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CRITICAL BUCKLING LOAD BASED ON MODAL ANALYSIS BY RAYLEIGH METHOD AND FINITE ELEMENT METHOD OF A NON-PRISMATIC SLENDER REINFORCED CONCRETE COLUMN

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Abstract. In this study, an analytical and computational procedure were developed for determining the critical buckling load. The analytical solution was based on the Rayleigh method and the computational one on the finite element method (FEM). Rayleigh method preconize that one equation named as shape (or trial) function should be defined to represent the vibrational movement of the system. Therefore, the result obtained by this method is entirely conditioned to the correct choice of this equation. Different equations even respecting the boundary conditions of the problem can lead to different results. Four mathematical expressions as shape function were used in the present study: a trigonometric, two polynomials and a potential equation. All these functions obey to the boundary conditions of the problem and were valid in the whole domain. Therefore, the integrals obtained by the Rayleigh method were solved considering the structural geometry. With comparative purpose the results obtained on the analytical procedure were compared with those yielded by computational modelling using a finite element modal analysis. The structure analyzed was a 46-m-high reinforced concrete pole, including its foundation, which has geometry and reinforcement arrangement varying along its length. For both solutions, three important items were considered: the geometric nonlinearity, due to the slenderness of the system; the material nonlinearity and the creep of the concrete. The last one aspect was introduced into the analysis by means of Eurocode criteria. Significant differences on the absolute value of the critical load were found in comparison with the adopted procedures, being possible to observe that the potential equation led to results too distant from the other equations. Analysis considering an elapsed time of 4000 days revealed an average decreasing of 22% on the intensity of the critical buckling load. The FEM presented the biggest percentual difference, 28%.

Keywords: Buckling load, Modal analysis, Rayleigh method, Shape functions, Analytical solution, Finite element method.

1 Introduction

Designing slender structures has been a permanent challenge for structural engineering, because involves the equilibrium of mechanical systems. Even after the advent of modern digital computers and their diffusion as a working tool among engineers, analytical solutions continue to attract the interest of scientist and practitioners. In particular, the elastic buckling of slender columns has been the object of constant study since the first formulations presented by Euler in 1744 [1], who developed a solution based on the static equilibrium of a bended section. Import to mention that Euler had difficulty to initially include the self-weight of the column, being this problem solved later by another mathematician, points out Timoshenko [2]. A column represents a continuous structural member that can be subjected to different types of compression loads; a conservative load (Euler's load) is one type of load affirm Uzny [3].

Although analytical solutions are usually taken as a reference for computational results, it is very difficult to apply them to complex systems with variable geometry and with large degrees of freedom. According to Cook [4], Courant was the first researcher to create a numerical model for structural analysis using the principle of stationary potential energy by interpolating a triangular region to the Saint-Venant torsion problem [5]. He assumed a linear distribution of functions for distortion on these elements, whose approximation extends to the Rayleigh-Ritz (RR) model.

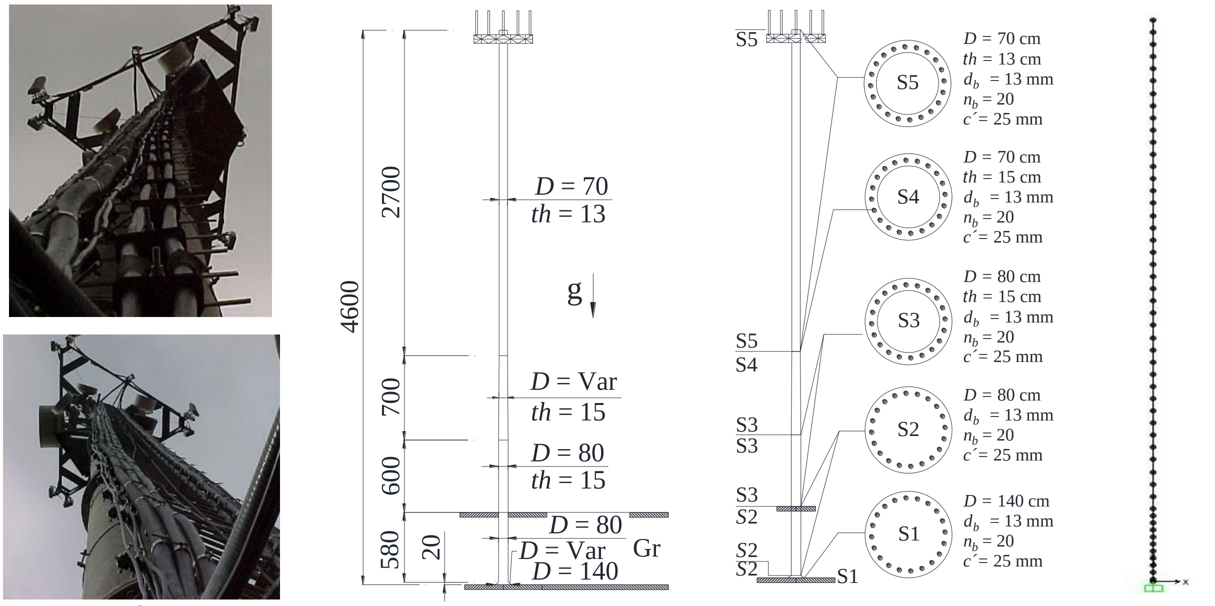
Rayleigh studied vibration problems and presented his postulates in 1877 [6]. In that book there are many examples of calculating the fundamental natural frequencies of free vibration of continuous systems (strings, bars, beams, membranes and plates). The fundament used by Rayleigh is the principle of conservation of energy, in which the maximum potential and kinetic energy generated by the vibratory movement is assumed, considering that it has a well-known and mathematically defined aspect. This procedure became known as the "Rayleigh Method". Rayleigh used his technique to solve a limiting problem, calculating the solution with a linear approximation of the basic functions, within the denominated variational calculus, whose objective was to minimize a special class of functions, called functional or "trial functions", which might to satisfy the boundary conditions of the problem, besides being differentiable in its domain. Therefore, it is evident that the result obtained by this method is conditioned to the correct choice of these functions. Different equations that meet the boundary conditions of the problem can even so lead to different results.

For comparative purposes, four mathematical expressions as shape (or trial) function were used in the present study: a trigonometric, two polynomials and a potential equation. These functions satisfied the boundary conditions of the problem and were valid throughout the domain of the structure, being the integrals obtained by the Rayleigh method solved within the limits defined in its geometry. In order to evaluate the results obtained for that analytical procedure, the values for the critical load of buckling were compared with those yielded by computational modelling using a modal analysis by finite element method (FEM), considering a nonlinear formulation based on geometric stiffness.

The structure analyzed was a real 46-meter-high concrete pole, including its foundation, which had geometry and reinforcement arrangement varying along its length. For both solutions, three important nonlinear aspects were considered: the geometric nonlinearity, due to the slenderness of the system; the material nonlinearity and the creep of concrete. The last one was considered by means of the existing criterium in Eurocode [7]. To the analytical and computational procedure, the ground was modeled as a set of springs distributed along the foundation, being the critical buckling load defined for modal analysis for different instants of time.

2 Description of the analyzed system

The evaluated problem involves calculating the critical buckling load of a slender reinforced concrete pole with variable geometry shown in Fig. 1, where g denotes gravitational acceleration; Gr means ground; *s* represents each structural segment; *S*, *D* and *th* are the type, the external diameter, and the wall thickness of the section; d_b, n_b and *c´* represent the diameter, quantity, and the concrete cover of reinforcing bars. The structure is 46-m-high, including a 40 m superstructure with a hollow circular section and a 6 m deep full-circular foundation. The modulus of elasticity adopted for the superstructure and foundation were 37566 MPa and 25044 MPa, and their density 2600 kg/m^3 and 2500 kg/m^3, respectively. The slenderness ratio of the tower is upper than 400. A set of antennas and a platform are usually installed on its top, constituting a concentrated mass, whose limit value in relation to the loss of stability for buckling needs to be determined. Cables and a ladder are attached along of the entire length, adding a distributed mass of 40 kg/m to the system.



(a) Photos (b) Geometry “cm” (c) Cross sections (d) FEM

Figure 1. Analyzed system

The foundation is a relatively deep shaft having a bell diameter and length of 140 cm and 20 cm, shaft diameter and length of 80 cm, and 580 cm, respectively. The lateral soil resistance is represented by an elastic parameter equal to 2669 kN/m^3. The physical nonlinearity of the material was computed by multiplying the product of flexural stiffness by 0.50 and the creep of concrete considered according to the EN 1992-1-1 [7] criteria. Because of that is a reinforcement concrete (RC) structure, it was necessary to account the presence of the reinforcing bars when calculating the moment of inertia, which was performed homogenizing the cross section. Then, according to the theorem of parallel axis, the factor Fh_s, which multiplied the nominal moment of inertia of the section in terms of the total moment of inertia of the reinforcing steel, in the homogenized section were $Fh_1 = 1.0199$, $Fh_2 = 1.0568$, $Fh_3 = 1.0811$, $Fh_4 = 1.0671$, and $Fh_5 = 1.0859$, respectively for intervals from 1 to 5.

3 Analysis by Rayleigh method

When developing the Rayleigh method for calculation of the critical buckling load to the present problem, the principle of virtual work (PVW) was written in terms of the generalized coordinate defined at the free end of the column and chosen to represent the first mode shape, in an undamped free vibration. For purposes of comparison, four expressions were used as a shape function: a trigonometric, Eq. (1); two polynomials, Eq. (2) and (3); and a potential, Eq. (4); with the exponent equal to 2.27, obtained in agreement with Wahrhaftig [8]. These functions were considered valid in the whole domain of the structure. The behavior each function along the structural domain (vertical axis) can be seen in Fig. 2.

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right), \tag{1}$$

$$\phi(x) = 3\left(\frac{x^2}{2L^2}\right) - \left(\frac{x^3}{2L^3}\right), \tag{2}$$

$$\phi(x) = 3\left(\frac{x^2}{L^2}\right) - 2\left(\frac{x^3}{L^3}\right), \tag{3}$$

$$\phi(x) = \left(\frac{x}{L}\right)^{\gamma}. \tag{4}$$

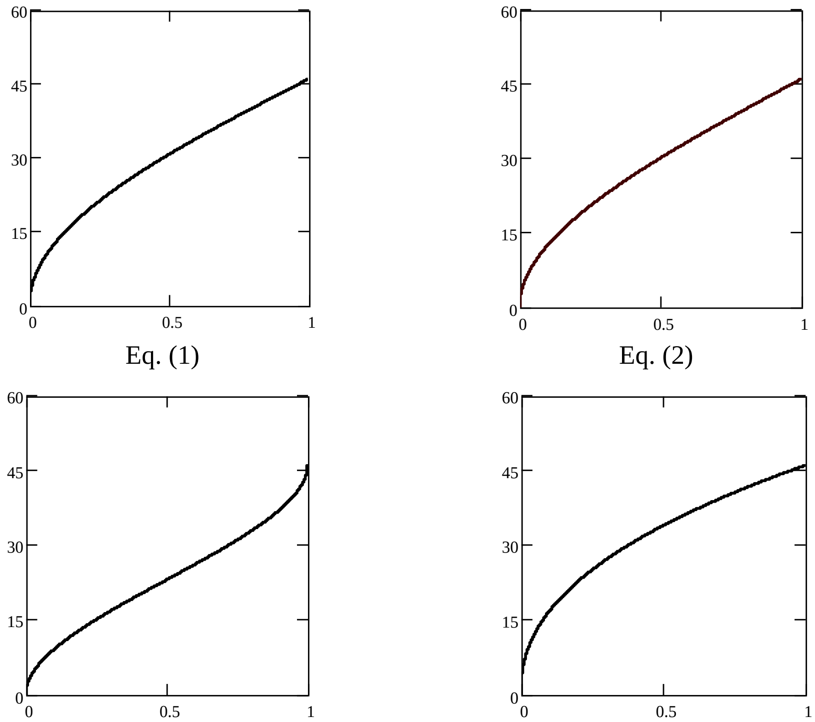


Figure 2. Shape functions along domain

Consider the model of Fig. 3, where *x* is the location of the calculation, originating at the base of the structure, and *L* is the length of the column, *t* indicates dependency of the time, *s* is a given segment, $\phi(x)$ a shape function and Gr delimits the buried part.

The system has constant and variable properties, which include an axial compressive force *N*(*x*), its geometry, elasticity or viscoelasticity and density. The viscoelasticity is related to the creep of concrete and its consideration can be made through rheological models, as done by Wahrhaftig [9]-[10], or utilizing normative criteria, as previously mentioned. Applied springs of variable stiffness $k_{so}(x)$ act as the lateral soil resistance until the foundation elevation. In this context, the column is under the action of gravitational forces, originating from distributed masses, including the self-weight, and a concentrated mass, m_0, at the upper end, whose limit value in terms of the critical buckling load will be determined.

To find the analytical solution of the problem, it is necessary to consider a shape function which restricts the problem to a system with a single degree of freedom (SDOF). Applying the PVW and its derivations, as in Wahrhaftig [11], the dynamic properties of the system are obtained. The conventional elastic/viscoelastic stiffness is given by:

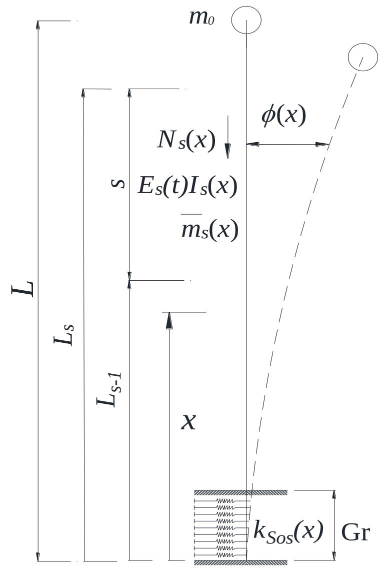


Figure 3. Frame element representing the structure in an undamped free vibration

$$k_{0s}\ \ t\ = \int_{L_{s-1}}^{L_s} E_s(t) I_s(x) \left(\frac{d^2\phi(x)}{dx^2} \right)^2 dx \text{, with, } K_0(t) = \sum_{s=1}^{n} k_{0s}(t)\,, \tag{5}$$

where for a segment s of the structure, $E_s(t)$ is the viscoelastic modulus of the material with respect to time; $I_s(x)$ is the variable moment of inertia of the section along the segment in relation to the considered movement, obtained by interpolation of the previous and following sections, already homogenized (if it is constant, it is simply I_s); $k_{0s}(t)$ is the temporal term for the stiffness; $K_0(t)$ is the final conventional stiffness varying over time; and n is the total number of segment intervals given by the structural geometry. If the material in a given interval is not a time-dependent, t vanishes of the formulation.

The geometric stiffness appears as a function of the axial load, including the self-weight contribution (Wahrhaftig [12]-[13]), and is expressed as:

$$k_{gs}(m_o) = \int_{L_{s-1}}^{L_s} \left[N_0(m_o) + \sum_{j=s+1}^{n} N_j + \bar{m}_s(x)\ \ L_s - x\ \ g \right] \left(\frac{d\phi(x)}{dx} \right)^2 dx \text{, with}$$

$$K_g(m_0) = \sum_{s=1}^{n} k_{gs}(m_0)\,, \tag{6}$$

where $k_{gs}(m_0)$ is the geometric stiffness in segment s, $K_g(m_0)$ is the total geometric stiffness of the structure with n as defined previously, and $N_0(m_0)$ is the concentrated force at the top, all of which are dependent on the mass m_0 at the tip. Further, N_j is the normal force from the upper segments, given by:

$$N_0(m_0) = m_0 g \text{, and } N_j\ = \int_{L_{s-1}}^{L_s} \bar{m}_s(x) g dx\,, \tag{7}$$

where $\bar{m}_s(x)$ is the mass per unit length. Then, the total generalized mass is given by

$$M(m_0) = m_0 + m\,, \tag{8}$$

considering that:

$$m = \sum_{s=1}^{n} m_s \text{ , with, } m_s = \int_{L_{s-1}}^{L_s} \bar{m}_s(x) \ \phi(x)^{\ 2} \, dx \text{ and } \bar{m}_s(x) = A_s(x)\rho_s \, , \tag{9}$$

where $A_s(x)$ represents the cross-sectional area and ρ_s the density of the material. If the cross section has a constant area along the interval, $A_s(x)$ will be only A_s, consequently, the mass distribution will also be constant. Similarly, if the mass m_0 does not vary, all other parameters that depend on it will also be constant. To consider the participation of the ground in the vibration of the system it is necessary to consider it as a series of vertically distributed springs along the foundation. With $k_{Sos}(x)$ denoting the spring parameter, the effective stiffness of the soil as a function of the variable x along the length can be defined as:

$$K_{So} = \sum_{s=1}^{n} k_s \text{ , with } k_s = \int_{L_{s-1}}^{L_s} k_{Sos}(x)\phi(x)^2 \, dx \text{ , and } k_{Sos}(x) = S_{ops} D_s(x) \, , \tag{10}$$

where the parameter K_{So} is an elastic characteristic consisting of the sum of $k_{Sos}(x)$ along the foundation depth, which dependents on the geometry of the foundation, $D_s(x)$, and the elastic soil parameter S_{ops}, considered constant, in this case, in each layer of soil. Considering the normal force as positive, the total structural stiffness appears as a function of two variables:

$$K(t, m_0) = K_0(t) - K_g(m_0) + K_{So} \, . \tag{11}$$

Therefore, the circular natural frequency, as a function of the time and the mass at the tip, can be calculated by:

$$\omega^2(t, m_0) = \frac{K(t, m_0)}{M(m_0)} \, . \tag{12}$$

The mathematical procedure described above constitute a modal analysis capable of calculating the critical buckling load of the structure once that all the generalized parameters are expressed as function of the mass at the top. After introducing the creep, the frequency becomes a temporal function because the modulus of elasticity varies over time. From this moment, the frequency passes to be written in terms of the time and the mass at the top, and the emerged expression from that process is sufficient to calculate the critical buckling load, determined when the frequency is zero at any arbitrary time after the structure to be placed in service. Details of this analytical procedure was presented by Wahrhaftig [14].

Taking all the previously explained in consideration, and making the mass at the top of the pole to vary, the force acting at the top also varies, as does the frequency of the structure varies according to Eq.(12). Thus, the critical buckling load, N_{buck}, is defined for a normal force at zero frequency as:

$$N_{buck} = N_0(m_0)\Big|_{\omega^2(t, m_0)=0} \, . \tag{13}$$

4 Computational modelling by FEM

It should be noted that, while the analytical solution presented in the previous section provides a single functional form for the entire problem domain, the FEM formulation establishes interpolation functions that are restricted to the domain of each finite element. In terms of modal analysis, the relevant eigenvalues and eigenvectors can be obtained by solving the following secular equation:

$$\left[K\right] - \omega^2 \left[M\right] \ \Phi = 0 \, , \tag{14}$$

where $[M]$ is the mass matrix and $[K]$ is the stiffness matrix, which includes the geometric stiffness term for nonlinear cases, formulated similarly to Eq. (11). In the FEM environment, ω^2 represents the eigenvalues and Φ represents the eigenvectors. The spring matrix that represents the soil-structure interaction is a 6 × 6 symmetric matrix of the spring coefficients, including all the translational and rotational degrees of freedom of a bar element. The components of this matrix are nodal springs. The mathematical development of the FEM system, as described above, is based on modal analysis performed by reducing the stiffness of the system using geometric stiffness matrix components and a more detailed description of it can be found in Wahrhaftig [15].

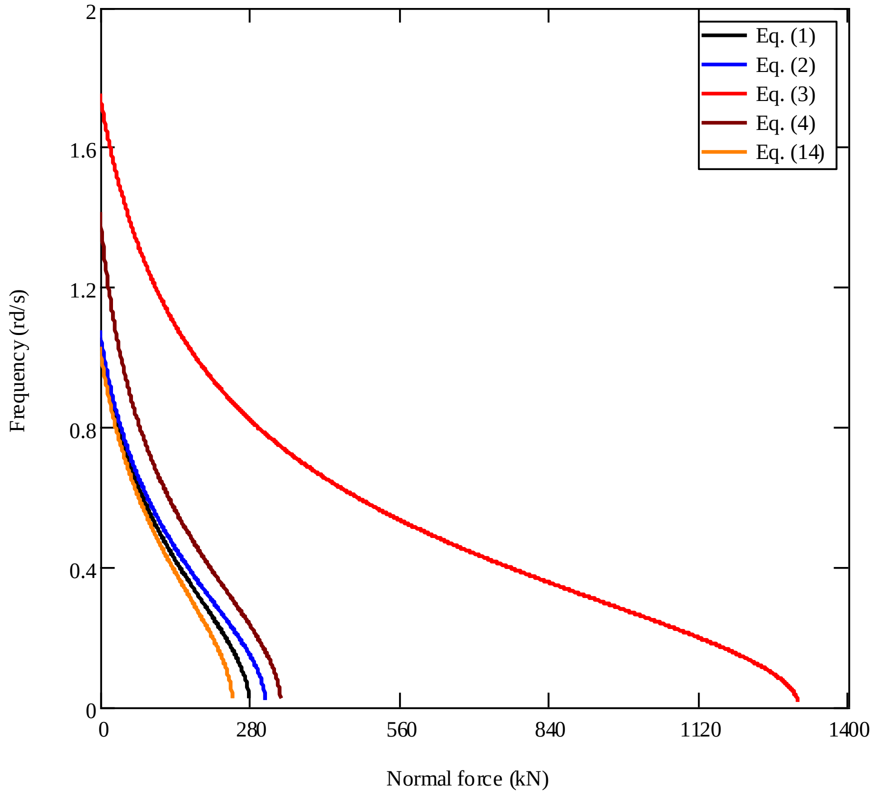
To assess the accuracy of the proposed shape equations for the critical buckling load by Rayleigh method, their results were compared with those given by an FEM computational model. The considered structure was modeled using frame elements with constant or variable cross sections, as appropriate. The varying mass was applied to the model with the corresponding axial forces, beyond of existing masses and forces present to the system. The spring factor was assigned to the foundation frame element as a linearly distributed parameter. Only a lateral spring was used to model the foundation system, and the spring stiffness matrix was thus considered to be null at all positions corresponding to the axial and rotational degrees of freedom, while the values extrapolated from the interior of each element joint were considered to be active components of the matrix. The model constructed using SAP2000 structural analysis software, such as the discretization of the structure in Fig. 1 was of 51 frame elements. Note that the global stiffness matrices of the structure were obtained automatically by the analysis software. The buckling load was determined by using the FEM and assuming an isotropic homogeneous material using the parameters employed in the analytical investigation and a Poisson's ratio of 0.2. It is important to observe that the interpolation functions used in FEM are third-degree polynomials as in Eq. (3).

5 Results and discussion

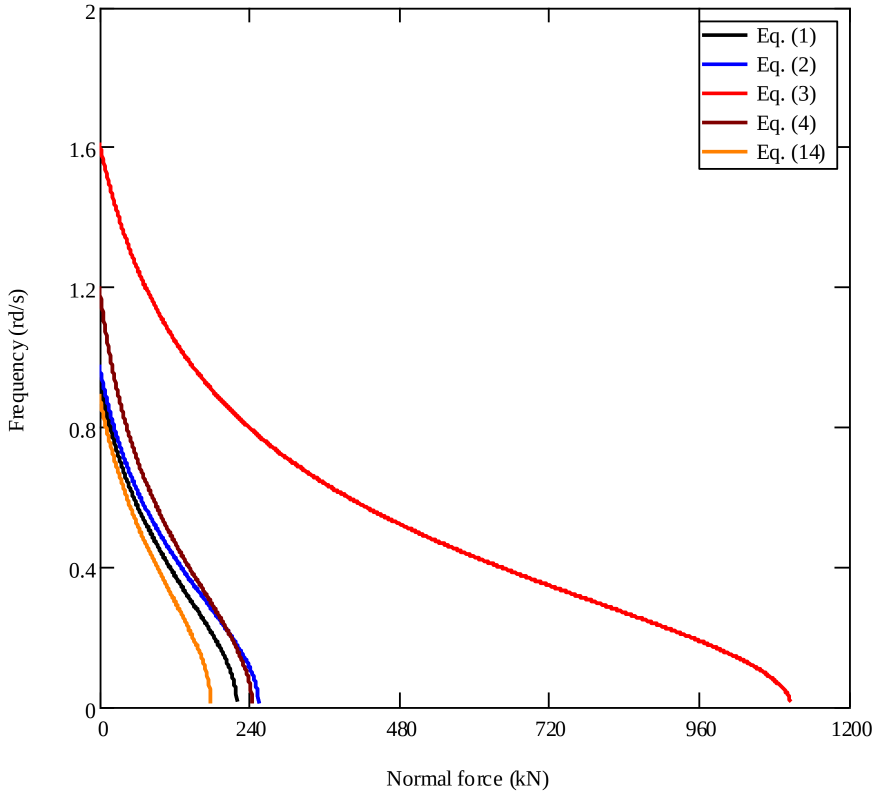
Figure 4(a) shows the values of the critical buckling load for the four adopted functions and FEM, analyzed at the time $t = 0$, instant on which the structure is loaded and for which the concrete creep has not produced any effects. Can be observed that the polynomial function given by Eq. (3) leads to a quite different result from the other equations. In order to evaluate values of the critical load at a different time of the initial one and to consider possible effects of the creep of concrete, a period of 4000 days after the structure got into service was stipulated. Figure 4(b) presents the results as analyzed for that instant. An important reduction in the vertical loading capacity of the system is observed for both instants of the time. It is also interesting to note the slight decrease in the intensity of the critical buckling load presented by the potential Eq.(4) with respect to the polynomial, Eq.(2) when analyzing for 4000 days. Table 1 summarizes the values found for each equation analyzed, considering a gravitational acceleration g = 9.80665 m/s², standard conditions of concrete production, and a 70% of environmental humidity.

Table 1. Values of the critical buckling load

Trial Function	Critical Load (kN) (t = 0)	Critical Load (kN) (t = 4000 days)	Difference (%)
Trigonometric - Eq. (1)	281.12	221.79	21.10
Polynomial - Eq. (2)	311.35	256.08	17.75
Polynomial - Eq. (3)	1308.11	1107.93	15.30
Potential - Eq. (4)	339.94	245.25	27.85
FEM - Eq.(14)	249.59	179.52	28.08



(a) $t = 0$



(b) $t = 4000$ days

Figure 4. Variation of the structural frequency and critical load

6 Conclusion

In this paper, a modal analysis has been performed to determine the critical buckling load by using an analytical process based on the Rayleigh method and finite element method (FEM). The analytical procedure was numerically solved for different shape functions while FEM was based on computational modelling processed through a software packed. A real slender reinforced concrete pole was analyzed. The structure presented variation of elastic parameters, density, reinforcement ratio and geometry throughout its length. The structural self-weight was directly considered on calculation, further of distribute mass and forces. The analytical procedure considered all the parameters necessary to a nonlinear calculation, such as the geometric, material and concrete creep. On it, four shape functions were used, all of them obeying to the boundary conditions of the problem and being differentiable on domain. For comparative porpoise, a modal analysis by FEM considering the structural stiffness obtained preliminary with all the nonlinear mentioned aspects was performed. In conclusion, is possible to state the following:

1 - For $t = 0$, instant on which the structure is loaded, the lowest critical load of 249.09 kN was obtained by using the finite element analysis, Eq. (14), while the largest of 1308.11 kN was provide by the polynomial function given by Eq. (3).
2 - When analyzing for $t = 4000$ days, the highest critical load allowed (1107.93 kN) was found through the polynomial function given by Eq. (3), while the lowest one (179.46 kN), was defined by a modal analysis through the FEM, Eq. (14).
3 - Still for 4000 days, an decrease of 4% on the intensity of the critical buckling load by Eq.(2), polynomial, in relation to Eq.(4), potential, was observed. This represents an inversion on behavior of the latter equation, once it turns back to Eq.(2) when the force on the top is approximately 215 kN.
4 - The FEM presented the biggest percentual difference between 0 and 4000 days, 28.08%.
5 - It was possible to observe that Eq. (3) led to results too distant from the other equations.

For future works comparison with other mathematical processes are expected.

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