

ASSESSMENT OF THE DYNAMIC STRUCTURAL RESPONSE OF BUILDINGS BASED ON EXPERIMENTAL MONITORING AND FEM SIMULATIONS

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Abstract. Numerical analyses based on the Finite Element Method (FEM) simulations are present in everyday routine of many academic centres and structural design offices around the world. However, one of the problems to be solved is the FEM reliability, in order to represent the actual response of the analysed structural model. Currently, it is possible to use the results of dynamic experimental monitoring to verify and adjust the numerical model and consequently improve accuracy of the results. The dynamic experimental monitoring allows the extraction of modal parameters (natural frequencies, vibration modes and damping coefficients). These parameters are relevant for a correct characterization of the investigated model and are calculated based on the experimental signals of accelerations, velocities and displacements. This way, in this work, the modal parameters, related to a real 5-story steel building constructed in laboratory are determined, based on dynamic experimental monitoring. In sequence, aiming to validate the motion equations of the structural system, both numerical and experimental dynamic responses of the physical model were correlated. The results are presented in terms of modal parameters, frequency response functions (FRFs) and time history. These modal parameters were used to calibrate the steel building FEM. The proposed computational model, developed for the investigated steel building dynamic analysis, adopted the usual mesh refinement techniques present in FEM simulations implemented in the ANSYS program. In this numerical model, the steel columns were represented by three-dimensional beam elements, where flexural and torsional effects are considered. The slabs were represented by shell elements. After that, a forced vibration analysis was carried out and the dynamic response of the building, when subjected to impact loads, was compared with the actual structural response of the steel building model. The conclusions emphasize the relevance of the dynamic experimental monitoring to characterize and adjust the developed FEM of the investigated building. The good agreement between the numerical and experimental results can corroborate the adequate use of the developed numerical model.

Keywords: Buildings, Dynamic structural analysis, Dynamic experimental monitoring, Finite element modelling.

1 Introduction

The construction of tall buildings has emerged as a constructive trend around the world, which is due to several factors, such as: population growth, urbanization of large centres, reducing the useful areas of construction, the recent technological evolution of construction materials and calculation methods adopted in recent years, enabling the construction of buildings with increasingly slender elements, among others.

Nahum and Oliveira [1] draw attention to the scarcity and high costs of spaces as the main reason for the growing trend of construction of these buildings. According to Borges et al. [2], “In the last four decades, large urban centres have shown a significant increase in the construction of multi-storey residential buildings. During this period, we observed the evolution of these buildings, which went from 20 floors in 1970 to 50 floors today”. Coupled with these needs, there is also the fact that the modern conception of the construction industry has regarded structures as true works of art, making designs increasingly challenging for engineers and architects.

Along with this trend, problems of excessive vibration or oscillation, cracking of structural elements and other damage to architectural elements are increasingly common, as well as human discomfort caused by various dynamic actions such as wind action, thermal expansion, retraction, earthquakes, repression and dynamic actions of external agents.

In Japan, in 1979, after a typhoon, Goto [3] studied the reaction of occupants of six tall buildings and found that 90% of occupants were able to feel the vibrations caused by the wind. In 1982, after another typhoon case, Goto [3] could go further, and studied the acceleration of three tall buildings, managing to relate the occupants’ perception of the building with the measured acceleration. Through his study, he found that with accelerations of 0.05 m/s^2 it was possible to feel its effects by the users of the building, and set a limit of 0.8 m/s^2 which, if exceeded, could cause extreme discomfort such as nausea, difficulty walking and problems performing routine work tasks.

More recently, the effects of wind-induced vibrations on buildings have been studied by Lamb et al. [4]. In their research, it was studied the reaction of 53 occupants of 47 commercial buildings for eight months. During the investigation it was concluded that vibration can cause loss of performance at work through feeling sick, tired, low motivation and loss of concentration. The study warns of the duration of vibration, which may cause the limit for perception to occur at lower acceleration amplitude.

Such studies based on real cases of excessive vibrations arising from dynamic actions in tall buildings show the need for dynamic experimental monitoring of these structures, because in order to know the effects of dynamic actions on structures it is necessary to follow their displacements, also finding directly or indirectly their natural frequencies.

Antunes [5] emphasizes buildings should be dynamically monitored over time, in order to prevent disasters and also to ensure user comfort and safety.

Figueiredo et al. [6] points out that the monitoring of structures is essential for a structural assessment when it is necessary to determine characteristics and properties of the structural system. However, this monitoring must take place throughout life, as mentioned by Palazzo [7], “the guarantee of the useful life is provided by monitoring over time”.

Slender architectural designs with lightly braced structures make buildings more flexible, with a low fundamental frequency of vibration, between 0 Hz and 10 Hz and therefore, in most cases susceptible to excitations due to various low frequency sources, commonly found in urban regions.

According to Moreira [8], “the monitoring and experimental measurements can be used to identify the dynamic characteristics of the affected structure, as well as to qualitatively and quantitatively identify those resulting from the dynamic actions”. Mathematical and computational models can then be elaborated and calibrated from these experimental results to initiate a dynamic analysis of the problem.

Numerical analyses based on the Finite Element Method (FEM) simulations are present in everyday routine of many academic centres and structural design offices around the world. However, one of the problems to be solved is the FEM reliability, in order to represent the actual response of the analysed structural model (Bastos [9], Bastos [10] and Barile [11]).

Ewins [12] says that, currently, it is possible to use the results of dynamic experimental monitoring to verify and adjust the numerical model and consequently improve accuracy of the results. The dynamic

experimental monitoring allows the extraction of modal parameters, such as: natural frequencies, vibration modes and damping coefficients. These parameters are relevant for a correct characterization of the studied model and calculated based on the acceleration and velocity experimental signals obtained by investigating the structure, and measured using accelerometers or laser vibrometry (Bastos [9], Bastos [10] and Barile [11]).

This way, in this work, the modal parameters, related to a real 5-story steel building constructed in laboratory (Miranda [14]) are determined, based on dynamic experimental monitoring, and used to calibrate the developed FEM (Miranda [13], Bastos and Barile [14]). After that, a forced vibration analysis is carried out and the dynamic response of the building, when subjected to impact loads, is compared to the actual structural response of the steel structural model (Miranda [13], Bastos and Barile [14]). The conclusions emphasize the relevance of using the dynamic experimental monitoring aiming to characterize and adjust the developed FEM of the investigated building.

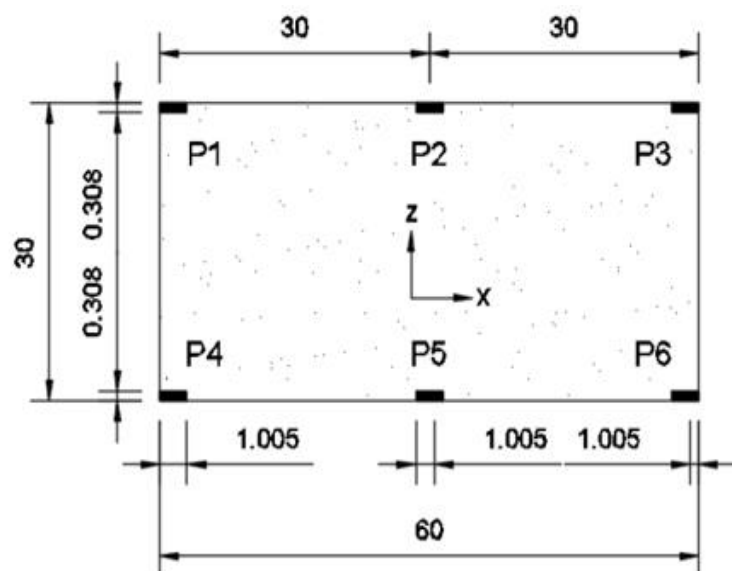
As can be seen from reading this article, the dynamic responses of the structure obtained through dynamic experimental monitoring tests, both in terms of natural frequencies and vibration modes, have a significant accuracy when compared with the responses obtained through of numerical computational analysis. Regarding the damping coefficients obtained through the tests, they present a growing behaviour, starting from the first vibration mode onwards, a tendency that is corroborated by the literature about that subject.

2 Investigated structural model

The investigated structural model is related to a 5-story steel building constructed in laboratory, see Fig. 1.a. This model simulates a building presenting indoor height between floors of 29.7cm, floor dimensions of 30cm x 60cm and total height of 150cm. The columns present dimensions of 0.34cm x 1.005cm and the slabs are 0.308cm thick. The steel presents a density equal to 7700kg/m³ and Young modulus of 210GPa, see Fig. 1.b.



a) Steel structural model



b) Floor dimensions of the investigated structural model
Units: cm

Figure 1. Investigated structural model

3 FEM of the building

The proposed computational model, developed for the steel building dynamic structural analysis, adopted the usual mesh refinement techniques present in finite element method simulations implemented in the ANSYS computational program (Ansys [15]). In this numerical model, the steel columns were represented by three-dimensional beam elements (BEAM44), where flexural and torsional effects are considered. The slab was represented by shell elements (SHELL63). Regarding the boundary conditions, the rigid support hypothesis is used, where the columns are connected to a highly rigid base plate, which is restricted in its entire contour. The developed building FEM presents an appropriate degree of refinement, allowing a good representation of the dynamic behaviour of the investigated building model, see Fig 2. The model has 2,506 nodes, 2,304 elements, which 360 are beam elements (BEAM44) and 1,944 are shell elements (SHELL63), totalizing 14,484 degrees of freedom.

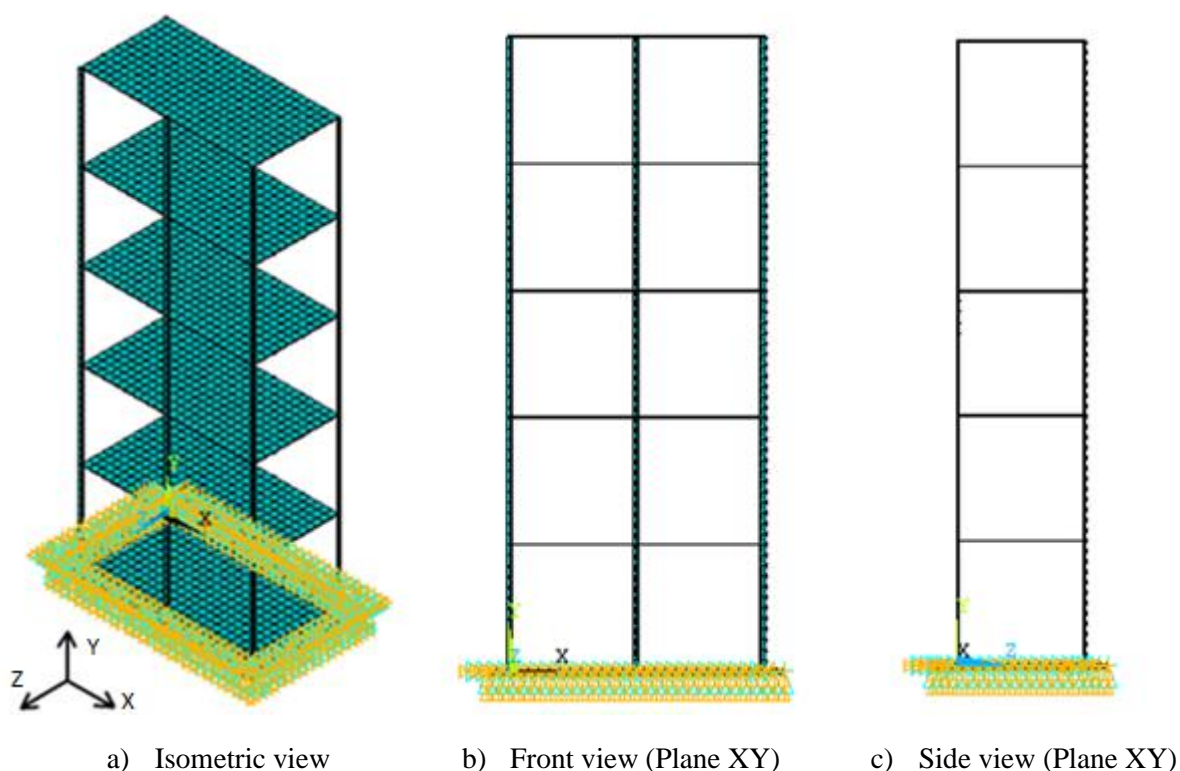


Figure 2. Finite elements model of the building

4 Modal analysis

The natural frequencies (eigenvalues) and the vibration modes (eigenvectors) of the investigated structural model were obtained by using the numerical methods of extraction (modal analysis), based on a free vibration analysis, performed with the use the program ANSYS (Ansys [15]).

It must be emphasized that the investigated structural model can vibrate in many different ways and these different mode shapes of vibration present their own natural frequency. In Figure 3, the first five bending vibration modes of the analysed structural model and their respective natural frequencies are presented. It is worth noting that the torsional vibration modes are not our object of study, therefore they will not be presented.

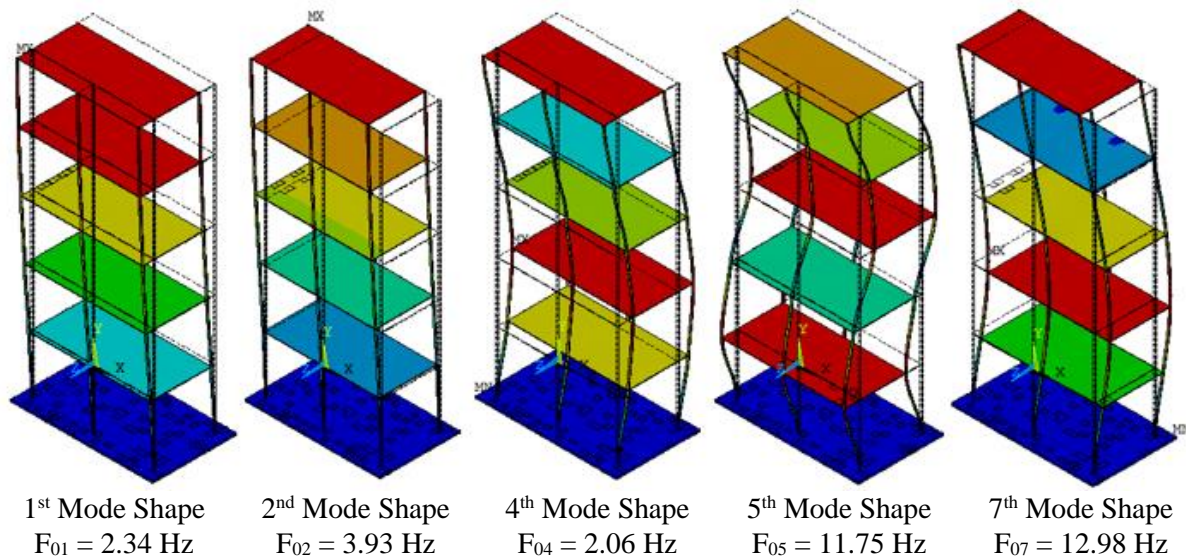


Figure 3. Bending vibration modes and natural frequencies of the building

5 Dynamic experimental monitoring

The dynamic experimental monitoring was performed based on the use of four unidirectional accelerometers, aiming to obtain the natural frequencies and vibration modes of the model. The accelerometers were positioned, each one in the centre of the slab, at the upper 4 floors of the building, at the heights of 150cm; 120cm; 90cm and 60cm, modifying the position of the accelerometers in each experimental test, in order to obtain the mode shapes of the building at the X and Z directions. In sequence, Table 1 and Table 2 present the general characteristics of the accelerometers, and their respective positions on the building.

Table 1. Resistive accelerometers


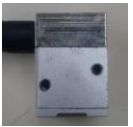
	Model of the accelerometers	Frequency range (Hz)	Shunt Eng	Height on the building (cm)
	KYOWA FU5900022	0 - 41	6.501486	100
	KYOWA FU5900024	0 - 45	7.233545	70

Table 2. Capacitive accelerometers

	Model of the accelerometers	Frequency range (Hz)	Measurement range (mV/g)	Height on the building (cm)
	DYTRAN 7521A1 S/N 3053	0 - 1500	530.62	160
	DYTRAN 7521A1 S/N 3051	0 - 1500	549.27	130

Initially, two free vibration experimental tests were performed related to X and Z directions. The test consists of causing a small horizontal displacement on the model and then let it vibrate freely. Each time domain accelerometer signal was recorded for analysis.


In order to confirm the frequency results, the experimental tests were repeated, measuring the velocity through laser vibrometry (PDV System).

The forced vibration tests were performed using an impact hammer with a coupled force cell, and its dynamic force was applied at a height of 150cm at the X and Z directions. The velocity was captured by laser vibrometry (PDV System), at a height of 150cm. The characteristics of the equipment are presented in Table 3 and Table 4, respectively.

Table 3. Characteristics of the Impact Hammer

	Model of the equipment	Frequency range (Hz)	Measurement range (mV/N)
	ICP® IMPACT HAMMER Model: 086C03	0.5 Hz – 22 kHz	2.25

Table 4. Characteristics of the PDV System

	Model of the equipment	Frequency range (Hz)	Measurement range (mV/g)
	PDV 100	0.5 Hz – 22 kHz	125

5.1 Free vibration: accelerometers

The free vibration test is performed in order to reproduce as close as possible the numerical analysis. Thus, the structure excitation was performed in order to avoid unwanted eccentricities, although this is a very difficult factor to avoid. Figure 4 shows the acceleration signal in time domain of the DYTRAN 3053 accelerometer, located at the top of the building, at a height of 150cm, and Figure 5 shows the acceleration signal in frequency domain, both in Z direction. You can also see the acceleration signal in time domain and acceleration signal in frequency domain, both related to the X directions, in Fig. 6 and Fig. 7, respectively.

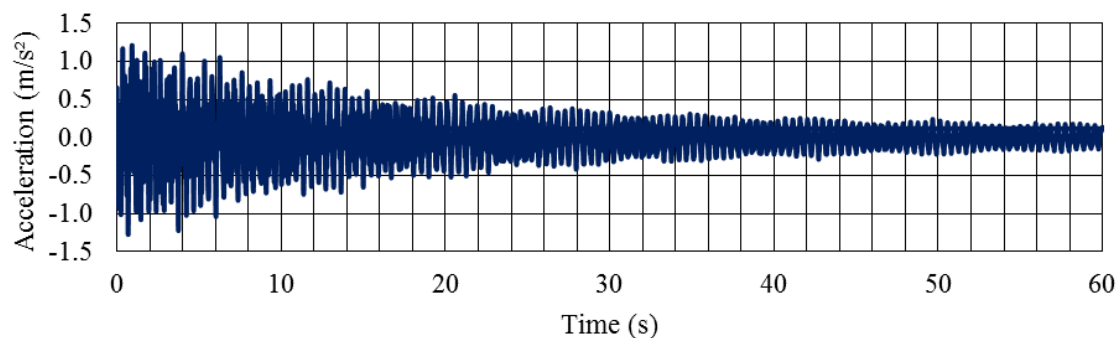


Figure 4. Experimental test using accelerometers: acceleration signal in time domain (Z direction)

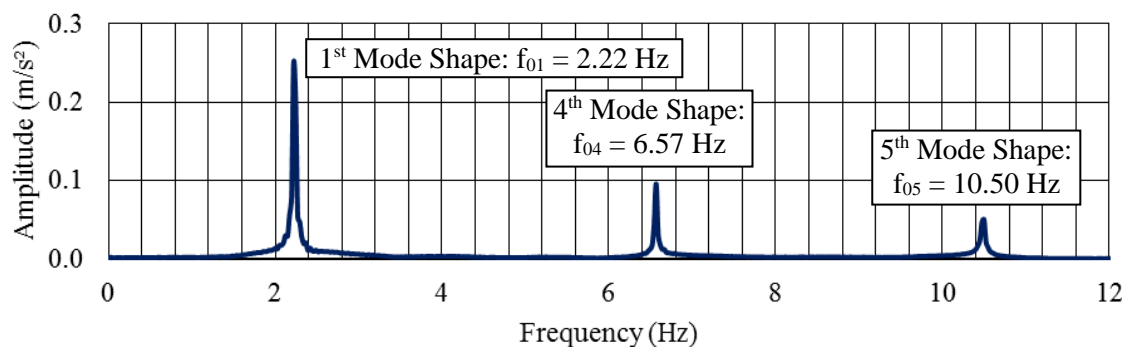


Figure 5. Experimental test using accelerometers: frequency spectrum (Z direction)

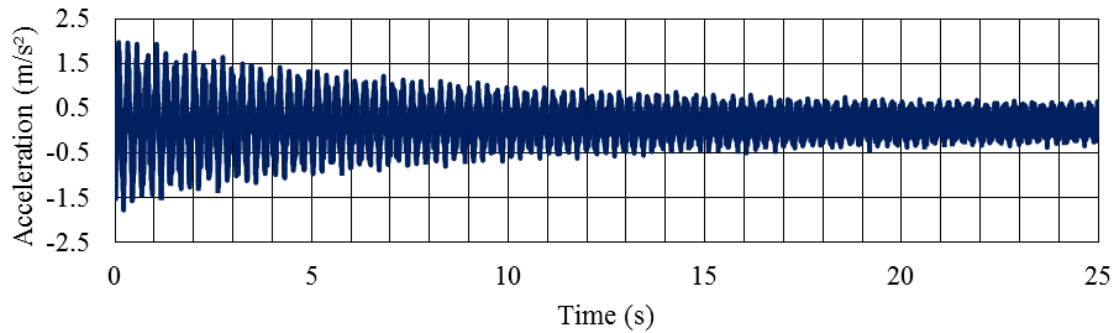


Figure 6. Experimental test using accelerometers: acceleration signal in time domain (X direction)

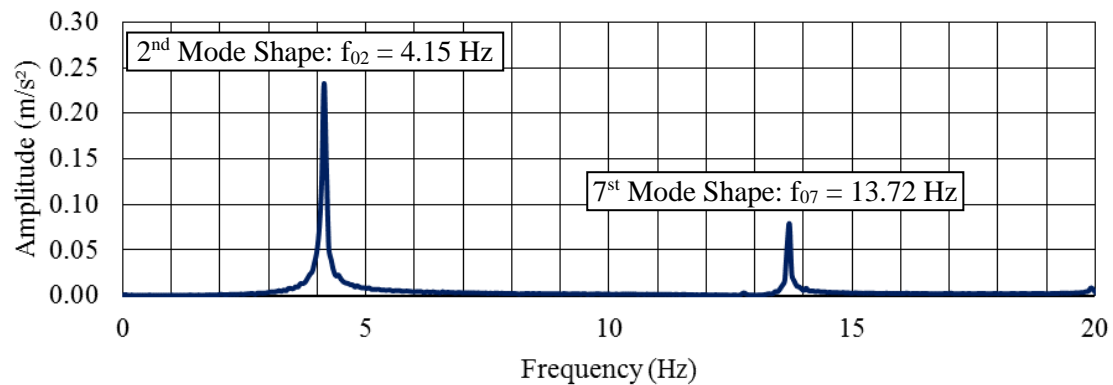


Figure 7. Experimental using with accelerometers: frequency spectrum (X direction)

It is possible to observe a very significant approximation in the values of the natural frequencies in both Z (see Fig. 5) and X (see Fig. 7) directions, when compared to the numerical model, see Fig. 3.

5.2 Free vibration: PDV System

In order to verify the reliability of the experimental results obtained based on the use of accelerometers, the same free vibration tests were also carried out using the PDV system. This way, Fig. 8 and Fig. 10 presents the velocity signals in time domain, at a height of 150cm, considering the model in free vibration parallel to the Z and X directions, respectively. The frequency spectrums is presented in Fig. 9 and Fig. 11, and it is possible to identify the natural frequencies related to the bending modal vibrations.

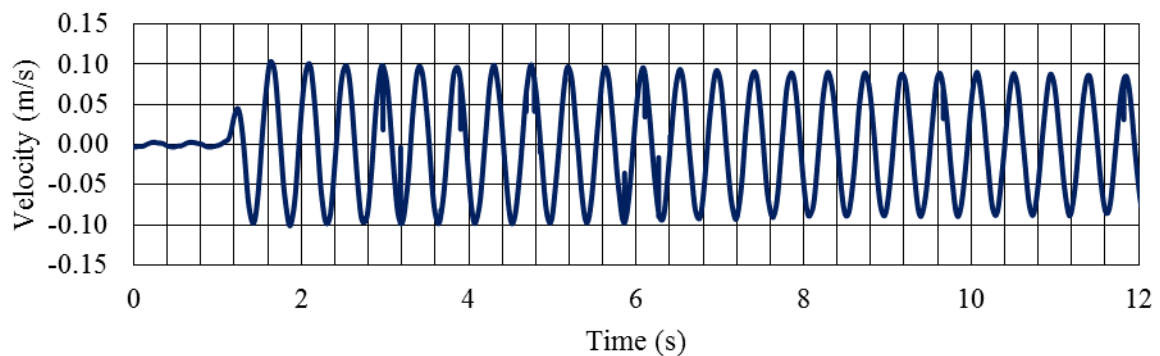


Figure 8. Experimental test using the PDV System: velocity signal in time domain (Z direction)

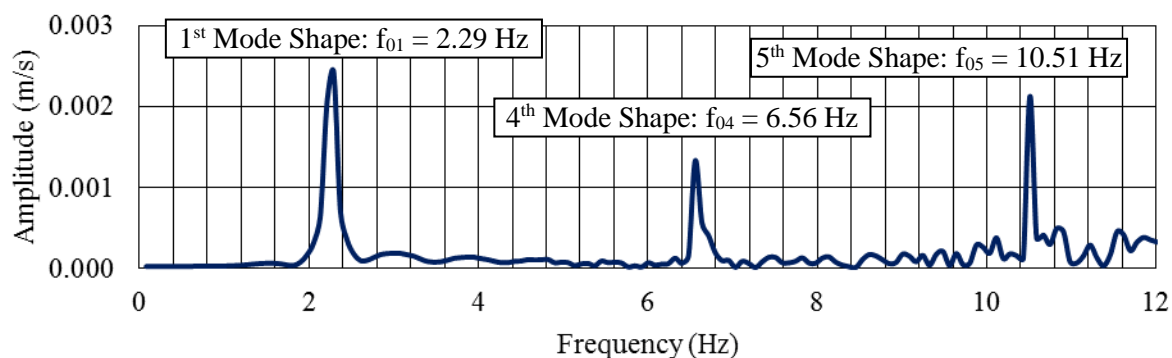


Figure 9. Experimental test using the PDV System: frequency spectrum (Z direction)

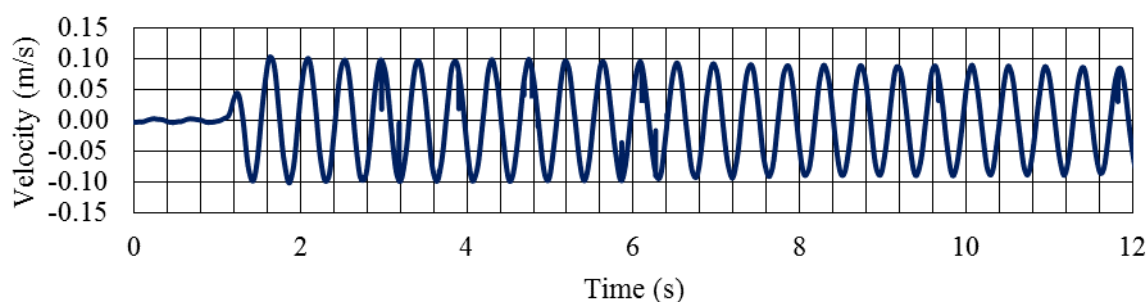


Figure 10. Experimental test using the PDV System: velocity signal in time domain (X direction)

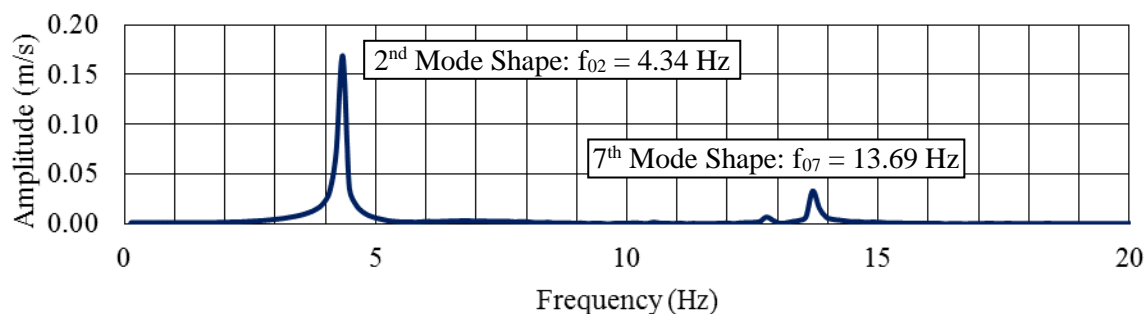
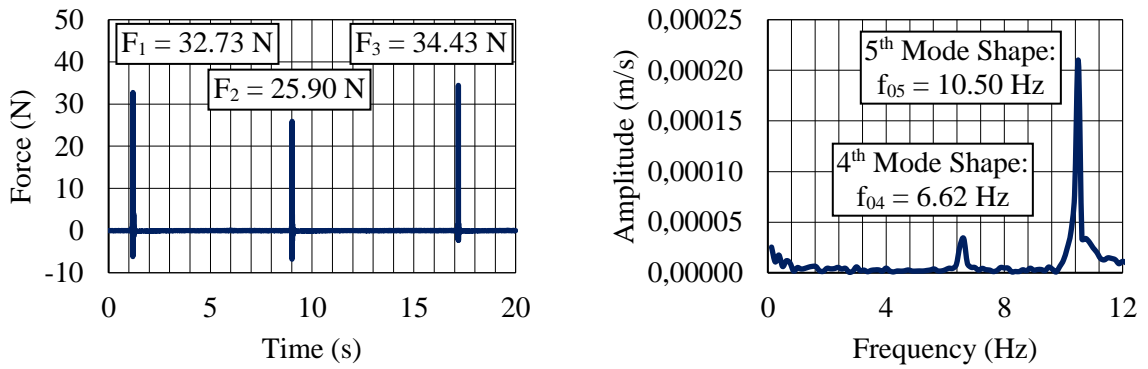


Figure 11. Experimental test using the PDV System: frequency spectrum (X direction)

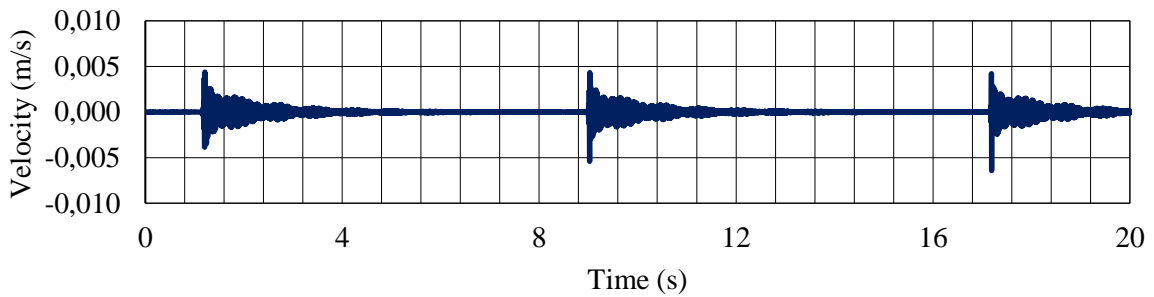
5.3 Forced vibration: PDV System

In sequence of the investigation, the structural model was subjected to forced vibration tests, based on the use of impulsive loads. The impulsive load was applied at a selected point on the structural model ($h = 150\text{cm}$), at the global Z direction, based on the use of an impact hammer connected to a force cell. In sequence, Fig. 12.a presents the impulsive load applied on the model using the impact hammer. The investigated model dynamic response, in time and frequency domains is presented in Fig. 12.c and Fig. 12.b, respectively. The same experimental test was also carried out considering the global X direction. The impulsive load is presented in Fig. 13.a and the building dynamic response in time and frequency can be observed in Fig. 13.c and Fig. 13.b, respectively. It is worth noting that the available hammer for the experiment did not have enough force to mobilize the first two vibration modes of the building. This would require the use of a larger hammer (for more details, see section 6.3).



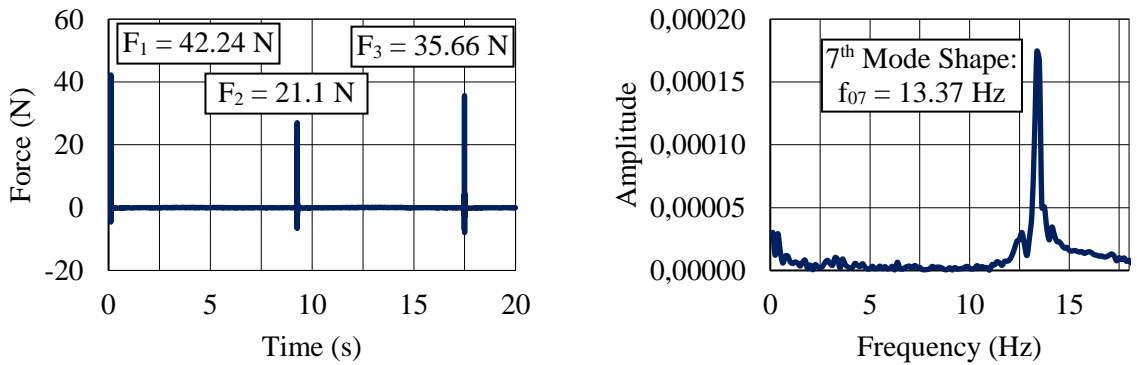
a) Force signal in time domain

b) Force Signal in frequency domain



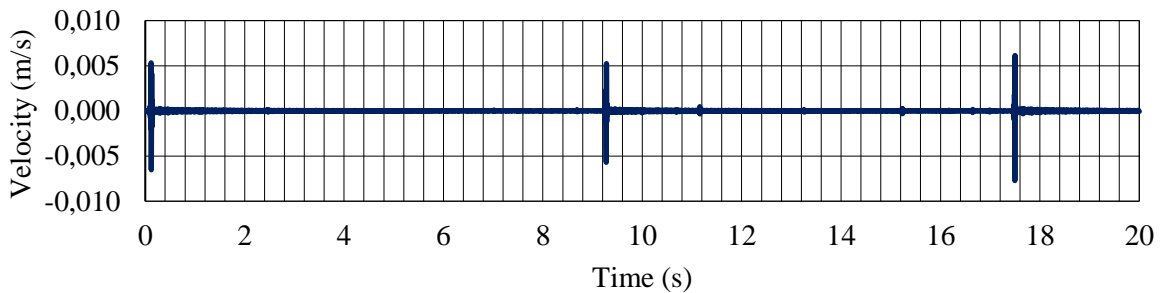
c) Velocity signal in time domain

Figure 12. Forced vibration response (Z direction)



a) Force signal in time domain

b) Force signal in frequency domain



c) Velocity signal in time domain

Figure 13. Forced vibration response (X direction)

5.4 Damping coefficients

The dynamic structural response of the investigated building is influenced by the damping coefficients. These damping coefficients determine the amplitude of vibration at the resonance and also the time of persistence of this vibration after there are no dynamic excitations. Thus, in this research work, the damping coefficients were determined for the following vibration modes based on the logarithmic decrement method, see Table 5.

Table 5. Damping coefficients

Vibration modes of the building	$\xi\%$
1 st	0.22
2 nd	0.55
4 th	0.68
5 th	0.93
7 th	0.42

6 Analysis results

In this section of the paper, the numerical results calculated using the developed FEM model (see Fig. 3) are compared with those obtained through the experimental tests (see Fig. 4 to Fig. 13).

6.1 Natural frequencies

In sequence, Table 6 presents the first five bending vibration modes and the respective natural frequencies values, obtained through the experimental tests and calculated based on the numerical modelling.

Table 6. Natural frequencies (Hz)

Mode Shape	Vibration Mode	FEM of the Building	Experimental Monitoring		Differences (%)	
			Accelerometers	PDV System	FEM/ Accelerometers	FEM/ PDV
1 st	Bending Z direction	2.34	2.22	2.29	5.12%	2.13%
2 nd	Bending X direction	3.93	4.15	4.34	5.30%	9.44%
4 th	Bending Z direction	7.06	6.57	6.56	6.94%	7.08%
5 th	Bending Z direction	11.75	10.57	10.51	10.04%	10.55%
7 th	Bending X direction	12.98	13.72	13.69	5.39%	5.19%

Based on the results presented in Table 6, it is possible to verify, quantitatively, that the order of magnitude of the natural frequencies values numerically calculated is close to those experimentally obtained. A maximum difference of the order of 10% may be noted in the results related to the fifth natural frequency.

These small differences are considered to be normal, especially when compared to other studies conducted on the subject by other authors. The highest differences are probably related to the fact that

the support conditions of the constructed steel building model were considered fixed in the FEM, but the experimental modal analysis indicated that in fact this support conditions are semi-rigid.

Another factor that may explain these differences is that the tests were performed using different methods and equipment, with different degrees of accuracy. A third factor that certainly had an influence on the final results is the possible mechanical imperfections of the devices, due to the time of use without the proper periodic calibration.

6.2 Vibration modes

The vibration modes in the Z and X directions are shown in Figures 14 and 15, respectively. A noticeable similarity is observed in all vibration modes analysed. This factor corroborates the validity of the results obtained in both experimental tests and numerical analysis of free vibration.

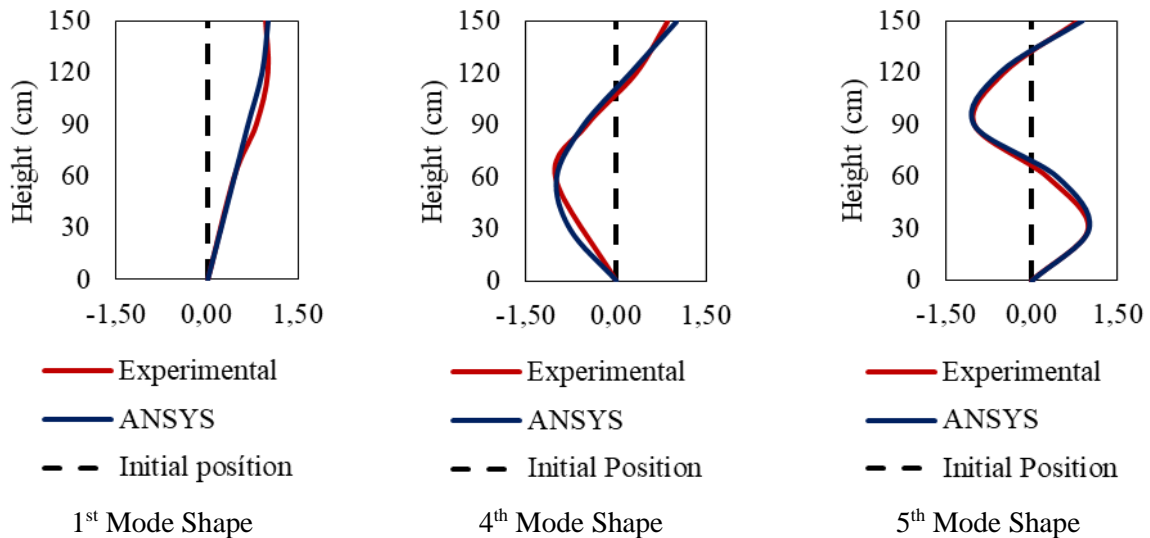


Figure 14. Mode shapes (Z Direction)

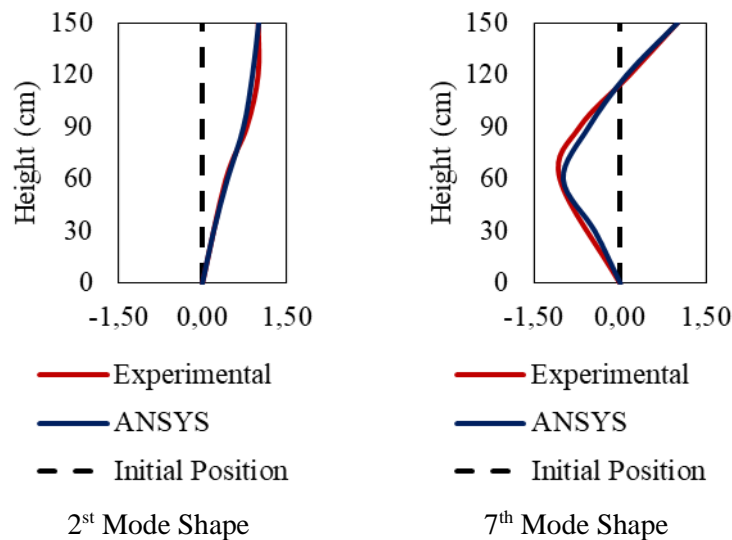


Figure 15. Mode shapes (X direction)

6.3 Forced vibration tests

Following the same strategy, the numerical dynamic structural response of the building was calculated and compared with the experimental results. In order to do this task, an impulsive load was applied at a selected point on the investigated building model, at the global Z direction ($h = 150\text{cm}$), based on the use of an impact hammer connected to a force cell (see Fig. 12.a).

The same dynamic force was applied on the FEM model and a transient analysis was carried out. The horizontal translational displacement response, along the time, was derived aiming to obtain the velocity and acceleration and in sequence to compare these values with those obtained based on the experimental tests.

This way, Fig. 16 and Fig. 17 present the experimental and numerical structural responses in time and frequency domains in the Z direction, respectively. It can be observed that, although the results calculated on the basis of a transient dynamic analysis, using the developed MEF model, are close to the experimental results obtained by the dynamic experimental monitoring in terms of acceleration in time domain, it can be observed that the experimental tests with the hammer failed to find the first two frequencies.

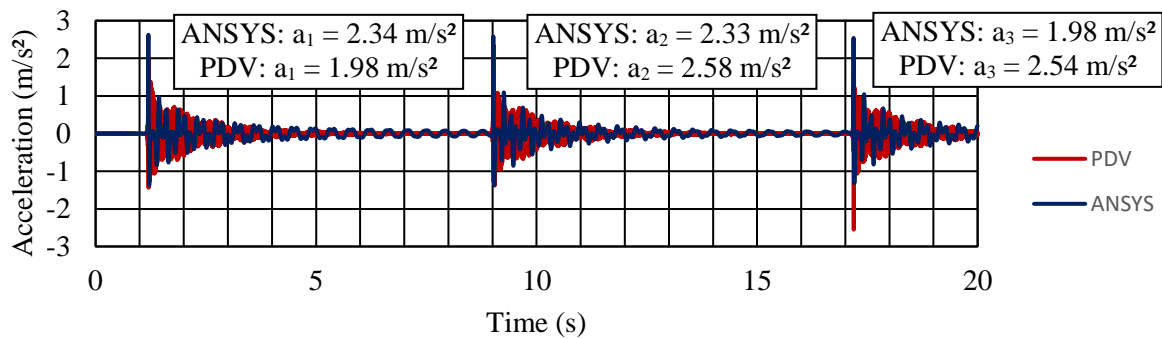


Figure 16. Velocity signals in time domain (Z direction)

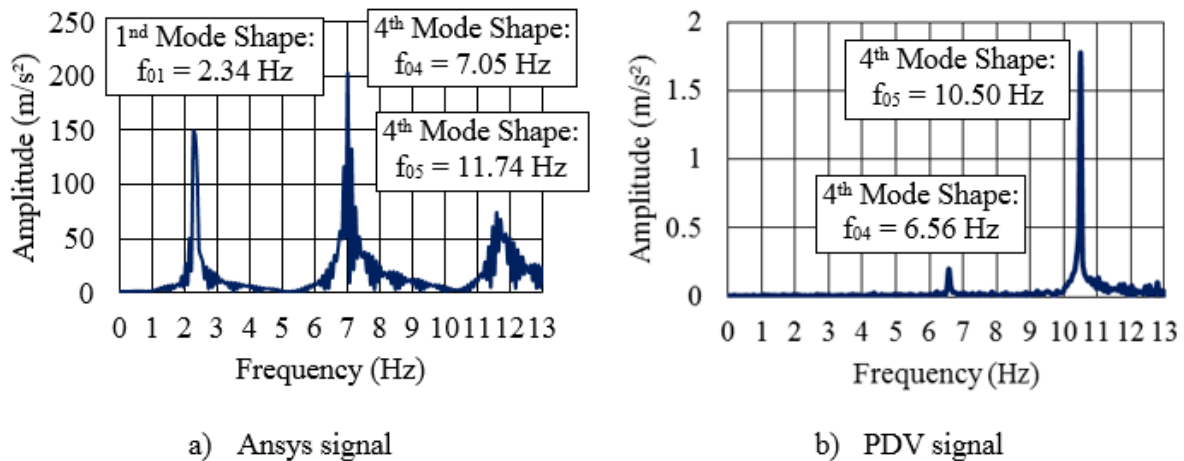


Figure 17. Frequency spectrum (Z direction)

During the performance of the tests using the impact hammer, for all attempts at frequencies between 0 Hz and 15 Hz, an unsatisfactory “coherence” between the input results performed by the hammer and the output results from the PDV was found. This “coherence” was satisfactory only for frequencies above 15 Hz. This factor negatively impacted the results, making them unsatisfactory. The hypothesis to justify this problem is that the impact hammer used for the tests was small and did not have enough energy to mobilize the modal mass of the lower natural frequencies of the structure.

Another probable hypothesis is that the structure has a geometry that was not adequate to perform this test with the hammer, presenting during the tests a different oscillatory behaviour when compared to the oscillatory behaviour obtained in the free vibration tests, presenting more torsional characteristics than of flexion.

7 Conclusions

In this work, the dynamic structural behaviour of a real 5-story steel building, constructed in laboratory, was investigated, based on dynamic experimental monitoring and FEM simulations. The main purpose of this work is to develop procedures, aiming to adjust and calibrate the numerical model, in order to represent the real structural behaviour of the building.

The results obtained along this analysis, regarding the natural frequencies values comparisons between the numerical versus experimental results, presents small differences, with the maximum difference around 10%. The highest differences are probably related to the fact that the support conditions of the constructed steel building model were considered fixed in the FEM, but the experimental modal analysis indicated that in fact this support conditions are semi-rigid. Another factor that may explain these differences is that the tests were performed using different methods and equipment, with different degrees of accuracy. A third factor that certainly had an influence on the final results is the possible mechanical imperfections of the devices, due to the time of use without the proper periodic calibration.

Based on the structural response of the accelerometers positioned at different heights of the model it was possible to obtain the vibration modes related to bending in X and Z directions. These vibration modes obtained experimentally coincided with those obtained through the numerical modal analysis. This way, it can be concluded that the developed building FEM represents the dynamic structural response of the constructed steel building model, with a very good degree of reliability.

Additionally, based on the dynamic experimental monitoring, it was possible to obtain the structural damping coefficients of the model, which are relevant to the analysis and also necessary for the adjustment and calibration of the FEM building.

Finally, considering the use of the calibrated numerical model, the dynamic structural response (forced vibration analysis) was calculated and compared with the experimental tests, and as a result, although the experimental results are close to the results of the numerical model in terms of acceleration in time domain, it can be observed that the experimental tests with the hammer did not find the first two frequencies. This event can be explained by the fact that the hammer used in the experimental tests does not have sufficient energy to mobilize the modal mass of the first and second vibration modes of the structure. Another probable hypothesis is that the structure has a geometry that was not adequate to perform this test with the hammer, presenting during the tests a different oscillatory behaviour when compared to the oscillatory behaviour obtained in the free vibration tests, presenting more torsional characteristics than of flexion.

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