulus, tensile strength, Poisson's ratio, and fracture energy were equal to 10GPa, 1.2MPa, 0.2, and 1.707N/mm for beam #1, 6GPa, 3.7MPa, 0.2, and 2.853N/mm for beam #2, 6GPa, 7.2MPa, 0.2 and 9.893N/mm for beam #3, respectively. Fig. 4 shows the damaged condition of the beams and the load-displacement curves.

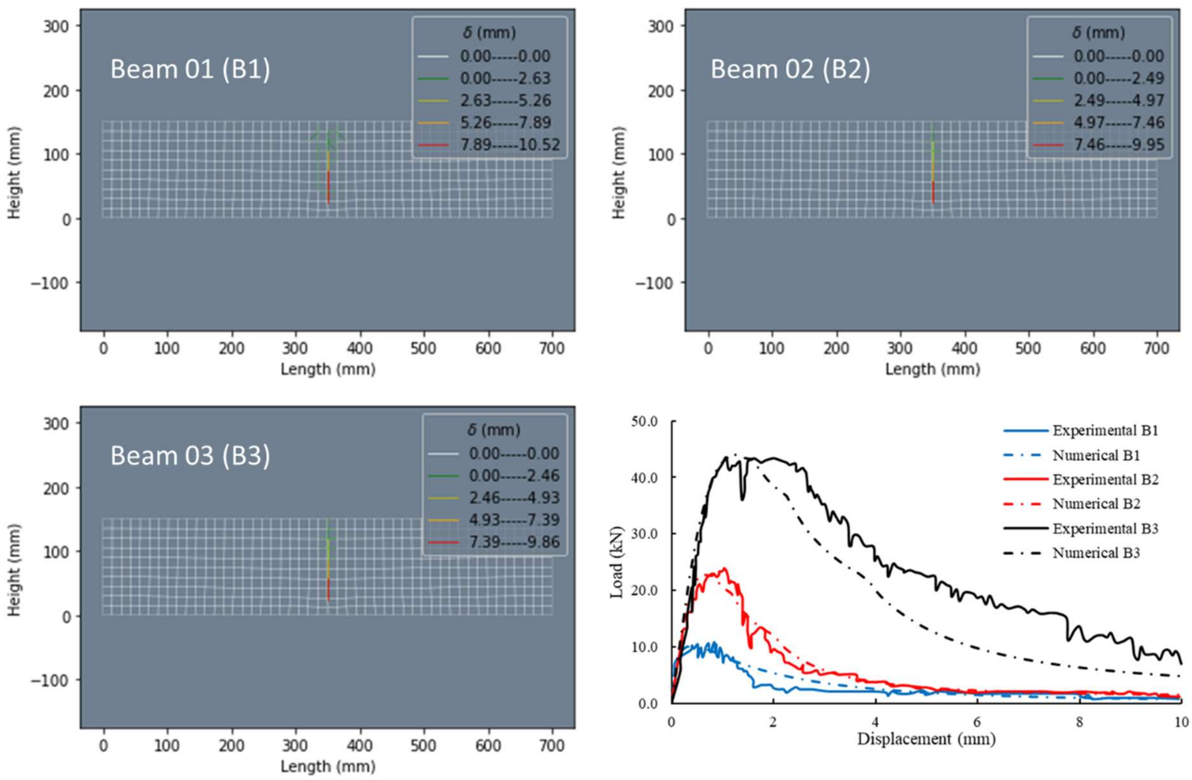

Figure 4. Force-displacement curves and analyzed meshes for beams.

The simulations led to cracking conditions that agree with the experimental results, with a single crack in the center of the beam. Moreover, note that the equilibrium trajectories are quite close to the experimental ones, and the bearing capacity and post-peak are also close to the experimental values.

4 Conclusions

As discussed in this paper and some references, lumped damage mechanics may have an intrinsic regularization technique in its formulation for two-dimensional media. The first example analyzed shows mesh independence and agrees with other papers in similar analyses.

This paper also presents the reproducibility of the size effect phenomenon with an academic cantilever beam. This finding is important since this phenomenon is experimentally observed for several materials. Finally, the lumped damage model presented in this paper shows good accuracy by using the ratio between the tensile strength and fracture energy as an internal parameter that dictates the softening phase. Such characteristics might ensure that this lumped damage model is a reliable numerical approach to be implemented in commercial finite element programs.

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