XL CILAMCE
IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING
NOVEMBER 11-14, 2019
Praiamar Natal Hotel & Convention
Natal, RN-BRAZIL

STABILITY ANALYSIS OF SOLUTIONS OBTAINED BY THE LTSN METHOD FOR A RADIATIVE TRANSFER PROBLEM IN A HIGHLY NON-HOMOGENEOUS MEDIUM

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Abstract. Heat transfer can be a result of a radiative process. In such case, it is modeled by the radiative transfer equation (RTE), which is an integro-differential mathematical model that simulates the movement of photons in a medium. However, many other applications are also modeled with the RTE. Here, the RTE outputs the radiance distribution in the considered medium given the boundary conditions, source term, and inherent optical properties, such as the absorption and scattering coefficients as well as the scattering phase function, in the case of a translucid medium. The domain is discretized into the azimuthal, polar and vertical dimensions. The azimuthal discretization is obtained by the finite expansion of the Legendre polynomial of the scattering phase function, and by the radiance expansion using the Fourier decomposition of cosines. The resulting number of azimuthal modes is equivalent to the order of anisotropy of the medium. The discretization in polar direction domain is made by an approximation of the corresponding integral in the RTE, and the number of polar angles denotes the employed quadrature order. Consequently, the RTE is expressed by a set of linear differential equations, one for each azimuthal mode. The selected case study tackles an anisotropic and non-homogeneous medium, where the vertical domain is discretized in 80 regions, with 50 polar angles, and the number of azimuthal modes is 174. Therefore, each azimuthal mode requires to solve a linear system of differential equations with 80x50, or 4000 unknowns. The chosen solver is the LTS$_N$ method. It emerged in the early 1990s in the neutron transport research, being further extended to solve radiative transfer problems. The aim of this work is to optimize the number of azimuthal modes and of the quadrature nodes while obtaining stable solutions with accurate values of radiance. Another approach that is presented here is to solve only one linear system for all azimuthal modes, in a single step, with a much larger number of unknowns.

Keywords: Radiative transfer equation, Laplace transform discrete ordinate method, Sparse direct solver

1 Introduction

The radiative transfer equation (RTE) is a mathematical model for the study of absorption, transmission, and scattering of photons in a medium. This work employs the RTE in a hydrologic optics problem that involves the determination of the radiance distribution in a body of water, given the boundary conditions, source term, inherent optical properties, such as the absorption a and scattering b coefficients, and the scattering phase function. A brief description of the RTE components is given in the beginning of Section 2.

This work also employs bio-optical models that correlate coefficients a and b to the chlorophyll-a concentration profile. The spatial domain is discretized in a number of regions (R), being the chlorophyll-a concentration assumed as constant in each region. The discrete chlorophyll-a profile is then defined by $(R+1)$ points.

One method for solving the RTE is given by the discrete ordinates equations, or S$_N$ equations [1]. Here, it is employed the LTS$_N$ method [2, 3], which applies the Laplace transform to the S$_N$ ordinates. The LTS$_N$ method emerged in the early 1990s as a result of research on transport of neutrons, and was further extended to radiative transfer problems. Details of the discrete ordinate formulation, and how LST$_N$ method solves it, are presented in Subsection 2.1 and in Subsection 2.2.

In a medium with high degree of anisotropy like a natural body of water, there is a big number of possible scattering directions that require radiance calculations. In the case of a non-homogeneous medium containing many regions with different optical properties according to the depth, such calculation of radiances is replicated to all discrete depth levels of the vertical domain. In such a scenario, the LTS$_N$ formulation yields a sparse matrix system, in which the radiances are calculated in different directions, for each depth level.

Due to the ill-conditioned nature of the LTS$_N$ sparse matrix, the obtained radiance values can become severely unstable. In Section 3, two alternative approaches are presented in order to overcome this issue, based on the sparse structure of the matrix. A comparison of the computational performance between these two proposed strategies is also discussed.

2 Radiative Transfer Equation

The RTE models the transport of photons through a medium. Light intensity is given by a directional quantity, the radiance *L*, that measures the rate of energy being transported at a given point and in a given direction.

Considering a horizontal plane, this direction is defined by a polar angle μ (relative to the normal of the plane) and an azimuthal angle φ (a possible direction in that plane). At any point of the medium, light can be absorbed, scattered or transmitted, according to the coefficients a and b and to a scattering phase function that models how light is scattered in any direction.

An attenuation coefficient c is defined as $c = a+b$ and the geometrical depth is mapped to an optical depth τ that embeds c. Assuming a plane-parallel geometry, and a single wavelength, the unidimensional integral-differential RTE, can be written as:

$$\begin{aligned} & \mu\frac{\partial}{\partial\tau}L(\tau,\mu,\varphi) + L(\tau,\mu,\varphi) = \\ & \frac{\varpi_0(\tau)}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\beta(\mu,\varphi;\mu',\varphi')L(\tau,\mu',\varphi')d\varphi' d\mu' \\ & + S(\tau,\mu,\varphi) \end{aligned} \tag{1}$$

subject to boundary conditions

$$L(0,\mu,\varphi) = F\delta(\mu-\mu_0)\delta(\varphi-\varphi_0) \tag{2a}$$

$$L(\zeta, -\mu, \varphi) = 0, \tag{2b}$$

for $\mu \in (0, 1]$ and $\varphi \in [0, 2\pi]$, the cosine of the incident polar angle θ and the incident azimuthal angle, respectively. $\varpi_0(\tau) = b(\tau)/c(\tau)$ is the single scattering albedo.

The scattering phase function $\beta(\mu, \varphi; \mu', \varphi')$, gives the scattering beam angular distribution, mapping the incident beam direction (μ, φ) to the scattered direction (μ', φ'), and the source term is $S(\tau, \mu, \varphi)$.

There are several methods to solve the RTE, most of them adopting the Chandrasekhar's decomposition on the azimuthal angle [1]. In a medium with anisotropy degree N_g, that decomposition generates $N_g + 1$ integral-differential equations, each one with no dependence on φ.

2.1 The discrete ordinates method

Using the addition theorem of the spherical harmonics, the phase function can be expressed as

$$\beta(\mu, \varphi; \mu', \varphi') = \sum_{m=0}^{N_g} (2 - \delta_{0,m}) \left[\sum_{l=m}^{N_g} \beta_l^m P_l^m(\mu) P_l^m(\mu') \right] \cos m(\varphi - \varphi') \tag{3}$$

the radiance and the source term are also expanded as a Fourier decomposition,

$$L(\tau, \mu, \varphi) \;=\; \sum_{m=0}^{N_g} L^m(\tau, \mu) \cos m\varphi \tag{4a}$$

$$S(\tau, \mu, \phi) \;=\; \sum_{m=0}^{N_g} S^m(\tau, \mu) \cos m\varphi \tag{4b}$$

The heterogeneous medium is modeled as a plane-parallel geometry, with a set of R homogeneous parallel finite layers with boundary conditions between layers. Each layer is denoted as being a region r of the multiregion domain, as shown in Figure 1. Thus, the properties of a heterogeneous medium are splitted into a multiregion domain composed by a set of homogeneous regions:

$$\varpi_0(\tau) \;=\; \varpi_r \;\; r = 1, 2, \ldots, R \tag{5}$$

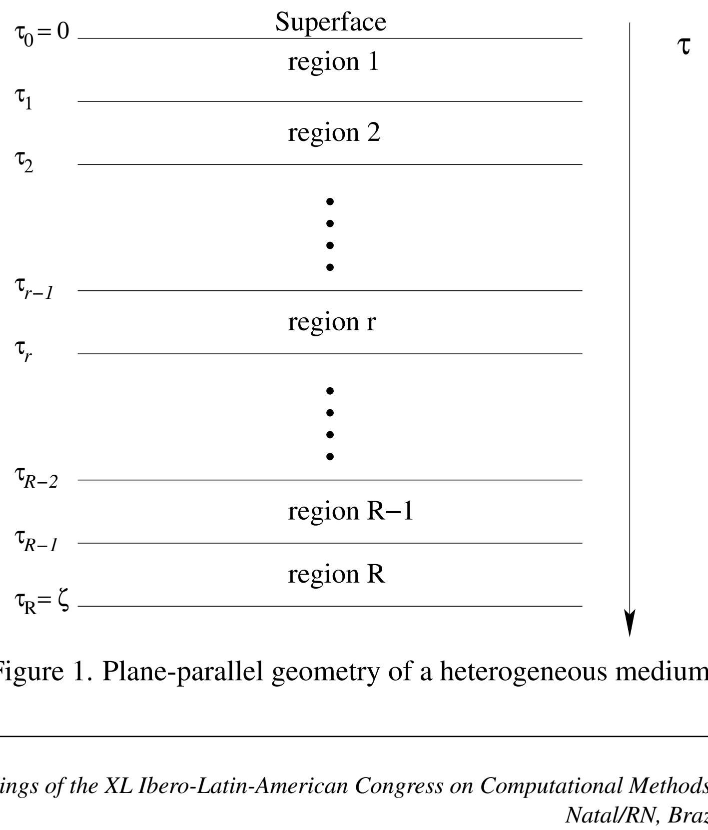


Figure 1. Plane-parallel geometry of a heterogeneous medium.

The substitution of Equations (3)-(4)-(5) in Equation (1) yields

$$\begin{aligned}
&\mu\frac{\partial}{\partial\tau}L_r^m(\tau,\mu) + L_r^m(\tau,\mu) = \\
&\frac{\varpi_r}{2}\sum_{l=m}^{L}\beta_l^m P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')L_r^m(\tau,\mu')d\mu' \\
&+ S_r^m(\tau,\mu)
\end{aligned} \tag{6}$$

subjected to the following boundary conditions, for $\mu \in (0,1]$

$$L_1^m(0,\mu) = f^m \tag{7a}$$

and

$$L_R^m(\tau_R,-\mu) = g^m \tag{7b}$$

and to the interface conditions, for $r = 1,2,\ldots,R-1$

$$L_r^m(\tau_r,\pm\mu) = L_{r+1}^m(\tau_r,\pm\mu) \tag{8}$$

Radiance was decomposed into azimuthal modes, while the phase function was replaced by the associated Legendre function expansion with degrees of anisotropy (N_g).

An approximation of the integral equation (1) is obtained using a quadrature of order $N_\mu = 2n$, with nodes $\{\mu_j\}$ and weights $\{\eta_j\}$. The value of μ is then discretized in μ_j, with $j = 1,2,\ldots,N$, that are the discrete ordinate directions.

In a non-homogeneous medium with N_g degrees of anisotropy, the scattering angle is then discretized into (N_g+1) azimuthal modes, with N polar angles, while the domain is splitted into R homogeneous regions. The radiate transfer equation is then expressed as the discrete ordinate equations, also known as S_N equations, given by

$$\begin{aligned}
&\mu_j\frac{d}{d\tau}L_r^m(\tau,\mu_j) + L_r^m(\tau,\mu_j) = \\
&\frac{\varpi_r}{2}\sum_{l=m}^{N_g}\beta_l^m P_l^m(\mu_j)\sum_{i=1}^{N}\eta_i P_l^m(\mu_i)L_r^m(\tau,\mu_i) \\
&+ S_r^m(\tau,\mu_j), \\
&j = 1,2,\ldots,N \\
&m = 0,1,2,\ldots,N_g \\
&r = 0,1,2,\ldots,R
\end{aligned} \tag{9}$$

The boundary conditions are

$$L_1^m(0,\mu_j) = 0 \qquad j = 1,2,\ldots,n \tag{10a}$$

$$L_R^m(\tau_R,-\mu_j) = 0 \qquad j = n+1,n+2,\ldots,N. \tag{10b}$$

For the discrete ordinate method, the above equations are approximated by a colocation method, where the μ integral is computed by the Gauss-Legendre quadrature formula. This yields a set of N differential equations for each azimuthal mode.

Each set (discretized RTE) is solved by the LTS$_N$ method, that generates a system of linear equations of order $R \times N$.

2.2 The LTS$_N$ method

The LTS$_N$ method [2, 3] applies the Laplace transform on the radiative transfer discrete ordinates equations, given by Eqs. (9) and (10). This yields a system of symbolic algebraic equations on s:

$$\begin{aligned} & s\overline{L}_{j,r}^{\,m}(s) + \frac{1}{\mu_j}\overline{L}_{j,r}^{\,m}(s) - \\ & \frac{\varpi_r}{2\mu_j}\sum_{l=m}^{N_g}\beta_l^m P_l^m(\mu_j)\sum_{i=1}^{N}\eta_i P_l^m(\mu_i)\overline{L}_{i,r}^{\,m}(s) = L_{j,r}^m(0) + \frac{1}{\mu_j}\overline{S}_{j,r}^{\,m}(s) \end{aligned} \tag{11}$$

with $\overline{L}_{j,r}^m(s) \equiv \int_0^\infty L_{j,r}^m(\tau)e^{-s\tau}\,d\tau$. The matrix form of equation (11) becomes

$$\overline{M}_{N,r}^{\,m}(s)\overline{L}_r^{\,m}(s) = L_r^m(0) + \overline{Q}_r^{\,m}(s). \tag{12}$$

where the N-order matrix $\overline{M}_{N,r}^{\,m}(s)$, called the LTS$_N$ matrix, is expressed as

$$\overline{M}_{N,r}^{\,m}(s) = s\mathbf{I} + A_r^m \tag{13}$$

and **I** is the N-order identity matrix, while the A^m matrix is given by

$$a_r^m(i,j) = \begin{cases} \dfrac{1}{\mu_j} - \dfrac{\varpi_r}{2\mu_j}\displaystyle\sum_{l=m}^{L}\beta_l^m P_l^m(\mu_j)\eta_j P_l^m(\mu_j), \\ \hfill \text{if } i = j, \\ \\ -\dfrac{\varpi_r}{2\mu_j}\displaystyle\sum_{l=m}^{L}\beta_l^m P_l^m(\mu_j)\eta_i P_l^m(\mu_i), \\ \hfill \text{if } i \neq j. \end{cases} \tag{14}$$

and vectors $\overline{L}_r^{\,m}(s)$, $L_r^m(0)$ and $\overline{Q}_r^{\,m}(s)$ are defined as

$$\begin{aligned} \overline{L}_r^{\,m}(s) &= \left[\overline{L}_{1,r}^{\,m}(s)\ \overline{L}_{2,r}^{\,m}(s)\ \ \dots\ \ \overline{L}_{N,r}^{\,m}(s)\right], \\ \overline{L}_r^{\,m}(0) &= \left[\overline{L}_{1,r}^{\,m}(0)\ \overline{L}_{2,r}^{\,m}(0)\ \ \dots\ \ \overline{L}_{N,r}^{\,m}(0)\right], \\ \overline{Q}_r^{\,m}(s) &= \left[\frac{\overline{S}_{1,r}^{\,m}(s)}{\mu_1}\ \ \frac{\overline{S}_{2,r}^{\,m}(s)}{\mu_2}\ \ \dots\ \ \frac{\overline{S}_{N,r}^{\,m}(s)}{\mu_N}\right]. \end{aligned}$$

In order to solve the matrix equation (12), it must be multiplied by the inverse matrix of $\overline{M}_{N,r}^{\,m}(s)$, as follows

$$\overline{L}_r^{\,m}(s) = \left[\overline{M}_{N,r}^{\,m}(s)\right]^{-1} L_r^m(0) + \left[\overline{M}_{N,r}^{\,m}(s)\right]^{-1}\overline{Q}_r^{\,m}(s), \tag{15a}$$

$$\overline{L}_r^{\,m}(s) = \overline{B}_r^{\,m}(s)L_r^m(0) + \overline{B}_r^{\,m}(s)\overline{Q}_r^{\,m}(s). \tag{15b}$$

Applying the Laplace inverse transform

$$L_r^m(\tau) = B_r^m(\tau)L_r^m(0) + H_r^m(\tau) \tag{16}$$

where

$$B_r^m(\tau) = \mathcal{L}^{-1}\left[\overline{B}_r^{m}(s)\right] \tag{17}$$

and

$$H_r^m(\tau) = B_r^m(\tau) * Q_r^m(\tau)\ . \tag{18}$$

The operation $B_r^m * Q_r^m$ in the above equation denotes the convolution between the two functions.

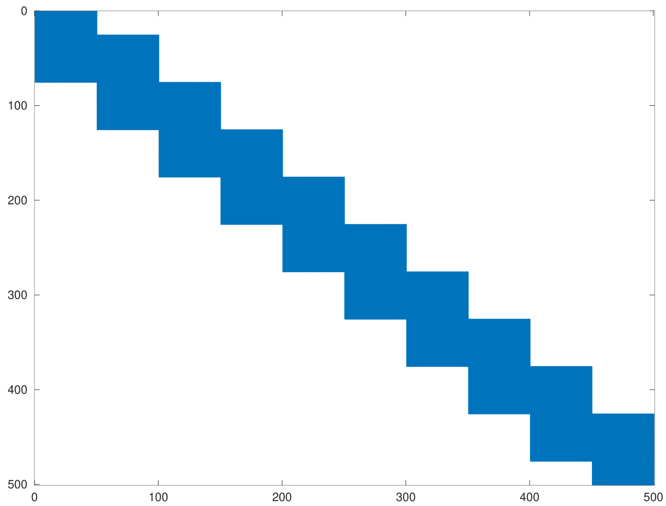


Figure 2. Sparse structure of the linear system matrix, considering a non-homogeneous medium with 10 regions and 50 polar directions ($N_s = 500$ and $nnz = 47,500$).

The implementation of the LTS$_N$ method solves each m-th azimuthal-mode system of order $R \times N$ shown in Equation (16), for $m = 0, 1, 2, \ldots, N_g$, and R regions, using N-th order of quadrature. Applying in Equation (16), the boundary conditions and a criterion of continuity between adjacent regions, i.e., $L^m_r(\tau) = L^m_{r+1}(\tau)$, yields to the systems given by the Equation (19)

$$\left[\begin{array}{r|l|l|c|l} \mathrm{B}^{(1)}_{11}(0)\mathrm{B}^{(1)}_{12}(0) & & & & \\ \mathrm{B}^{(1)}(\tau_1) & -\mathrm{B}^{(2)}(0) & & & \\ & \mathrm{B}^{(2)}(\tau_2) & -\mathrm{B}^{(3)}(0) & & \\ & & & \ddots & \\ & & & \mathrm{B}^{(R-1)}(\tau_{R-1}) & -\mathrm{B}^{(R)}(0) \\ & & & & \mathrm{B}^{(R)}_{21}(\tau_R)\mathrm{B}^{(R)}_{22}(\tau_R) \end{array}\right] \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \vdots \\ L_{R-1} \\ L_R \end{bmatrix} = \tag{19}$$

$$= \begin{bmatrix} -H^d_1(0) \\ H_2(0) - H_1(\tau_1) \\ H_3(0) - H_2(\tau_2) \\ \vdots \\ H_R(0) - H_{R-1}(\tau_{R-1}) \\ -H^u_R(\tau_R) \end{bmatrix}$$

For instance, Figure 2 shows a representation of the sparse structure from the linear system matrix given by the Equation (19), for a non-homogeneous medium with 10 regions ($R = 10$) and 50 polar directions ($N = N_\mu = 50$). From these discretization parameters, the resulting matrix has 500 rows and columns ($N_{system} = 500$, or $N_s = 500$ for short), containing 47,500 nonzero entries ($nnz = 47,500$), meaning more than 90% are zero in the matrix. Note that the number of nonzeros elements for the matrix of can be obtained simply by using $nnz = (2R - 1) \times N \times N$.

3 RTE for Hydrological Optics solved by LTS$_N$

The Radiative Transfer Equation problem in hydrologic optics involves the determination of the radiance distribution in a body of water, given the boundary conditions, source term, inherent optical properties, such as the absorption a and scattering b coefficients, and the scattering phase function.

This work employs bio-optical models that correlate absorption and scattering coefficients of each region to the chlorophyll-*a* concentration. These coefficients are assumed to be constant in each region. Therefore discrete values a_r and b_r can be estimated for each region from the discrete values C_r. A real chlorophyll-*a* concentration profile, presented in Figure 3, was employed.

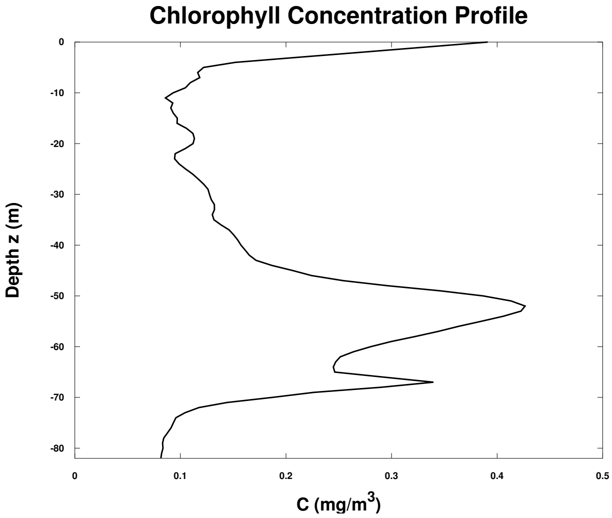


Figure 3. Profile of chlorophyll-a concentration.

The adopted bio-optical model for the absorption coefficient was formulated by [4]:

$$a_r = \left[a^w + 0.06\, a^c\, C_r^{0.65}\right] \left[1 + 0.2\, e^{-0.014(\lambda-440)}\right] \tag{20}$$

where a^w is the pure water absorption and a^c is a non-dimensional, statistically derived chlorophyll-specific absorption coefficient, and λ is the considered wavelength. The adopted bio-optical model for the scattering coefficient was formulated by [5]:

$$b_r = \left(\frac{550}{\lambda}\right)\, 0.30\, C_r^{0.62} \tag{21}$$

As previously mentioned, the number of independent azimuthal modes is equivalent to the degree of anisotropy of the medium. The radiance values are then obtained summing up the results of these linear systems for all azimuthal modes.

A case study for a non-isotropic, with degree of anisotropy $N_g = 174$, in a non-homogeneous medium is presented, where the domain is divided into 80 regions ($R = 80$) and 50 polar directions ($N_\mu = 50$). The resulting sparse linear system has $4,000$ unknowns radiance values ($N_s = 4000$), and its linear system matrix has $397,500$ nonzero entries ($nnz = 397,500$).

The higher the azimuthal mode, the smaller are the values of the results of its linear system, and less relevant is its contribution to overall sum. Due to this reason, for this case study, only the firsts sixteen ($N_m = 16$) azimuthal modes were considered in order to calculate the radiance values. The algorithm SMALLERSYSTEMS, presented in Figure 4, summarizes the sequence of steps to obtain radiance values.

Using a direct solver from LAPACK solver [6] in the SOLVESYSTEM step, the obtained radiance values are showed in Figure 5a, and the corresponding sparse structure of the linear systems is showed in Figure 5b.

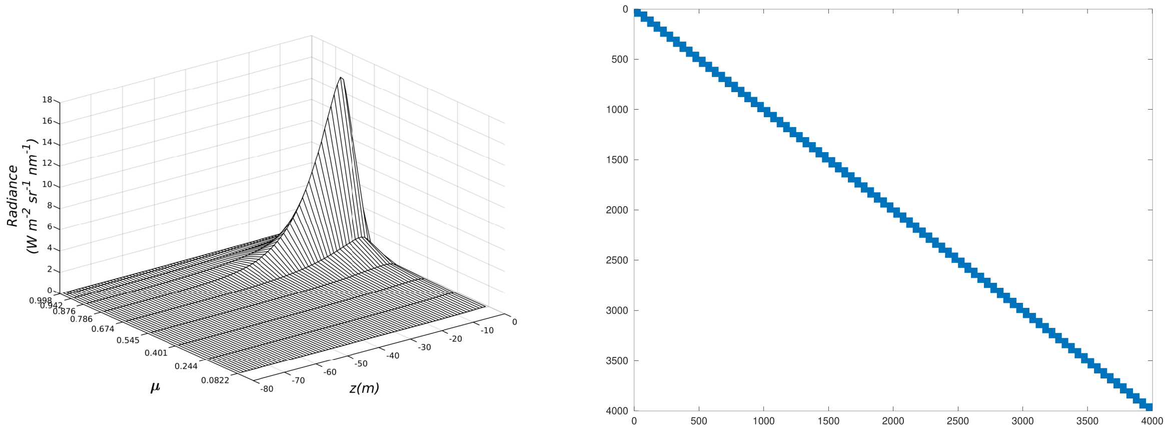
Once the LAPACK solver was implemented for dealing with dense matrices, it results in a waste of processing time when it is used in sparse system matrices. Therefore, we adopted a sparse direct solver for computing the radiance values, from the Intel® MKL PARDISO library [7].

However, some unexpected huge values of radiance were obtained, as seen in the Figure 6. Probably, these singular values are due to the ill-conditioned nature of the LTS$_N$ matrix, compromising the stability of the solution.

Algorithm: SMALLERSYSTEMS(N_m,R,N_μ,L)

Data:
N_m: number of independent azimuthal modes
R: number of regions
R: number of polar angles
Result: L: radiances
begin
  $L = 0$
  for $m \leftarrow 1$ **to** N_m **do**
    CREATEMATRIX(R,N_μ,A^m)
    CREATERHS(R,N_μ,b^m)
    SOLVESYSTEM(A^m,b^m,L^m)
    $L = L + L^m$
  end
end

Figure 4. SMALLERSYSTEMS algorithm.



(a) (b)

Figure 5. (a) Radiance values in all depths/regions (z) for 10 selected polar directions (μ); (b) Sparse structure of the linear system matrix, considering a non-homogeneous medium with 80 regions and 50 polar directions ($N_s = 4,000$ and $nnz = 397,500$)

It is important to note that the LAPACK library has a function that computes the row and column scalings in order to balance the coefficients matrix and reduce its condition number. Consequently, there are no unstable value of radiance in the solution, as already observed in Figure 5a.

The first approach employed to overcome this issue was simply to increase the number of polar directions (N_μ) and, consequently, the number of nodes in the quadrature, until obtaining stable values of radiance. The values of radiances remained unstable for some increasing numbers of polar angles (N_μ): 70, 90, 110 and 130, as shown in Figure 7 and in Figure 8. Finally, for ($N_\mu = 150$), stable values of radiance were reached (Figure 9a)implying in solving the corresponding sparse linear system with $12,000$ unknowns ($N_s = 12,000$) and more than 3.5 million of non-zero values ($nnz = 3,577,500$) in the sparse matrix system.

A second approach was adopted to treat the unstable radiance values. It consists on simultaneously calculate the radiance values for all azimuthal modes by solving a single sparse linear system that embeds all the linear systems of the azimuthal modes. The algorithm ONESYSTEM, presented in Figure 10, summarizes the sequence of steps to obtain radiance values with this alternative approach.

The sparse structure of the matrix generated with this approach is presented in Figure 11a. The horizontal and vertical black lines in this figure do not correspond to matrix values. These are imaginary lines that delimit the sub-matrices of each azimuthal mode.

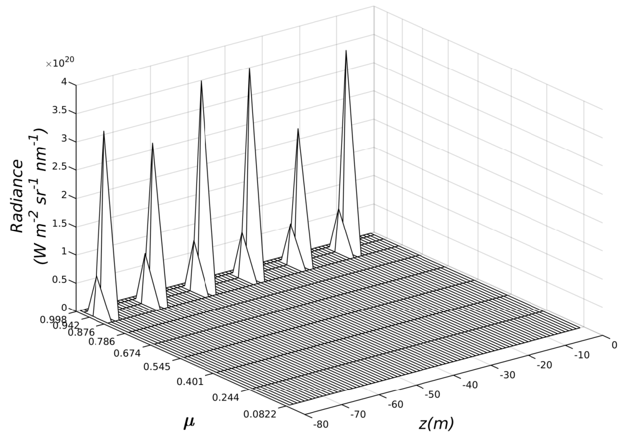
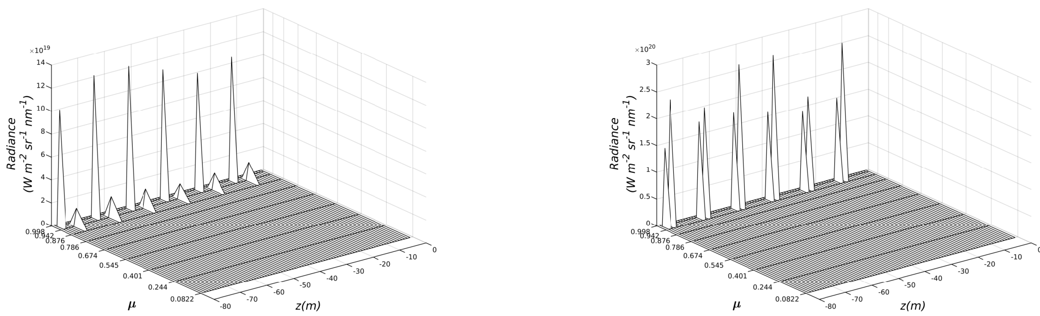


Figure 6. Unstable values of radiance obtained solving the linear system given by Eq. (19), using a sparse direct solver from MKL PARDISO library, for with $R = 80$ and $N_\mu = 50$.



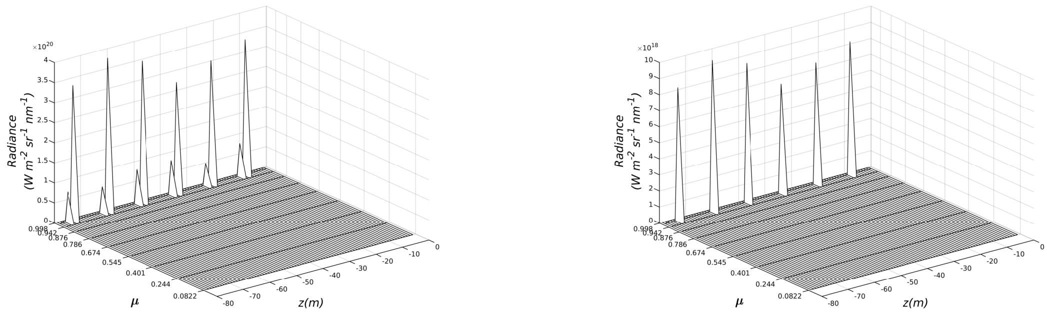
(a) 70 polar directions

(b) 90 polar directions

Figure 7. Unstable values of radiance obtained by solving the linear system given by Eq. (19), using the sparse direct solver from MKL PARDISO library, for with $R = 80$ and: (a) $N_\mu = 70$; (b) $N_\mu = 90$

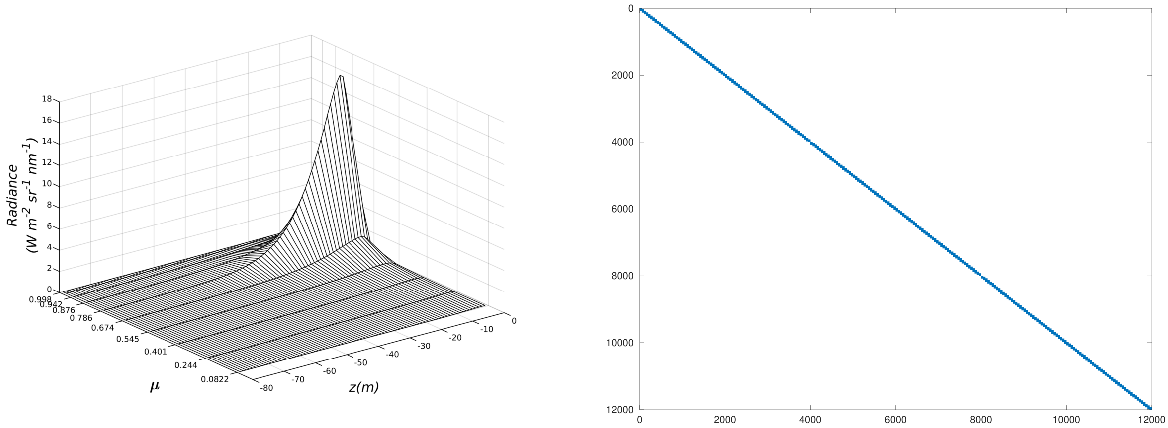
Note that this new approach employed the original number of polar direction, i.e., $N_\mu = 50$, for $R = 80$ regions. Thus, each sub-matrix has exactly 4000 rows and columns. Since there are 16 azimuthal modes, the size of the whole matrix is $N_s = 4,000 \times 16 = 64,000$. Such linear system was solved, and the resulting values of radiance are shown in Figure 11b. No unstable values appear.

Table 1 shows a brief comparison between the SMALLERSYSTEMS and ONESYSTEM approaches. Additional information about the corresponding linear systems and computing performances is shown in Table 2 for the numeric factorization evaluated by the MKL PARDISO sparse direct solver. Note that, although the number of non-zeros values in the L and U matrices from SMALLERSYSTEMS approach is nearly 30% smaller, the number of float-point operations (*flop*) to perform the numeric factorization is almost 2.7 times higher than the system of ONESYSTEM approach. This greater performance cost is most likely due to the ill-conditioned nature of SMALLERSYSTEMS matrices, even with an increased number of polar angles. Indeed, an extra computational effort (more *flop*) is required to find the right pivot in order to reach a stable solution.



(a) 110 polar directions (b) 130 polar directions

Figure 8. Unstable values of radiance obtained by solving the linear system given by Eq. (19), using sparse direct solver from MKL PARDISO, for with $R = 80$ and: (a) $N_\mu = 110$; (b) $N_\mu = 130$



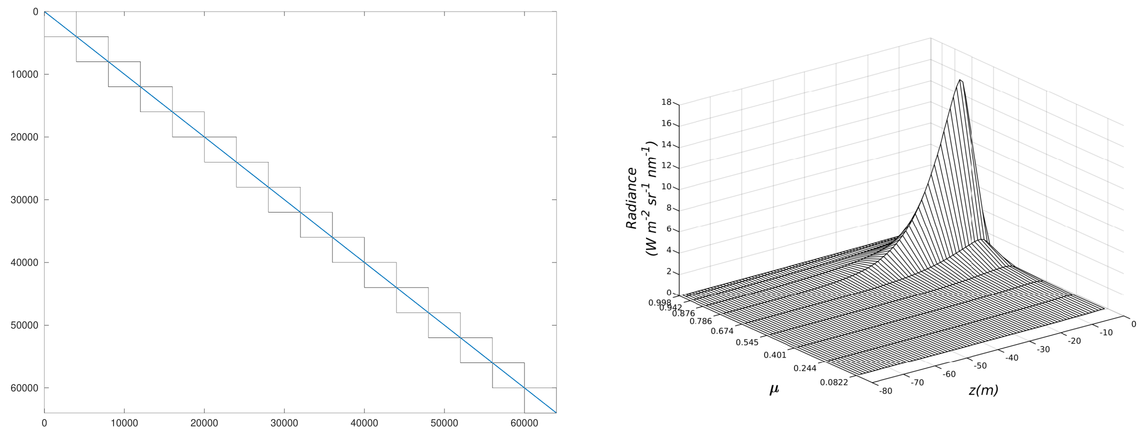
(a) (b)

Figure 9. (a) Radiance values in all depths (z) and for 10 selected polar directions (μ); (b) Sparse structure of the linear system matrix, considering a non-homogeneous medium with 80 regions and 150 polar directions ($N_s = 12,000$ and $nnz = 3,577,500$)

Algorithm: ONESYSTEM(N_m,R,N_μ,L)

Data:
N_m: number of independent azimuthal modes
R: number of regions
R: number of polar angles
Result: L: radiances
begin
  for $m \leftarrow 1$ **to** N_m **do**
    CREATEMATRIX(R,N_μ,A^m)
    CREATERHS(R,N_μ,b^m)
    ADDTOONESYSTEM(A^m,b^m,A,b)
  end
  SOLVESYSTEM(A,b,L)
end

Figure 10. ONESYSTEM algorithm.



(a) (b)

Figure 11. (a) Only one system for all $N_m = 16$ azimuthal modes, considering a non-homogeneous medium with 80 regions and 50 polar directions ($N_s = 64,000$ and $nnz = 6,360,000$); (b) Radiances values in all depths (z) and for 10 selected polar directions (μ).

Table 1. SMALLERSYSTEMS *vs.* ONESYSTEM comparison - I

STRATEGY	R	N_μ	N_m	N_s	Nonzeros	Systems
SMALLERSYSTEMS	80	150	16	12,000	3,577,000	16
ONESYSTEM	80	50	16	64,000	6,360,000	1

Table 2. SMALLERSYSTEMS *vs.* ONESYSTEM comparison - II

STATISTICS	SMALLERSYSTEMS (per azimuthal mode)	ONESYSTEM (all azimuthal modes)
number of unknowns (radiances)	12,000	64,000
number of non-zeros in A	3,577,500	6,360,000
number of non-zeros in L	4,506,418	6,860,000
number of non-zeros in U	3,323,582	4,140,000
number of non-zeros in L+U	7,830,000	11,000,000
Gigaflop for the numerical factorization	2.69	1.01

4 Final Remarks

This work analyzed the stability of solutions obtained by the LTS$_N$ method for a radiative transfer problem. It was detected that this method can generate inconsistent values of radiance due to the ill-conditioned nature of the corresponding linear system, making such solutions very unstable. Two different approaches were then tried in order to obtain stable solutions in these particular cases. The first, called SMALLERSYSTEMS, is based on gradually increasing the number of polar angles until achieving consistent radiance values, which required a minimum value of $N_\mu = 150$. In the second approach, called ONESYSTEM, instead of solving one linear system for each azimuthal mode, a single linear system was employed for all azimuthal modes, which also resulted in consistent radiance values.

These approaches present quite different computational costs. In the SMALLERSYSTEMS approach, each one of the 16 linear systems has order $N_s = 12,000$ (that corresponds to 80×150), while for the ONESYSTEM approach, the system order is $N_s = 64,000$ (that corresponds to $80 \times 150 \times 16$). However, even comparing one system of the first approach (12,000 unknowns) with the single system of the second approach (64,000 unknowns), the number of floating point operations of the numeric factoring is approximately 2.7 higher for the first approach ($2.69GFlop \times 1.01GFlop$). Probably, this higher cost of the SMALLERSYSTEMS is due to the additional effort of finding the pivot during numeric factorization in the many ill-conditioned systems. In future work, it is intended to further investigate this issue in order to better understand the ill-conditioned nature of linear systems generated by the LTS$_N$ method, and also propose a new approach for obtaining stable solutions without increasing the number of polar directions in SMALLERSYSTEMS and, consequently, its computational cost.

This work is aimed at the stability of solutions and computational performance of a RTE solver applied to a hydrologic optics problem. It is part of a research that involves the use of the RTE solver for solving an inverse problem that estimates the optical properties of the medium from the observed/measured light field. This inverse problem is solved implicitly, being formulated as an optimization problem. The corresponding iterative scheme implies in successively generating and evaluating candidate solutions (values of the optical properties). Each candidate solutions serves as input to the RTE in order to generate a light field, which is compared to the observed one by means of a quadratic difference. This metric is then employed by a stochastic optimizer to generate the candidate solution for the next iteration [8–10]. Such implicit solver may demand hundreds or even thousands of iterations, and each one demands solving the RTE, making the corresponding processing time unfeasible, particularly in the RTE problem described here, that involves anisotropic and non-homogeneous medium with many vertical levels [11, 12]. Therefore, the optimization of the considered RTE solver, the LTS$_N$ method, is a critical issue.

Acknowledgements

Author Haroldo F. Campos Velho thanks the Brazilian support agencies CNPq (Proc. 312924/2017-8) and FAPESP (Proc. Proc. 2016/14393-2) for the corresponding research grants.

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