HIGH ORDER COMPACT METHOD USING EXPONENTIAL DIFFERENCE SCHEMES IN THE SOLUTION OF THE CONVECTIVE DIFFUSION EQUATION

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Abstract. We will present in this paper scheme for the numerical solution of the convective-diffusive equations in incompressible, inviscid, stationary and transient flows by the technique known as *High-Order Exponential Finite Difference Schema*. Today, it is generally accepted that a realities arise in the various branches of science, such as physics, biology, chemistry, materials, engineering, ecology, bio-mechanical economics, combustion, computer science, epidemiology, finance, groundwater pollution, heat transfer, neurosciences, physiology, infiltration flow, solids mechanics, and turbulence are modeled by EDP typically similar to the RDC equation - Reaction-Diffusion-Convection Equation. In the last decades, many known numerical techniques have been applied to solve this problem. RDC equation: finite differences, finite volumes, finite elements and spectral or meshless, to name a few.
In this respect, a general-purpose numerical methodology still does not seem to be available actually. In general, the methods cited are successful when convection, reaction or combination of both acting together are largely dominated by diffusion, tending to purely diffusive process. The situation is drastically altered when convection, reaction, or a combination of both overload diffusion. In such situations, numerical instability arises in cases where diffusion becomes less predominant. Thus, the purpose of this paper is to present the above scheme.

Keywords: High-Order Exponential Difference Schemas, Convective-Diffusion Equations, Finite Difference Schemas, Computational Fluid Dynamics

1 Introduction

Here is introduced a systematic overview of the treatment of high order compact numerical methods applied to convection-diffusion equations. Covers 1D problems with coefficients of convective and diffusive constant or variable terms solved in uniform meshes. Amaral [1] e Polyakov et al. [2] states that the advection - diffusion equations form the basis of many mathematical models and are considered as one of the main components of the numerical problem in hydrodynamics.

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} = F(x); \tag{1}$$

with the boundary conditions

$$u\,(x, 0) = \varphi\,(x)\,, \qquad x \in [0, 1],$$

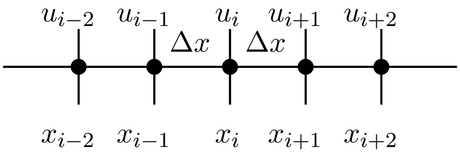
and

$$u\,(0, t) = g_1(t), \qquad u\,(1, t) = g_2(t), \quad t \in (0, T].$$

In order to present the main ideas, we will take the equation of stationary convection-diffusion to develop the concepts of high order schemes and later we expand to the transient equation.
Thus, a typical limit value between two points of a boundary value problem (PVC) that models the representation of a class of stationary transport problems: (stationary convection-diffusion equation - CDE as stated above), taking the variable u as the variable to be carried (sometimes also referred to in the technical literature by the Greek letter ϕ) of:

$$-a\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} = F(x); \tag{2}$$

with the coefficient c constant and, the source term F and the boundary conditions of Dirichlet $u(0) = 0$ and $u(1) = 1$. Note that it is required that the functions u and F have strong assumptions of regularity over the solutions and data, which increases the mathematical complexity to deduce the discrete model.
Let the space domain Ω where $x \in [0, 1]$ will be divided into evenly spaced cells of length $\delta_x = h$; using central finite differences, the discretization of above equation (2) becomes:



Estencil-x

Knowing that the series expansion of the derived terms for the finite difference discretization process is given by:

$$\frac{\partial u}{\partial x} = u_{x_i} = \delta_h u_i - \sum_{n=1}^{\infty} \frac{h^{2n}}{(2n+1)!}\delta_x^{2n+1} u_i;$$

and

$$\frac{\partial^2 u}{\partial x^2} = u_{xx_i} = \delta_h^2 u_i - \sum_{n=1}^{\infty} \frac{2h^{2n}}{(2n+2)!}\delta_x^{2n+2} u_i = \delta_h^2 u_i + O\left(h^4\right);$$

where u_{x_i} and u_{xx_i} respectively represents the first and second derivatives of u with respect to the variable x applied at the discrete point i; $\delta_x^n u_i$ is the derivative of the order n in x of the function u applied at the point i; $h = \Delta x$ is the step in the direction of the x axis; and $\delta_h^n u_i$ is the local discrete derivative of order n in x of function u applied at point i.
Developing the diffusive and convective terms of (2), order $O(h^2)$ using the above expressions and

truncating the terms of the expansion of order equal to or greater than $O(h^5)$ and, furthermore, replacing F by the original equation, we find the new fourth order equation with a residual term of order $O(h^4)$:

$$- a\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + c\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) + u_i - \varepsilon_i = F\left(x_i\right), \tag{3}$$

where

$$\varepsilon_i = \frac{h^2}{12}\left(2c\frac{d^3u}{dx^3} - a\frac{d^4u}{dx^4}\right) + O\left(h^4\right). \tag{4}$$

It is seen that ε_i is the truncation error introduced in the development of discretization by the Taylor series expansion. Note that u_i represents the approximation of the value the variable $u(x_i)$ and is indicative of the coordinate of a typical node.
In order to have a compact high-order scheme (**HOC** - **H**igh **O**rder **C**ompact Scheme) $O\left(h^4\right)$, it is necessary to retain the term of the truncation error in the discretization expression. For this, if you have to take CDE (3) without the error (4):

$$\frac{d^2u}{dx^2} = \frac{c}{a}\frac{du}{dx} + \frac{1}{a}\left(u - F\right). \tag{5}$$

Deriving the above expression in relation to x it has:

$$\frac{d^3u}{dx^3} = \frac{c}{a}\frac{d^2u}{dx^2} + \frac{1}{a}\left(\frac{du}{dx} - \frac{dF}{dx}\right). \tag{6}$$

Deriving again in relation to x,

$$\frac{d^4u}{dx^4} = \frac{c}{a}\frac{d^3u}{dx^3} + \frac{1}{a}\left(\frac{d^2u}{dx^2} - \frac{d^2F}{dx^2}\right). \tag{7}$$

Replacing (6) in (7) we have:

$$\frac{d^4u}{dx^4} = \frac{c}{a}\left(\frac{c}{a}\frac{d^2u}{dx^2} + \frac{1}{a}\left(\frac{du}{dx} - \frac{dF}{dx}\right)\right) + \frac{1}{a}\left(\frac{d^2u}{dx^2} - \frac{d^2F}{dx^2}\right), \tag{8}$$

and

$$\frac{d^4u}{dx^4} = \left(\frac{c^2 - a}{a^2}\right)\frac{d^2u}{dx^2} + \frac{c}{a^2}\left(\frac{du}{dx} - \frac{dF}{dx}\right) - \frac{1}{a}\frac{d^2F}{dx^2}. \tag{9}$$

By doing so, the above formulation retains the increased convergence rate for $O\left(h^4\right)$ as the desired hypothesis, and the resulting scheme remains compact.The system generated by the discretized scheme is clearly tridiagonal, motivated by the stencil used in the discretization of central finite differences with 3 points.
The term source $F(x)$ plays an important role in approximation of the derivatives of order 3 and 4 as seen above. If these derivatives of F are known analytically, this knowledge can be used in the discretized equations. However, if only a discrete approximation of F is known, its central differences can be used which still results in a $O\left(h^4\right)$ as Cui [3], Mishra and Yedida [4], Amaral [1], Spotz and Carey [5] and Fu et al. [6].
Note that we can deduce compact schemes of higher order by repeating the above procedure to obtain schemes of some order higher. It is important to note that high order schemes are particularly relevant in detecting pressure disturbances associated with acoustic waves whose order of magnitude is much smaller than hydrodynamic pressure fluctuations in compressible flows, for example.

2 High Order Compact Formulation - HOC

We present in this section, the development of a compact high order scheme for the advection-diffusion equation, as an introductory mode to the exponential difference schemes that will be developed in later sections.
The basic approach to high order compact difference methods is to introduce the standard compact difference approximations for the differential equations, and then by repeated differentiation and associated compact differentiation, a new high order compact scheme will be developed incorporating the effect of the main truncation.
Recently we have seen a growing development of compact finite difference methods of high order in computational dynamics of fluids, computational acoustics and electromagnetism.
To introduce the compact fourth-order finite difference scheme for the spatial derivatives of (2), can be represented by the dimensionless central difference scheme as:

$$-\frac{d^2u}{dx^2}+c\frac{du}{dx}=f\left(x\right); \tag{10}$$

where $f=\left(\frac{L^2}{au^*}\right)F\left(x\right)$; Discretizing it and neglecting truncation ϵ at point i we see that the main term is of the order $O\left(h^2\right)$:

$$-\delta_x^2u_i+c\delta_xu_i-\varepsilon_i=f_i, \tag{11}$$

where ε_i is the truncation error given in 4 e 6. The basic idea behind the HOC approach is to find compact approximations for the derivatives in the above equation by differentiating the governing equation from the problem. According to the equations (of truncation) these can be rewritten as follows Spotz and Carey [5]:

$$\left.\frac{d^3u}{dx^3}\right|_i=\left(c\frac{d^2u}{dx^2}+\frac{dc}{dx}\frac{du}{dx}-\frac{df}{dx}\right)_i=c_i\delta_x^2u_i+\delta_xc_i\delta_xu_i-\delta_xf_i+O\left(h^2\right) \tag{12}$$

and in a similar way one can write

$$\left.\frac{d^4u}{dx^4}\right|_i=\left(c\frac{d^3u}{dx^3}+2\frac{dc}{dx}\frac{d^2u}{dx^2}+\frac{d^2c}{dx^2}\frac{du}{dx}-\frac{d^2f}{dx^2}\right)_i \tag{13}$$

or as

$$\left.\frac{d^4u}{dx^4}\right|_i=c_i\left.\frac{d^3u}{dx^3}\right|_i+2\delta_xc_i\delta_x^2u_i+\delta_xc_i\delta_x^2u_i+\delta_x^2c_i\delta_xu_i-\delta_x^2f_i+O\left(h^2\right). \tag{14}$$

The equations (13) and (14) must be combined with (4) to produce a new expression for the truncation error. Thus by substituting in (4) the above expressions, all at any generic point any i of domain Ω, have:

$$\varepsilon_i=\frac{h^2}{12}\left[(c_i^2-2\delta_xc_i)\delta_x^2u_i+(c_i\delta_xc_i-\delta_x^2c_i)\delta_xu_i-c_i\delta_xf_i+\delta_x^2f_i\right]+O\left(h^4\right). \tag{15}$$

The above equation (15) is clearly of a higher order compared to the equation (4) and will be used to improve the accuracy of the solution of the advection-diffusion equation. In this way the compact high order scheme can be derived from the above equation by adding to the coefficients of the original equation analyzed (2) the respective coefficients of the derivative and source term found in (15) . Thus, in a simplified way, one can write the HOC scheme as Spotz and Carey [5]:

$$-A_i\delta_x^2u_i+C_i\delta_xu_i=F_i+O\left(h^4\right); \tag{16}$$

making the necessary algebrisms, now using the new error (15) if one has:

$$\delta_x^2u_i+c_i\delta_xu_i+\epsilon_i=f_i,$$

$$\left[1+\frac{h^2}{12}\left(c_i-2\delta_xc_i\right)\right]\delta_x^2u_i+\left[c_i+\frac{h^2}{12}\left(c_i\delta_xc_i-\delta_x^2c_i\right)\right]\delta_xu_i=f_i+\frac{h^2}{12}\left(\delta_x^2f_i-c_i\delta_xf_i\right), \tag{17}$$

comparing the last expression (17) above with (16) we see that the new coefficients are:

$$A_i=1+\frac{h^2}{12}(c_i^2-2\delta_xc_i);C_i=c_i+\frac{h^2}{12}\left(c_i\delta_xc_i-\delta_x^2c_i\right);F_i=f_i+\frac{h^2}{12}\left(\delta_x^2f_i-c_i\delta_xf_i\right). \tag{18}$$

2.1 Compact High Order Scheme for ECD Transient

In the transient case (1), we take the solution for the stationary case - see Fu et al. [6], Amaral [1], Tian and Ge [7] and Spotz and Carey [8] simply write $g = F - \frac{\partial u}{\partial t}$ applied at any point i. Remember that, without loss of generality, the deductions were made considering the diffusion coefficient $a = 1$, but to observe the temporal effect of both the diffusion and the convection, the coefficient of the convection with $\frac{c_i}{a}$. Like this,

$$\left\{\frac{\partial u}{\partial t}\right\}_i = -\frac{h^2}{12a}\frac{\partial}{\partial t}\left(c\delta_x u_i - a\delta_x^2 u_i\right) + c\delta_x u_i - \left(a + \frac{c^2h^2}{12a}\right) = f_i - \frac{h^2}{12a}\left(c_i\delta_x f_i - a\delta_x^2 f_i\right) + O\left(h^4\right), \tag{19}$$

one must now proceed to integration in time by some solver. High-order solving algorithms are quite complicated, but a second-order Runge-Kutta algorithm can be used when stability and convergence are desired instead of precision and agility.

By taking a two-stage solver involving the more classical methods, one can apply the differentiation at time t^n of the form $t_\zeta = (1-\mu)t^n + \mu t^{n+1}$. In doing so, we find the following scheme:

$$\delta_t^+ u_i^n - \frac{h^2}{12a}\left(c\delta_t^+\delta_x u_i^n - a\delta_t^+\delta_x^2 u_i^n\right) + (1-\mu)\left[c\delta_x u_i^n - \left(a + \frac{c^2h^2}{12a}\right)\delta_x^2 u_i^n\right] + \mu\left[c\delta_x u_i^{n+1} - \left(a + \frac{c^2h^2}{12a}\right)\delta_x^2 u_i^{n+1}\right] = (1-\mu)\left[f_i^n - \frac{h^2}{12a}\left(c\delta_x f_i^n - a\delta_x^2 u_i^n\right)\right] + \mu\left[f_i^{n+1} - \frac{h^2}{12a}\left(c\delta_x f_i^{n+1} - a\delta_x^2 f_i^{n+1}\right)\right] + O\left(h^4\delta t\right), \tag{20}$$

where $\delta_x^2 u_i^{n+1} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}$; and $\delta_t^+ u_i = \frac{u_i^{n+1} - u_i^n}{\Delta t}$ and so on.

3 Exponential Compact High Order Scheme - EHOC

Recently there have been published several articles that deal with this topic, among which Tian and Yu [9], Polyakov [10], Mishra and Yedida [4], Fu et al. [6], Tian and Ge [7], Cui [3], Jha and Kumar [11], Cao et al. [12].

The exponential schemes of finite differences applied in uniform meshes modifies the finite difference operator according to the behavior of the differential equation to be solved. Here we take into account that the fields between the nodal points are locally approximated using a set of exponential base functions, type $\{1, x, exp[\pm(\nu_x x + \nu_y y + \nu_z z)]\}$ for the most general (3D) case, where ν_x, ν_y, ν_z are complex exponents in the directions x , y and z.

This method has some important properties such as non-oscillation in convection-diffusion problems, and that it tends to the traditional finite difference method, when the exponents of the exponential functions tend to zero, that is, when $\nu_x \to 0, \nu_y \to 0, \nu_z \to 0$. Although the method has a good numerical stability structure, in cases where the problem has convective predominance or cells with a high Reynolds number, the scheme may produce a slight spurious, non-physical behavior in the diffusion.

The solution of the generic transient convection-diffusion equation begins with the solution of the associated permanent equation Spotz and Carey [8], Tian and Yu [9] and Mahdi [13]. Thus, in the case of equation 1D, we have $\nu_y \to 0, \nu_z \to 0$. Then the equation in steady state, in the domain $\Omega = (0, 1)$ with the boundary conditions $u(0) = \varphi_1$ and $u(1) = \varphi_2$; thus, discretizing has:

$$-a\delta_x^2 u_i + c\delta_x u_i = F_i \tag{21}$$

For the schema to be exponential, we introduce in any interval any $[x_{i-1}, x_{i+1}]$ that the solution of the problem is of the form $exp(-\frac{c_i x}{a})$ so that the equation (10) can be rearranged as follows:

$$-a\frac{d}{dx}\left(e^{-\frac{c_i x}{a}}\frac{du}{dx}\right) = e^{-\frac{c_i x}{a}}f. \tag{22}$$

Integrating the expression above on the interval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and remembering that using the finite central difference scheme the approximation of first derivative at a point i is given by:

$$\frac{du_i}{dx} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} + O\left(h^2\right),$$

it has

$$c_i \left[e^{-\frac{c_i h}{2a}} \left(\frac{du}{dx}\right)_{i+\frac{1}{2}} - e^{\frac{c_i h}{2a}} \left(\frac{du}{dx}\right)_{i-\frac{1}{2}} \right] = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} e^{-\frac{c_i x}{a}} f_i dx; \tag{23}$$

making $\eta = \frac{c_i h}{2a}$ and $u_x = \frac{du}{dx}$ result in

$$c_i \left[e^{-\eta} \left(u_x\right)_{i+\frac{1}{2}} - e^{\eta} \left(u_x\right)_{i-\frac{1}{2}} \right] = \left(e^{-\frac{c_i h}{2a}} - e^{-\frac{c_i h}{2a}} \right) f_i. \tag{24}$$

Using the approximation in central differences for the points $u_{i+\frac{1}{2}}$ and $u_{i+\frac{1}{2}}$ in (24):

$$c_i \left[e^{-\eta} \left(\frac{u_{i+1} - u_i}{h}\right) - e^{\eta} \left(\frac{u_i - u_{i-1}}{h}\right) \right] = \left(e^{\eta} - e^{-\eta}\right) f_i.$$

In the first portion of the term the left we multiply and divide by $2h$ and adding and subtracting u_{i-1} in the second portion of the term the left of the expression above multiplying and dividing by 2 and adding and subtracting u_{i+1}, you have:

$$\frac{c_i h}{2} \left[e^{-\eta} \left(\frac{2u_{i+1} - 2u_i + u_{i-1} - u_{i-1}}{h^2}\right) \right] - c_i \left[e^{\eta} \left(\frac{2u_i - 2u_{i-1} + u_{i+1} - u_{i+1}}{2h}\right) \right] = \left(e^{\eta} - e^{-\eta}\right) f_i.$$

Rearranging the terms one has

$$\begin{aligned}
&\frac{c_i h}{2} \left[u_{i+1} \left(e^{\eta} + e^{-\eta}\right) - 2u_i \left(e^{\eta} + e^{-\eta}\right) + u_{i-1} \left(e^{\eta} + e^{-\eta}\right) \right] + \\
&\frac{c_i}{2h} \left[u_{i+1} \left(e^{\eta} - e^{-\eta}\right) - u_{i-1} \left(e^{\eta} - e^{-\eta}\right) \right] = \left(e^{\eta} - e^{-\eta}\right) f_i, \\
-&\frac{c_i h}{2} \left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \right] \left[\frac{e^{\eta} + e^{-\eta}}{e^{\eta} - e^{-\eta}} \right] + \\
&c_i \left[\frac{u_{i+1} - u_{i-1}}{2h} \right] \left[\frac{e^{\eta} + e^{-\eta}}{e^{\eta} - e^{-\eta}} \right] = f_i, \\
-&\frac{c_i h}{2} \coth(\eta) \delta_{h_x}^2 u_i + c_i \delta_{h_x} u_i = f_i.
\end{aligned} \tag{25}$$

Then the expression of the second-order exponential scheme for the convection-diffusion equation can be finally written as

$$-\alpha \delta_h^2 u_i + c \delta_h u_i = F_i; \tag{26}$$

remember that

$$\alpha = \left(\frac{ch}{2}\right) \coth\left(\frac{ch}{2a}\right) \Longleftrightarrow c \neq 0; \quad \text{e} \quad \alpha = a \Longleftrightarrow c = 0; \tag{27}$$

because (23) has exact solution for $e^{\frac{cx}{a}}$.

For the development of the high order compact exponential scheme, we have several ways of combining the source factors F_i, according to different authors. Thus, we will see in the following subsections some of these stationary and transient forms .

In order to make the high precision order scheme (23), F_i can be set as a linear combination of source terms and their derivatives on the chosen stencil. This scheme is suggested, among other authors, as Mishra and Yedida [4] and others.

Since the stencil we are working on is 3-points, then by making the above-mentioned linear combination, one has an approximation of the exponential scheme in finite differences at a point x_i with step h: $-\alpha\delta_h^2 u_i + c\delta_h u_i = F_i$.
As in the case of HOC, here the combination of F_i to make the high-order precision schema (19) is, according to Tian and Ge [7], Tian and Yu [9], Fu et al. [6]:

$$\begin{aligned} F_i &= \left(\gamma_1 + \gamma_2\delta_x + \gamma_3\delta_x^2\right) f_i \\ &= \gamma_1 f_i + \gamma_2 f_{x_i} + \gamma_3 f_{xx_i}. \end{aligned} \tag{28}$$

3.1 EHOC when the ECD coefficients are constant

In this variant of the exponential scheme, the convection coefficient is constant and we try to obtain the coefficients of f so that we have the exact solution of (24). Thus, one has:

$$\begin{aligned} -\alpha\delta_h^2 u_i + c\delta_h u_i &= \left(\gamma_1 + \gamma_2\delta_x + \gamma_3\delta_x^2\right) f_i \\ &= \gamma_1 f_i + \gamma_2 f_{x_i} + \gamma_3 f_{xx_i}. \end{aligned} \tag{29}$$

Rewriting (29), so as to substitute in the values of f_i, f_{x_i} and f_{xx_i} to its equation $-au_{xx}+cu_x$ of equality, its first derivative and its second derivative respectively, always applied at the point i has:

$$-\alpha\delta_h^2 u_i + c\delta_h u_i = \gamma_1\left(-au_{xx} + cu_x\right)_i + \gamma_2\left(-au_{xx} + cu_x\right)_{x_i} + \gamma_3\left(-au_{xx} + cu_x\right)_{xx_i} \tag{30}$$

Performing the operations indicated in (30) and developing the terms $\delta_h^2 u_i = u_{xx_i}$ and $\delta_h u_i = u_{x_i}$ by the expressions below the result of its development by Taylor series, truncating in the desired order):

$$u_{x_i} = \delta_h u_{xi} - \sum_{n=1}^{\infty} \frac{h^{2n}}{(2n+1)!}\delta_x^{2n+1} u_i; \tag{31}$$

$$u_{xx_i} = \delta_h^2 u_i - \sum_{n=1}^{\infty} \frac{2h^{2n}}{(2n+2)!}\delta_x^{2n+2} u_i = \delta_h^2 u_i\left(h^4\right); \tag{32}$$

By doing the substitutions and the algebraic operations, and zeroing one finds each of the coefficients of the terms u_{x_i}, u_{xx_i}, $\delta_x^3 u_i$ and $\delta_x^4 u_i$, that will appear, one has:

$$\left(\gamma_1 - 1\right)c = 0 \longrightarrow \gamma_1 = 1; \tag{33}$$

$$-\gamma_1 a + \gamma_2 c + \alpha = 0 \longrightarrow \gamma_2 = \begin{cases} \frac{a-\alpha}{c} \Leftrightarrow c \neq 0; \\ 0 \Leftrightarrow \quad c = 0; \end{cases} \tag{34}$$

$$-\gamma_2 a + \gamma_3 c - \frac{ch^2}{6} = 0 \longrightarrow \gamma_3 = \begin{cases} \frac{a(a-\alpha)}{c} + \frac{h^2}{6} \Leftrightarrow c \neq 0; \\ \frac{h^2}{12} \qquad\qquad \Leftrightarrow c = 0; \end{cases} \tag{35}$$

Taking these results in (29) we obtain the high order exponential scheme, for constant and nonzero coefficients: $a, c =$ constants.

3.2 EHOC when the ECD coefficients are variable

For the case of variable coefficients, we rewrite (30) with the coefficients α and c applied at a point i:

$$-\alpha_i\delta_h^2 u_i + c_i\delta_h u_i = \gamma_1\left(-au_{xx} + c_i u_x\right)_i + \gamma_2\left(-au_{xx} + c_i u_x\right)_{x_i} + \gamma_3\left(-au_{xx} + c_i u_x\right)_{xx_i}. \tag{36}$$

Applying the same concepts and formulations applied to equation with constant coefficients, we find:

$$\left(\gamma_1 - 1\right)c_i = 0 \longrightarrow \gamma_1 = 1; \tag{37}$$

$$-\gamma_1 a + \gamma_2 c_i + \alpha_i = 0 \longrightarrow \gamma_2 = \begin{cases} \frac{a-\alpha_i}{c_i} \Leftrightarrow c_i \neq 0; \\ 0 \Leftrightarrow \quad c_i = 0; \end{cases} \tag{38}$$

$$-\gamma_2 a + \gamma_3 c_i - \frac{c_i h^2}{6} = 0 \longrightarrow \gamma_3 = \begin{cases} \frac{a(a-\alpha_i)}{c_i} + \frac{h^2}{6} \Leftrightarrow c \neq 0; \\ \frac{h^2}{12} \qquad \Leftrightarrow c_i = 0; \end{cases} \tag{39}$$

In view of this, using the equation (10) and its Taylor series expansion, we can find an equation similar to the schema found above:

$$-au_{xx} + cu_x - 2\gamma_3 c_x u_{xx} - (\gamma_2 c_x + \gamma_3 u_{xx}) u_x + O\left(h^4\right) = f, \tag{40}$$

where $c_x = -\frac{c_i h^2}{12a}$; and $c_{xx} = \frac{h^2}{12}$, and note that now (27) needs to be rewritten applied at a point i as follows:

$$\alpha_i = \left(\frac{c_i h}{2}\right) \coth\left(\frac{c_i h}{2a}\right) \Longleftrightarrow c_i \neq 0; \quad \text{e} \quad \alpha_i = a \Longleftrightarrow c_i = 0; \tag{41}$$

Rearranging the equation (40) has:

$$-(a - 2\gamma_3 c_x) u_{xx} + (c + \gamma_2 c_x + \gamma_3 c_{xx}) u_x + O\left(h^4\right) = f$$

using equalities $A_m = (a - 2\gamma_3 c_x)$ and $C_m = (c + \gamma_2 c_x + \gamma_3 c_{xx})$ we have:

$$-A_m u_{xx} + C_m u_x = f \tag{42}$$

It is seen that the equations 10 and 42 have the same shape and are $O\left(h^4\right)$, to construct the EHOC scheme for the studied problem. Thus, if applying at a point i, it can be written that:

$$-A_{f_i} u_{xx_i} + C_{f_i} u_{x_i} = \Gamma_1 f_i + \Gamma_2 f_{x_i} + \Gamma_3 f_{xx_i}, \tag{43}$$

which resolving the similarity of what has been done previously, is:

$$A_{f_i} = \begin{cases} \frac{C_{f_i} h}{2} \coth\left(\frac{C_{f_i} h}{2A_m}\right) & \Leftrightarrow C_{f_i} \neq 0; \\ A_m & \Leftrightarrow \qquad C_{f_i} = 0; \end{cases}, \tag{44}$$

where

$$\Gamma_1 = 1; \tag{45}$$

$$\Gamma_2 = \begin{cases} \frac{A_m - A_{f_i}}{C_{f_i}} & \Leftrightarrow C_{f_i} \neq 0; \\ 0 \qquad \Leftrightarrow & \qquad C_{f_i} = 0; \end{cases} \tag{46}$$

$$\Gamma_3 = \begin{cases} \frac{A_m\left(A_m - A_{f_i}\right)}{c} + \frac{h^2}{6} & \Leftrightarrow C_{f_i} \neq 0; \\ \frac{h^2}{12} \qquad \Leftrightarrow & \qquad C_{f_i} = 0; \end{cases} \tag{47}$$

Tian and Ge [7] ensures that the scheme formed by the equations of (43) to (47) produces a diagonally dominant tridiagonal system, is an EHOC whose solution incorporates an additional artificial perturbation in the diffusive coefficient of $a\left[\frac{C_m h}{2a} \coth \frac{C_m h}{2A_m} - 1\right]$, a perturbation in the coefficient of the same convection as $\gamma_2 c_x + \gamma_3 c_{xx}$ and an additional perturbation in the source term equal to $\Gamma_2 f_x + \Gamma_3 f_{xx}$.

4 Extension of EHOC Schemes for transient ECD

In order to extend the high order exponential schemes deduced in the previous section to the transient convection-diffusion equation, Tian and Yu [9], Tu et al. [14], Cui [3], Mahdi [13] and Amaral [1] where the function u it's not just space dependent but also time dependent, like $u(x,t)$. For this, simply change in (29), when the coefficients are constant,$u(x)$ by $u(x,t)$ and $f(x)$ by $-\frac{\partial u}{\partial t}+g(x,t)$. therefore

$$-\alpha\delta_h^2 u_i^n + c\delta_h u_i^n = \left(\gamma_1+\gamma_2\delta_x+\gamma_3\delta_x^2\right)\left(-\frac{\partial u}{\partial t}+g(x,t)\right). \tag{48}$$

Developing the terms, it has:

$$\begin{aligned}\left(\frac{\gamma_3}{h^2}-\frac{\gamma_2}{2h}\right)\left[\frac{\partial u}{\partial t}-g(x,t)\right]_{i-1}^n &+ \left(1-\frac{2\gamma_3}{h^2}\right)\left[\frac{\partial u}{\partial t}-g(x,t)\right]_i^n \\ &+ \left(\frac{\gamma_3}{h^2}+\frac{\gamma_2}{2h}\right)\left[\frac{\partial u}{\partial t}-g(x,t)\right]_{i+1}^n \\ &= \left(\frac{\alpha}{h^2}+\frac{c}{2h}\right)u_{i-1}^n - \frac{2\alpha}{h^2}u_i^n + \left(\frac{\alpha}{h^2}-\frac{c}{2h}\right)u_{i+1}^n \\ &+ O\left(h^4\right).\end{aligned} \tag{49}$$

See the equation above, it is verified that it forms a system along with the initial condition:

$$u\left(x,0\right)=\varphi\left(x\right), \qquad x\in[0,1], \tag{50}$$

and with boundary conditions like as pure Dirichlet type:

$$u\left(0,t\right)=p_1, \qquad u\left(1,t\right)=p_2, \quad t\in(0,T], \tag{51}$$

or boundary conditions as pure Neumann type:

$$\left(\frac{\partial u}{\partial x}\right)_{x_0}=g_1\left(t\right) \quad \text{em} \quad x=x_0=0; \qquad \left(\frac{\partial u}{\partial x}\right)_{x_N}=g_2\left(t\right) \quad \text{em} \quad x=x_N=1; \tag{52}$$

or mixed contour conditions of the Neumann and Dirichlet type:

$$u\left(0,t\right)=p_1; \qquad \left(\frac{\partial u}{\partial x}\right)_{x_N}=g_2\left(t\right) \quad \text{em} \quad x=x_N=1; \tag{53}$$

5 Treatment of the Initial and Boundary Conditions

In both transient and stationary ECD cases, the initial and boundary conditions must be treated in order for the resulting EHOC scheme to have the desired order.

Now we have to incorporate the boundary conditions in the EHOC scheme. If pure Dirichlet boundary conditions are established at both ends, the insertion is automatic since we only have to replace the given values at the ends.

However, when the conditions are Neumann (or mixed), which involves derivatives, one has to guarantee its high order compactness (Spotz and Carey [8]). For this, we take each boundary condition of the Neumann type and develop its expansion in series, similar to what was done in the equation (36). Thus, considering both ends with Neumann conditions (see Tian and Yu [9], Polyakov et al. [2], Fu et al. [6] and Cui [3]), we have the following serial expansions of the boundary conditions given at the extreme $x=x_0=0$:

$$\begin{aligned}\frac{\partial u\left(x_0,t\right)}{\partial x} &= \frac{u\left(x_1,t\right)-u\left(x_0,t\right)}{h}+\sigma_1\frac{h}{2}\frac{\partial^2 u\left(x_0,t\right)}{\partial x^2} \\ &+ \sigma_2\frac{h}{2}\frac{\partial^2 u\left(x_1,t\right)}{\partial x^2}+\sigma_3\frac{h^2}{6}\frac{\partial^3 u\left(x_0,t\right)}{\partial x^3}+O\left(h^4\right),\end{aligned} \tag{54}$$

and in the extreme $x = x_N = 1$:

$$\begin{aligned} \frac{\partial u\left(x_N.t\right)}{\partial x} = \frac{u\left(x_N,t\right)-u\left(x_{N-1},t\right)}{h} + \sigma_4 \frac{h}{2}\frac{\partial^2 u\left(x_N,t\right)}{\partial x^2} \\ + \sigma_5 \frac{h}{2}\frac{\partial^2 u\left(x_{N-1},t\right)}{\partial x^2} + \sigma_6 \frac{h^2}{6}\frac{\partial^3 u\left(x_N,t\right)}{\partial x^3} + O\left(h^4\right). \end{aligned} \tag{55}$$

In order to determine the coefficients σ_i in (54) and (55) it will be necessary to apply to each of the above equations a value a $u(x,t)$, which is an exact solution locally (boundary conditions are local); for both local solutions are normal and exponential monomials, as well as trivial solutions.
Thus, by replacing in the above formulas the variable $U(x,t)$ by the set of solutions $\left\{1, x, x^2, x^3, e^{cx/a}\right\}$ of each instead of taking into account that the derivatives that appear in (54) and (55) need to be developed, it has:

$$\begin{aligned} \frac{\partial^2 u\left(x_0,t\right)}{\partial x^2} &= \frac{1}{a}\left(c\frac{\partial u}{\partial x} - F\right)_{x_0} = \frac{c}{a}g_1 - \frac{1}{a}F_0; \\ \frac{\partial u\left(x_N,t\right)}{\partial x} &= \frac{1}{a}\left(c\frac{\partial u}{\partial x} - F\right)_{x_N} = \frac{c}{a}g_2 - \frac{1}{a}F_N; \end{aligned} \tag{56}$$

and

$$\begin{aligned} \frac{\partial^3 u\left(x_0,t\right)}{\partial x^3} &= \frac{1}{a}\left(c\frac{\partial^2 u}{\partial x^2} - \frac{\partial F}{\partial x}\right)_{x_0} = \frac{c^2}{a^2}g_1 - \frac{c}{a}F_{x_0} - \frac{1}{a}\left(\frac{\partial F}{\partial x}\right)_{x_1}; \\ \frac{\partial^3 u\left(x_N,t\right)}{\partial x^3} &= \frac{1}{a}\left(c\frac{\partial^2 u}{\partial x^2} - \frac{\partial F}{\partial x}\right)_{x_N} = \frac{c^2}{a^2}g_1 - \frac{c}{a}F_{x_N} - \frac{1}{a}\left(\frac{\partial F}{\partial x}\right)_{x_N}. \end{aligned} \tag{57}$$

Taking the expressions (56) and (57) into (54) and (55) and replacing u with the set of solutions, $\eta = ch/a$ is the coefficients, first those due to the Neumann condition on the left edge:

$$\begin{aligned} \sigma_1 = \frac{1}{2} + \beta_1, \quad \sigma_2 = -\beta_1, \quad \sigma_3 = -\frac{1}{6} + \beta_1, \\ \therefore \beta_1 = \begin{cases} \frac{1}{\eta^2} + \frac{3+\eta}{6(\eta+1-e^{\eta})} & \Leftrightarrow c \neq 0; \\ \frac{1}{12} & \Leftrightarrow c = 0; \end{cases} \end{aligned} \tag{58}$$

and now the values of the coefficients due to Neumann boundary condition on the right edge:

$$\begin{aligned} \sigma_4 = \frac{1}{2} - \beta_2, \quad \sigma_5 = \beta_2, \quad \sigma_6 = \frac{1}{6} - \beta_2, \\ \therefore \beta_2 = \begin{cases} \frac{1}{\eta^2} + \frac{3-\eta}{6(-\eta+1-e^{-\eta})} & \Leftrightarrow c \neq 0; \\ \frac{1}{12} & \Leftrightarrow c = 0; \end{cases} \end{aligned} \tag{59}$$

For the sake of simplification, we can substitute the coefficients in (54) and in (55); Thus, let Taylor serially develop the parameters β_1 e β_2; taking as variable the term η, since, being the parameters c and a constant, the fact variable is the parameter h. Thus, there are respectively:

$$\begin{aligned} \beta_1 &= \frac{1}{12} - \frac{\eta}{90} - \frac{\eta^2}{2160} + \frac{\eta^3}{11340} \\ &+ \frac{17\eta^4}{1360800} - \frac{\eta^5}{2041200} - \frac{23\eta^6}{97977600} + O\left(h^7\right); \\ \beta_2 &= \frac{1}{12} + \frac{\eta}{90} - \frac{\eta^2}{2160} - \frac{\eta^3}{11340} \\ &+ \frac{17\eta^4}{1360800} + \frac{\eta^5}{2041200} - \frac{23\eta^6}{97977600} + O\left(h^7\right). \end{aligned} \tag{60}$$

Truncating the above series in order to have $O\left(h^2\right)$ due to the fact that the expressions (54) and (55) are already $\mathbf{0}\left(h^4\right)$, we have the rewritten coefficients σ_i, just for comprehension purposes, as (since in the final formulation the former expression of σ_i generic):

$$\sigma_1 = \frac{1}{2} + \beta_1 = \frac{7}{12} - \frac{c}{90a}h; \quad \sigma_2 = -\beta_1 = -\frac{1}{12} - \frac{c}{90a}h;$$
$$\sigma_3 = -\frac{1}{6} + \beta_1 = -\frac{1}{12} - \frac{c}{90a}h; \quad \sigma_4 = \frac{1}{2} - \beta_2 = \frac{5}{12} - \frac{c}{90a}h;$$
$$\sigma_5 = \beta_2 = \frac{1}{12} + \frac{c}{90a}h; \quad \sigma_6 = \frac{1}{6} - \beta_2 = \frac{1}{12} - \frac{c}{90a}h.$$

Before replacing in (54) and (55), it is necessary to find the definitive expressions of the second and third derivatives that appear in these equations. Thus, namely, in the expression due to the left Neumann boundary conditions, the second derivatives appear in x_0 and x_1, and the third derivative in x_0, which below we detail:

$$\begin{aligned}\frac{\partial^2 u\left(x_0,t\right)}{\partial x^2} &= \frac{1}{a}\left(c\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right)_{x_0} = \frac{c}{a}g_1(t) + \frac{1}{a}\left(\frac{\partial u(x,t)}{\partial t}\right)_{x_0};\\ \frac{\partial^3 u(x_0,t)}{\partial x^3} &= \frac{1}{a}\left(c\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{a}\frac{\partial u}{\partial u \partial t}\right)_{x_0} = \frac{c^2}{a^2}g_1(t) + \frac{c}{a}\left(\frac{\partial u(x,t)}{\partial t}\right)_{x_0} + \frac{1}{a}g_1^{'}(t);\end{aligned} \tag{61}$$

To develop the second derivative at the point $x = x_1$, it is replaced by the remainder of the terms of the homogeneous ECD equation, where it contains a derivative $\frac{\partial u}{\partial x}$ and in this case it is necessary to develop the function $u(x,t)$ in series and to apply on it the derivative; In doing so, we see that

$$\begin{aligned}\frac{\partial^2 u\left(x_1,t\right)}{\partial x^2} &= \frac{1}{a}\left(c\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right)_{x_1}\\ &= \frac{c}{a}\left(c\frac{\partial}{\partial x}\left(u(x_[0],t) + hu^{'}(x_0,t) + \frac{h^2}{2}u^{''}(x_0,t)\right)\right) + \frac{1}{a}\frac{\partial u(x_1,t)}{\partial t}\\ &= \frac{c}{a}g_1(t) + \frac{ch}{a}\frac{\partial^2 u(x_0,t)}{\partial x^2} + \frac{ch^2}{2a}\frac{\partial^3 u(x_0,t)}{\partial x^3} + \frac{1}{a}\frac{\partial u(x_1,t)}{\partial t}.\end{aligned} \tag{62}$$

In the expression due to the boundary conditions of Neumann on the right, second derivatives appear in x_N and x_{N-1}, and a third derivative in x_N, which will be made below - similar to what was done on the left edge. Like this,

$$\begin{aligned}\frac{\partial^2 u\left(x_N,t\right)}{\partial x^2} &= \frac{1}{a}\left(c\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right)_{x_N} = \frac{c}{a}g_2(t) + \frac{1}{a}\left(\frac{\partial u(x,t)}{\partial t}\right)_{x_N};\\ \frac{\partial^3 u(x_0,t)}{\partial x^3} &= \frac{1}{a}\left(c\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{a}\frac{\partial u}{\partial u \partial t}\right)_{x_N} = \frac{c^2}{a^2}g_2(t) + \frac{c}{a}\left(\frac{\partial u(x,t)}{\partial t}\right)_{x_N} + \frac{1}{a}g_2^{'}(t);\end{aligned} \tag{63}$$

Similarly to what has already been done on the left edge, the second derivative must be developed at the point $x = x_{N-1}$, it is replaced by the equation, where it contains a derivative $\frac{\partial u}{\partial x}$ and in this case we need to develop the function $u(x,t)$ in series and apply the derivative on it; In doing so, we see that

$$\begin{aligned}\frac{\partial^2 u\left(x_{N-1},t\right)}{\partial x^2} &= \frac{1}{a}\left(c\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right)_{x_{N-1}}\\ &= \frac{c}{a}\left(c\frac{\partial}{\partial x}\left(u(x_[N,t) + hu^{'}(x_N,t) + \frac{h^2}{2}u^{''}(x_N,t)\right)\right) + \frac{1}{a}\frac{\partial u(x_{N-1},t)}{\partial t}\\ &= \frac{c}{a}g_2(t) + \frac{ch}{a}\frac{\partial^2 u(x_N,t)}{\partial x^2} + \frac{ch^2}{2a}\frac{\partial^3 u(x_N,t)}{\partial x^3} + \frac{1}{a}\frac{\partial u(x_{N-1},t)}{\partial t}.\end{aligned} \tag{64}$$

Now we are able to make the definite substitutions of (61) and (62) and other expressions required in (54) and (1):

$$A_e = B_e + C_e; \tag{65}$$

where the auxiliary variables are:

$$A_e = \left[\left(\frac{1}{2} - \beta_1\right)\frac{h}{a} + \frac{ch^2}{6a^2} + \beta_1 \frac{c^2h^3}{2a^3}\right]\left(\frac{\partial u}{\partial t}\right)_0^n + \beta_1 \frac{h}{a}\left(\frac{\partial u}{\partial t}\right)_1^n;$$

$$B_e = -\frac{1}{h}u_0^n + \frac{1}{h}u_1^n + \left(-1 - \frac{1}{2}\frac{ch}{a} - \frac{c^2h^2}{6a^2} - \beta_1 \frac{c^3h^3}{2a^3}\right) g_1(t);$$

$$C_e = \left[\left(-\frac{1}{6} + \beta_1\right)\frac{h^2}{a} - \beta_1 \frac{ch^3}{2a^2}\right] g_1^{'}(t).$$

Making similarly for the pertinent equations to the right end of the contour, we find the following expression:

$$A_d = B_d + C_d, \tag{66}$$

where the auxiliary variables are:

$$A_d = \beta 2 \frac{h}{a}\left(\frac{\partial u}{\partial t}\right)_{N-1}^n + \left[\left(\frac{1}{2} - \beta_2\right)\frac{h}{a} - \frac{ch^2}{a^2} + \beta_2 \frac{c^2h^3}{2a^3}\right]\left(\frac{\partial u}{\partial t}\right)_N^n;$$

$$B_d = \frac{1}{h}u_{N-1}^n - \frac{1}{h}u_N^n + \left(1 - \frac{ch}{2a} + \frac{c^2h^2}{6a^2} - \beta_2 \frac{c^3h^3}{2a^3}\right) g_2(t);$$

$$C_d = \left[\left(\frac{1}{6} - \beta_1\right)\frac{h^2}{a} - \beta_2 \frac{ch^3}{2a^3}\right] g_2^{''}(t).$$

These are the EHOC equations.

From the above equations, we can see that they form a system of first order ordinary differential equations, given by:

$$\begin{cases} \mathbf{A}\left(\frac{d\mathbf{U}(t)}{dt}\right) = \mathbf{B}\mathbf{U}(\mathbf{t}) + \mathbf{g}; \\ \mathbf{U}(\mathbf{0}) \qquad = \varphi_{\mathbf{0}}. \end{cases} \tag{67}$$

$$\begin{aligned} \mathbf{U_i^0} &= \varphi_{xi}, i = 0, 1, \ldots, N+1; \\ \mathbf{U_0^n} &= p_1(t_n), \\ \mathbf{U_{N+1}^n} &= p_2(t_n), \quad n = 1, 2, \ldots, M. \end{aligned} \tag{68}$$

The matrix A, without bound boundary conditions, has the following form:

$$\mathbf{A} = tridiag(\zeta_1, \zeta_2, \zeta_3) \begin{bmatrix} \zeta_2 & \zeta_3 & 0 & \ldots & \ldots & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & \ldots & 0 \\ 0 & \zeta_1 & \zeta_2 & \zeta_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ldots & 0 & \zeta_1 & \zeta_2 & \zeta_3 \\ 0 & \ldots & \ldots & 0 & \zeta_1 & \zeta_2 \end{bmatrix},$$

being that

$$\begin{aligned} \zeta_1 &= \left(\frac{\gamma_3}{h^2} - \frac{\gamma_2}{2h}\right); \zeta_2 = \left(1 - \frac{2\gamma_3}{h^2}\right); \zeta_3 = \left(\frac{\gamma_3}{h^2} + \frac{\gamma_2}{2h}\right); \\ a_1 &= \left[\left(\frac{1}{2} - \beta_1\right)\frac{h}{a} + \frac{ch^2}{6a^2} + \beta_1 \frac{c^2h^3}{2a^3}\right]; \\ a_{N+1} &= \left[\left(\frac{1}{2} - \beta_2\right)\frac{h}{a} + \frac{ch^2}{6a^2} + \beta_2 \frac{c^2h^3}{2a^3}\right] \end{aligned} \tag{69}$$

With the coefficients γ_2 and γ_3 given respectively by the equations (38) and (39). The matrix $\mathbf{B}$ is succinctly given by:

$$\mathbf{B} = tridiag\left(\chi_1, \chi_2, \chi_3\right) \begin{bmatrix} \chi_2 & \chi_3 & 0 & \dots & \dots & 0 \\ \chi_1 & \chi_2 & \chi_3 & 0 & \dots & 0 \\ 0 & \chi_1 & \chi_2 & \chi_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & \chi_1 & \chi_2 & \chi_3 \\ 0 & \dots & \dots & 0 & \chi_1 & \chi_2 \end{bmatrix},$$

being that

$$\chi_1 = \left(\frac{\alpha}{h^2} + \frac{c}{2h}\right); \quad \chi_2 = \left(1 - \frac{2\alpha}{h^2}\right); \quad \chi_3 = \left(\frac{\alpha}{h^2} - \frac{c}{2h}\right); \tag{70}$$

In Tian and Yu [9], Fu et al. [6] and Cui [15] we find that the matrices $\mathbf{A}$ and $\mathbf{B}$ are strictly diagonally dominant and thus are non-singular, admitting the existence of inverse. Then, from (67) one can do:

$$\begin{cases} \frac{d\mathbf{U}(t)}{dt} = \mathbf{A}^{-1}\mathbf{B}\mathbf{U}(t) + \mathbf{A^{-1}g}; \\ U(0) = \varphi_0; \end{cases} \tag{71}$$

The exact solution of this system is

$$\mathbf{U}\left(t\right) = -\mathbf{B^{-1}g} + exp(t\mathbf{A^{-1}B})\left(\mathbf{U}(0) + \mathbf{B^{-1}g}\right), \tag{72}$$

and, by similarity, obeys the following recurrence relation (substituting the terms $\mathbf{U}(t)$ by $\mathbf{U}(t_n + \Delta t)$ and $\mathbf{U}(0)$ by $\mathbf{U}(t_{tn})$):

$$\mathbf{U}\left(t_n + \Delta t\right) = -\mathbf{B^{-1}g} + exp(t\mathbf{A^{-1}B})\left(\mathbf{U}\left(t_n\right) + \mathbf{B^{-1}g}\right), \tag{73}$$

rearranging to put the term $\mathbf{U}(t_n)$ in evidence, so one has:

$$\mathbf{U}\left(t_n + \Delta t\right) = \left(exp(\Delta t\mathbf{A^{-1}B}) - \mathbf{I}\right)\mathbf{B^{-1}g} + exp(\Delta t\mathbf{A^{-1}B})\left(\mathbf{U}\left(t_n\right)\right). \tag{74}$$

By doing the serial expansion of the term $\mathbf{U}\left(\mathbf{t}\right)$ in time $t = t_n + \Delta t$,

$$\begin{aligned} \mathbf{U}\left(\mathbf{t_n} + \mathbf{\Delta t}\right) &= \left(1 - \Delta t\frac{d}{dt} + \frac{\Delta t^2}{2!}\frac{d^2}{dt^2} + \dots\right)\mathbf{U}\left(\mathbf{t}\right) \\ &= exp\left(\Delta t\frac{d}{dt}\right)\mathbf{U}\left(\mathbf{t}\right). \end{aligned} \tag{75}$$

In order to find the recurrence relation of the implicit system in its final form, the Padé approximation will be used (note that the exponential function is approximated by rational fractions, somewhat analogous to the approximation by polynomials with integer series; the polynomials form a sequence, the Padé approximations define a dual input matrix called the Padé table, in which the approximate index (p, q) is in the p-column and in the q-line, for the discretization of time - see Zhou et al. [16]:

$$\begin{aligned} exp\left(\Delta t\frac{d}{dt}\right) &= \frac{1 + \frac{1}{2}\left(\Delta t\frac{d}{dt}\right) + \frac{1}{12}\left(\Delta t\frac{d}{dt}\right)^2}{1 - \frac{1}{2}\left(\Delta t\frac{d}{dt}\right) + \frac{1}{12}\left(\Delta t\frac{d}{dt}\right)^2} \\ exp\left(\Delta t\mathbf{A^{-1}B}\right) &= \frac{\left[\mathbf{I} + \frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right) + \frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}{\left[\mathbf{I} - \frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right) + \frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}. \end{aligned} \tag{76}$$

Note that $\mathbf{I}$ is the identity matrix of order $N-1$. Taking (76 into (74), one has:

$$\mathbf{U}\left(\mathbf{t_n}+\mathbf{\Delta t}\right)=\left(\frac{\left[\mathbf{I}+\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}{\left[\mathbf{I}-\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}-\mathbf{I}\right)\mathbf{B^{-1}g} + \left(\frac{\left[\mathbf{I}+\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}{\left[\mathbf{I}-\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]}\right)\mathbf{U}\left(t_n\right). \tag{77}$$

By calling $\mathbf{U}\left(\mathbf{t_n}+\mathbf{\Delta t}\right)$ of $\mathbf{U}^{n+1}$ and $\mathbf{U}\left(\mathbf{t_n}\right)$ of $\mathbf{U}^n$ in the above formula, and rearranging it will have the final expression of the high-order exponential scheme for the convection- 1D transient, with truncation error of the order of $O\left(\Delta t^4, h^4\right)$:

$$\mathbf{U}^{n+1}=\left(\left[\mathbf{I}-\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]^{-1}\left[\mathbf{I}+\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]-\mathbf{I}\right)\mathbf{B^{-1}g} + \left(\left[\mathbf{I}-\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]^{-1}\left[\mathbf{I}+\frac{1}{2}\left(\Delta t\mathbf{A^{-1}B}\right)+\frac{1}{12}\left(\Delta t\mathbf{A^{-1}B}\right)^2\right]\right)\mathbf{U}\left(t_n\right). \tag{78}$$

6 Convergence and Stability

The importance of the convergence and stability of the schemes in finite differences is indisputable, without which it would not be possible to develop new schemes of any order of precision. Some definitions are fundamental to understanding convergence and stability, namely Thomas [17]:

Definicão 1. *A finite difference scheme $L_k^n = G_k^n$ approaching a partial differential equation $\mathcal{L}v = F$ is a convergent point-by-point schema when both $(k\Delta x, (n+1)\Delta t)$ and v_k^n converges simultaneously to (x,t) and to $v\left(x,t\right)$ for any x and t, and Δx and Δt converge to 0.*

Definicão 2. *A finite difference scheme $L_k^n = G_k^n$ approaching a partial differential equation $\mathcal{L}v = F$ is a convergent schema in time t, when $((n+1)\Delta t) \to t$, it's also valid,*

$$\|\mathbf{u}^{n+1}-\mathbf{v}^{n+1}\| \to 0, \tag{79}$$

when $\Delta x \to 0$ e $\Delta t \to 0$.

Note that in the above definition, the standard used has not been defined, because under different circumstances, specific standards may be used, but the definition, in a generic sense, remains valid. When one wants to discuss the speed with which a scheme converges, that is, what order of convergence, the following definition is used:

Definicão 3. *A finite difference scheme $L_k^n = G_k^n$ approaching a partial differential equation $\mathcal{L}v = F$ is a convergent schema of order (p,q) if for any time t, if when $((n+1)\Delta t) \to t$, is also valid.*

$$\|\mathbf{u}^{n+1}-\mathbf{v}^{n+1}\| = O\left(\Delta x^p\right)+O\left(\Delta t^q\right), \tag{80}$$

when $\Delta x \to 0$ and $\Delta t \to 0$.

When we have boundary value problems with initial conditions, convergence will depend on the partitioning of space. So, in general, if you set the space in a $[0,1]$ range, so that the partition $\{\Delta x_j\}$ to $j = 1, ...$ so that when $\Delta x \to 0$ has $j \to \infty$. In general, if $[0,1]$ has M discrete points, then $\Delta x = 1/M$. Now, if the space being worked is a normalized linear space of finite dimension, associated with the increment Δx_j, which can be denoted by X_j with associated norm $\|\cdot\|_j$, then the definition of convergence for this initial value problem with boundary values will be given by

Definicão 4. *A schema in differences approaching an initial value and boundary value problem is a time-convergent schema t for any partition sequences $\{\ Deltax_j\}$, when $((n+1)\Delta t) \to t$, is also valid.*

$$\|\mathbf{u}^{n}-\mathbf{v}^{n}\|_i \to 0, \tag{81}$$

when $i \to \infty$ and $\Delta t \to 0$.

Remember that, depending on the boundary conditions, the mesh must be adapted, for example:

1 - having Neumann-type boundary conditions at both ends, one must have a uniformly-meshed mesh in $[0, 1]$ with M discrete points, in the form

$$\left\{x_k : x_k = (k-1)\,\Delta x + \frac{\Delta x}{2}, k = 1, 2, ..., M\right\},$$

In this condition, phantoms (not belonging to the domain) are used at the ends: $x_0 = -\Delta x/2$ e $x_{M+1} = 1 + \Delta x/2$;

2 - if it is the case of having Neumann boundary condition at the extreme $x = 0$ and Dirichlet boundary condition at $x = 1$ there will be a partition of the form:

$$\left\{x_k : x_k = (k-1)\,\Delta x + \frac{\Delta x}{2}, k = 1, 2, ..., M\right\},$$

where $\Delta x = 2/(2M-1)$.

In order to study the consistency, before denoting the partial differential equation in $\mathcal{L}v = F$ by a finite difference scheme $L_k^n = G_k^n$ where G_k^n denotes the approximation that was performed in the source term, it is necessary to define:

Definicão 5. *The finite difference scheme $L_k^n = G_k^n$ is consistent point-to-point with the partial differential equation $\mathcal{L}v = F$ at a point (x,t) if for any smooth functions $\phi = \phi(x,t)$, the relation below is true*

$$(\mathcal{L}\phi - F)\_k^n - \left[L_k^n \phi\,(k\Delta x, n\Delta t) - G_k^n\right], \tag{82}$$

when $\Delta x \to 0$, $\Delta t \to 0$ e $(k\Delta x, (n+1)\Delta t) \to (x,t)$, it is also valid.

The consistency test by choosing ϕ to be v solution in the chosen partial differential equation, then

$$L_k^n v_k^n - G_k^n \to 0 \quad \text{quando} \quad \Delta x, \Delta t \to 0$$

With this, it can be written that the difference scheme (assuming that the EDP has derived in the first order time, as in the ECD) as:

$$\mathbf{u}^{n+1} = Q\mathbf{u}^n + \Delta t \mathbf{G}^n \tag{83}$$

with

$$\begin{aligned} \mathbf{u}^n &= \left(..., u_{-1}^n, u_0^n, u_1^n, ...\right)^T \\ \mathbf{G}^n &= \left(..., G_{-1}^n, G_0^n, G_1^n, ...\right)^T \end{aligned} \tag{84}$$

being Q an operator acting appropriately in space. In this way, thus understanding, one can define strongly consistency as follows:

Definicão 6. *The difference scheme (83) is said to be consistent with EDP in a norm $\|\cdot\|$ if the EDP solution, $\mathbf{v}$, satisfies:*

$$\mathbf{v}^{n+1} = Q\mathbf{v}^n + \Delta t \mathbf{G}^n + \Delta t \tau^n \quad \therefore \quad \|\tau^n\| = 0, \tag{85}$$

when $\Delta x, \Delta t \to 0$ and more $\mathbf{v^n}$ represents the vector whose k-th component is $v\,(k\Delta x, n\Delta t)$

It is convenient to remember that the truncation term τ^n contains both the truncation due to the approximation of $\mathcal{L}$ by L_k^n as the truncation due the approximation of the source term F.

Note that the approximation of the F function contributes the truncation term, and a bad or not very good approximation of F will always lead to the schema losing precision orders, which is very bad.

Definicão 7. *The difference scheme (83) is said to be precise of the order (p,q) to a given EDP if, in the limit:*

$$\|\tau^n\| = O\left(\Delta x^p\right) + O\left(\Delta t^q\right), \tag{86}$$

where τ^n or $\|\tau^n\|$ refers to the truncation error.

Note that by 5 the condition of functions with softness assumptions always needs to be checked, since this condition is a necessary condition for calculating the consistency of the finite difference schemes.
As seen, most of the schemes that are used in practice are consistent. The main problem with proving convergence is to achieve stability. Although stability is much easier to establish than convergence, it is still difficult to prove that a given scheme is stable. Stability is defined for a two-level difference scheme of the shape:

$$\mathbf{u}^{n+1} = Q\mathbf{u}^{n}, \quad n \geq 0 \tag{87}$$

which generally serves to solve a given initial value problem in which, in general, it includes a homogeneous linear partial differential equation.
Definicão 8. *The difference scheme (87) is said to be stable with respect to norm $\|\cdot\|$ if there are positive constants Δx_0 and Δt_0 non-negative constants K and β which:*

$$\|\mathbf{u}^{n+1}\| \leq Ke^{\beta t}\|\mathbf{u}^{0}\|, \tag{88}$$

to $0 \leq t = (n+1)\Delta t$, $0 < \Delta x \leq \Delta x_0$ and more $0 < \Delta t \leq \Delta t_0$
Note that, just like the definitions of convergence and consistency, the definition of stability is given in terms of a norm. As it was in the case of convergence and consistency, this standard may vary depending on the situation. Also note that the stability setting actually allows the solution to grow. We should note that the solution can grow over time, not with the number of time steps. Note also that stability is defined by a homogeneous behavior of the difference scheme.
As stated above all contributions of the non-homogeneous term will be and will be contained in the truncation term τ^n. Just to fix, any discussion about stability of a non-homogeneous scheme of differences, we must consider the stability of the associated homogeneous scheme.
Note that both the stability and the consistency of the finite difference schemes are required for the convergence of the numerical solution to the exact solution of a linear EDP. To see how these concepts are connected, there are two theorems, which will be presented without demonstration:
Teorema 1 (Lax-Richtmyer Equivalence Theorem)**.** *A two-level, consistent scheme for a well-situated linear problem of initial value is convergent if and only if it is stable.*
Teorema 2 (Lax's Theorem)**.** *If a finite difference scheme on two levels*

$$\mathbf{v}^{n+1} = Q\mathbf{v}^{n} + \Delta t\mathbf{G}^{n} \tag{89}$$

is consistent and accurate in order (p, q) in the norm $\|\cdot\|$ to a well-linear problem of initial value with respect to this same norm, then it converges to order (p, q) with respect to this same norm $\|\cdot\|$.
One of the hypotheses of the Lax-Richtmyer Equivalence Theorem and the Lax Theorem is that the initial value problem is well set (with its boundary conditions defined), so it is called if it depends continuously on its initial conditions. The proof of this theorem can be found in Thomas [17].
The Lax equivalence theorem does not ensure the consistency-convergence-stability relation for nonlinear partial differential equations, such as the Navier-Stokes equations. However, it provides us with the perception that satisfying stability and consistency is important for the development of convergent finite difference methods to solve fluid flow problems.

6.1 Stability of the Step-by-Step in Time

To characterize the stability of the step-by-step advance in time (or integration in time), be the model below, with f only function of time t Kajishima and Taira [18]:

$$\frac{df}{dt} = \lambda f, \qquad \text{com} \quad f(t_0) = 1 \tag{90}$$

where λ must be complex $= -\omega^2$, since it represents a possible natural frequency of the system. This model can capture the fundamental behavior of many differential equations, including the convection-diffusion equation. The solution of the model (90) is:

$$f(t) = e^{\lambda t} = e^{\Re(\lambda)t}\left[\cos\left(\Im(\lambda)\, t\right) + i \sin\left(\Im(\lambda)\, t\right)\right]. \tag{91}$$

Note that $\Re(.)$ and $\Im(.)$ mean the real part and the imaginary part of a complex, respectively. Analysis of the above expression shows that this solution can be stable when $\lim_{t\to 0}|f(t)| = 0, \leftarrow \Re(\lambda) < 0$; unstable if $\Re(\lambda) > 0$ or still neutral if $\Re(\lambda) = 0$. It is also seen that the part $\Re(\lambda)$ and the imaginary part $\Im(\lambda)$ respectively capture the diffusive and advective behaviors in the solution found. In order to continue the stability test, a time-forward method is chosen, which, for simplicity, will first of all choose the explicit Euler method; for this method the recurrence relation is:

$$f^{n+1} = (1 + \lambda\Delta t)\, f^n = (1 + \lambda\Delta t)^n\, f^0. \tag{92}$$

It is known that for this method to be stable it is necessary that

$$|1 + \lambda\Delta t| < 1 \tag{93}$$

which geometrically symbolizes that any values of $\lambda\Delta t$ that are inside a circle centered on -1 in the plane of the complexes, the method is stable. Doing the same for the textit Crank-Nicolson method, one has:

$$\begin{aligned} f^{n+1} &= f^n + \left(\frac{\lambda}{2}\Delta t\right)\left(f^n + f^{n+1}\right) \\ &= \frac{1 + \frac{\lambda}{2}\Delta t}{1 - \frac{\lambda}{2}\Delta t} f^n \end{aligned} \tag{94}$$

That for reasons identical to that used in the Euler method, the Crank-Nicolson Method requires that the following relationship be fulfilled to be stable:

$$\left|\frac{1 + \frac{\lambda}{2}\Delta t}{1 - \frac{\lambda}{2}\Delta t}\right| < 1 \quad \Rightarrow \quad \Re(\lambda\Delta t) < 0 \tag{95}$$

This, in fact, means that the Crank-Nicolson method will be stable for any value of $\lambda\Delta t$ that is to the left of the half plane of the plane of the complexes. By doing the same for, for example, the second-order Runge-Kutta method, we will have to be stable, the following relations respectively:

$$\begin{aligned} f^{n+1} &= \left[1 + \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2}\right] f^n \\ \Longrightarrow & \left|1 + \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2}\right| < 1 \end{aligned} \tag{96}$$

In Kajishima and Taira [18] (pp. 54) the regions where the various methods, among the most well-known, step-by-step methods of time are stable, are shown graphically. In its analysis it is observed that if advection is preponderant the method of integration in the time must cover the imaginary axis, being recommended methods of order greater or equal to 3; if on the other hand, diffusion is preponderant, we need methods that increase stability by covering the left side of the complex plane and then higher order methods are recommended.

6.2 Stability of EHOC

For the EHOC scheme, the amplification matrix, according to Fu et al. [6], is given by the matrix $\mathbf{\Phi}$, common in (78):

$$\mathbf{\Phi} = \left(\left[\mathbf{I} - \frac{1}{2}\left(\Delta t\mathbf{A}^{-1}\mathbf{B}\right) + \frac{1}{12}\left(\Delta t\mathbf{A}^{-1}\mathbf{B}\right)^2\right]^{-1}\left[\mathbf{I} + \frac{1}{2}\left(\Delta t\mathbf{A}^{-1}\mathbf{B}\right) + \frac{1}{12}\left(\Delta t\mathbf{A}^{-1}\mathbf{B}\right)^2\right]\right) \tag{97}$$

If λ are the eigenvalues of the matrix $\mathbf{A^{-1}B}$, then the eigenvalues of $\mathbf{\Phi}$ can be calculated by the following expression:

$$\mathbf{\Phi} = \left[1 - \frac{1}{2}\left(\Delta t\lambda_i\right) + \frac{1}{12}\left(\Delta t\lambda_i\right)^2\right]^{-1}\left[1 + \frac{1}{2}\left(\Delta t\lambda_i\right) + \frac{1}{12}\left(\Delta t\lambda_i\right)^2\right] \tag{98}$$

According to Dehghan and Mohebbi [19] if the real part of λ, $\Re(\lambda)$ is negative, then we can prove that the EHOC schema under analysis is *unconditionally stable*:

$$\max_{\lambda_i}\left|\frac{\left[1 + \frac{1}{2}\left(\Delta t\lambda_i\right) + \frac{1}{12}\left(\Delta t\lambda_i\right)^2\right]}{\left[1 - \frac{1}{2}\left(\Delta t\lambda_i\right) + \frac{1}{12}\left(\Delta t\lambda_i\right)^2\right]}\right| \leq 1 \tag{99}$$

In Tian and Yu [9] and Fu et al. [6] there are theorems and deductions from the above assertion.

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