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STATIC ANALYSIS OF INEXTENSIBLE SUSPENDED CABLES BY THE GFEM

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Abstract. Cable structures have high performance when required for traction and therefore provide lighter and thinner structures. On the other hand, the shape of the cable may vary according to its loading, which makes its analysis difficult. The Finite Element Method (FEM) has good results in the analysis of cable structures, but demands a high number of degrees of freedom to achieve a better accuracy. In order to evaluate the application of the Generalized Finite Element Method (GFEM) in suspended cable structures, a simplified cable model is presented, considering a static and inextensible analysis. The formulation considers the weak form of the inextensible cable problem and does the enrichment of the shape functions space for the conventional Finite Element Method. The model is implemented in Python language and tested with applications in the literature. In this work, second-degree Lobatto's polynomials are used and also hyperbolic enriching functions are proposed. The efficiency and convergence of the proposed model are verified and the matrix condition number is calculated to examine the numerical stability. The application of GFEM for inextensible cable problem, as presented here, is an original approach. Models based on parabolic or catenary configuration, when compared with others found in literature, have the closest results with the analytical solutions. GFEM proved to be an excellent method for cable problems. In conclusion, it is possible to solve several cable problems with a single element, surpassing the results presented by FEM.

Keywords: Generalized Finite Element Method; Cable Structures; Lobatto polynomial.

1 Introduction

Cable structures have numerous applications in Engineering, such as the development of roofing suspension structures, suspended or cable-stayed bridges, offshore structures, tower mooring, towing systems and power transmission structures, among others. They can reduce the cost of the structure due to their high performance when requested tensile and by providing lighter structures [9, 10, 11, 12].

Cables are very light elements and generate slender structures with lower dead load. In contrast, due the greater slenderness, they may lead to nonlinear behavior with large displacements. If the cables are untied, they experiment large displacements because the geometric nonlinearity. Cable elements have low flexural rigidity and in most analyzes it is not considered even neglected [16, 19, 20].

In the study of suspended cables, the analytical equation of the inextensible parabolic cable configuration can be considered as the simplest of the configurations. Even so, it is not always possible to solve it directly, requiring an iterative process [1, 14]. When considering the elastic elongation of the material, the problem becomes even more complex, since the cable adjust its configuration at each new deformation suffered. Then, it is necessary to solve it iteratively with respect to the total cable length.

The elongation and nonlinear effects of large span structures, such as the Akashi-Kaikyo Bridge [7], should certainly be evaluated. Nevertheless, for suspended steel cables with spans of up to 100 meters, the elongation caused by elasticity is of little importance and even the inextensible cable theory can be used [1, 2, 5, 8, 10, 11].

In this sense, the inextensible cable approach is suitable for validating the most robust models and can also be used as an initial solution for nonlinear model iterations. Therefore, the present paper discusses the inextensible cable model with its numerical implementation and shows its results and characteristics.

The initial geometric configuration of the cables is affected by the loading imposed on the structure. For example, evenly distributed loads relative to the cable span generate a cable parabolic shape. If the loads are distributed over the cable length, they generate a catenary shape. In addition, concentrated forces induce a polygonal shape with straight segments.

The Finite Element Method (FEM) is one of the main numerical methods for structural analysis. With the FEM, it is possible to obtain approximate solutions of differential equations of the studied problem [4, 9, 23]. The method uses the weak form of a boundary value problem that, from the discretization of the continuous medium and interpolating polynomial functions, seeks an approximate solution to the initial equation.

Despite the good performance of the FEM in most problems, it is still possible to improve the results by refining the mesh (*h*-refinement) as well as increasing the order of the interpolator polynomial (*p*-refinement). Due to certain particularities of the problem to be analyzed, the FEM needs a great refinement which can spend much time and grows up the analysis costs. For these situations, due for example to singularities or discontinuities, it is advantageous to use enriched methods that perform the addition of functions related to the nature of the problem solution [3, 4, 6, 24].

The Generalized Finite Element Method (GFEM) is an extension of FEM in the context of the Partition of Unit Method (PUM) [3, 4, 6, 24, 26]. Enrichment functions are added to the FEM shape functions space, widening the solution space, and incorporating information regarding the nature of the problem analyzed.

In this sense, for cable models, which have a priori information of the analytical solution, it is understood that the GFEM becomes an advantageous tool for the analysis of these models.

The authors did not find any research in the literature using GFEM in cable analysis, but for the FEM it is possible to cite several works, such as Antunes and Sampaio [2], Barbato [5], Costa [8], Desai and Punde [9], Lourenço [13], Morini [14], Negrão et al [15], Papini [17], Pauletti and Pimenta

[18], Pereira Jr. [19], Rente [20], Souza Jr. [21], Thai and Kim [22] and Wei, Bingnan and Jinchun [23] which makes this work an unprecedented contribution to the subject.

The general objective of this work is the application of the Generalized Finite Element Method in static analysis of inextensible suspended cable structures, trying to find their equilibrium configuration, the tension intensity and the maximum deflection. The GFEM was developed in a Python language program for solving the inextensible static cable configuration, which permits to compare solutions with different types of enrichment or different levels of enrichment. It is possible to compare the results between FEM and GFEM, and verify the numerical stability by the condition number of the stiffness matrix. In this work, the influence of the cable configuration type (parabolic or catenary) on the structure responses is also investigated as enrichment functions.

2 Cable Theories and Numerical Models

Consider the Figure 1 that illustrates an infinitesimal cable subjected to a uniformly distributed transversal load q(x) with respect to the X axis. The deformed configuration has slope θ, and μ is the weight per unit length of cable (s).

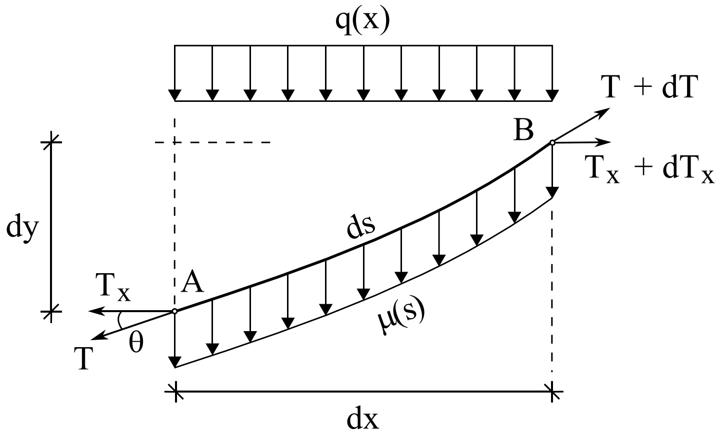

Figure 1 – Free body diagram of infinitesimal cable

Cables subjected predominantly to their own weight or evenly distributed loads in relation to their length are in the form of a catenary. The differential equation that governs the vertical displacement *y* of the cable is [10,11]:

$$\frac{d^2y}{dx^2} = \frac{q}{T_x} + \frac{\mu}{T_x} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1}$$

The Figure 2 illustrates a generic cable suspended on two points *A* and *B*, with gap *L*; the difference between the cable attachment points *h*; the abscissa x_v for the maximum deflection point; θ_A the slope of the cable at point *A* (negative); μ is a load evenly distributed over the cable length; *f* (negative) is the maximum deflection that occurs in x_v. The particular solution of Equation (1) is presented in Equation (2), whose constants C_1 and C_2 depend on the boundary conditions and the type of loading.

$$y(x) = \frac{\mu}{T_x} \cosh\left(\frac{\mu . x}{T_x} + C_1\right) + C_2 \tag{2}$$

The non-linear condition of the cable problem can be seen in the solution of Equation (2) where the vertical displacement *y(x)* depends on the constant value T_x and vice-versa. One can also observe that the analytical solution uses a hyperbolic function. This is important for GFEM, since the method can incorporate particular characteristics of the problem for enriching the solution space.

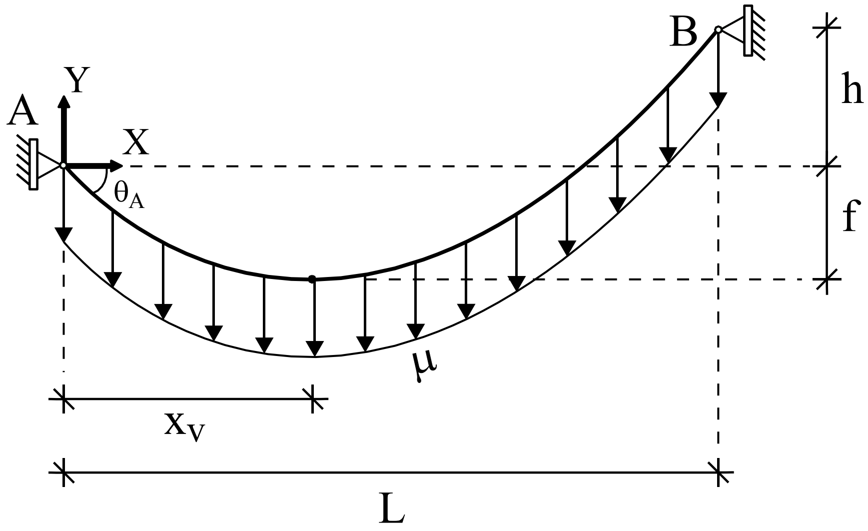

Figure 2 - Catenary suspension cable

2.1 The simplest cable finite element

It is common in FEM and GFEM to use a standard auxiliary element called a master element, to which all mesh elements are mapped. This device facilitates the integration process and later the enrichment itself in the GFEM. For the present work, a size 2 master element is adopted as shown in Figure 3.

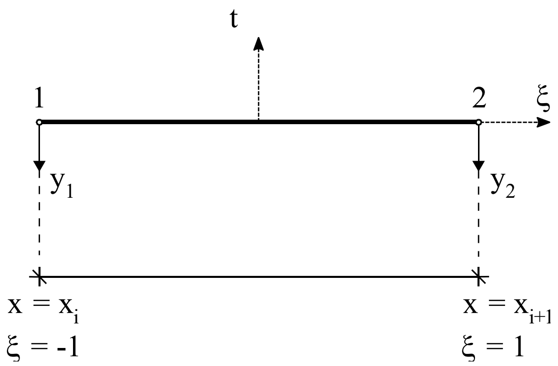

Figure 3 – Master element for FEM and GFEM

2.2 The weak form for inextensible cable problem

The weak form of inextensible cable problem can be developed starting Equation (1). As already mentioned before, there is a nonlinearity in the problem that can be taken into account by solving the power function of the cable. However, it is understood that this approach escapes the main scope of the present work and therefore, to circumvent the problem a simplification is adopted considering the first derivative of y for catenary suspension cable with level span $[C_1 = -\mu.L/(2.T_x)]$, expressed by:

$$\frac{dy}{dx} = \text{senh}\left(\frac{\mu.x}{T_x} - \frac{\mu.L}{2.T_x}\right) \tag{3}$$

Considering the hyperbolic identity $\cosh^2(x)$ - senh^2 (x) = 1 and with the help of Equation (3) it is possible to write:

$$\frac{d^2y}{dx^2} = \frac{q}{T_x} + \frac{\mu}{T_x}.\cosh^2\left(\frac{\mu.x}{T_x} - \frac{\mu.L}{2.T_x}\right) \tag{5}$$

From the Galerkin Method and rewriting Equation (1) one has:

$$T_x \int_\Omega N_j.\frac{d^2N_i}{dx^2}d\Omega\,.\,y_i^e - \int_\Omega [q + g(x)].N_j d\Omega = 0 \quad i,j = 1,2. \tag{6}$$

$$g(x) = \mu\,.\cosh^2\left(\frac{\mu.x}{T_x} - \frac{\mu.L}{2.T_x}\right) \tag{7}$$

Since N_j and N_i are functions of ξ, we use the Jacobian J and replace x by $x(\xi)$ to perform a matching of variables. Thus, the derivative of N with respect to x can be obtained by the Chain Rule using:

$$\frac{dN}{dx} = \frac{dN}{d\xi}.\frac{d\xi}{dx} = \frac{dN}{d\xi}.\frac{2}{L_e} \tag{8}$$

Replacing Equation (8) into (6), and integrating by parts the first term of Equation (6), one obtains:

$$T_x \int_{-1}^{1} \frac{dN_j}{d\xi}.\frac{dN_i}{d\xi}.\frac{2}{L_e}d\xi\,.\,\mathrm{y_i} = -\int_{-1}^{1}\left[q + g\big(x(\xi)\big)\right].N_j.\frac{L_e}{2}.d\xi + \mathrm{CC} \quad i,j = 1,2. \tag{9}$$

From Equation (9) it is possible to recognize that:

$$\boldsymbol{K^e} = T_x \int_{-1}^{1} \boldsymbol{B^T}.\boldsymbol{B}\,.\frac{2}{L_e}.d\xi \tag{10}$$

and

$$\boldsymbol{f^e} = -\int_{-1}^{1}\left[q + g\big(x(\xi)\big)\right].\boldsymbol{N^T}.\frac{L_e}{2}.d\xi \tag{11}$$

where, $\boldsymbol{K}^e$ is the element stiffness matrix and $\boldsymbol{f}^e$ is the element load vector; $\boldsymbol{N}$ is the vector with the shape functions and $\boldsymbol{B}$ is the vector with the derivatives of shape functions.

When the "q" parameter is null, the expression is equivalent to the catenary suspension cable; otherwise, when "g" is null, the expression is equivalent to parabolic cable [10, 11, 12].

Moreover, T_x force is obtained through iterations by the secant method, in which ${}^1T_x = -q.L^2/8.f$ and ${}^0T_x = 1{,}01.{}^1T_x$, with satisfatory results.

Once the local matrices are built, it is possible to assemble the global stiffness matrix of the structure. Then, the global system of equations is getting up and, after introducing of boundary conditions, the system can be solved.

2.3. The Generalized Finite Element Method

The usual and widespread known Finite Element Method is based mainly in Lagrangean polynomials, due to easy computational implementation of shape functions and their derivatives. But, in general, they are not hierarchical and for each time that a refinement is necessary, the whole system needs to be rebuilt. Hierarchical methods became more attractive since the refinement can be done without rebuilding the matrices. Among the hierarchical methods, those based in the Partition of Unit

Method (PUM) are recommended and the Generalized Finite Element Method (GFEM) is the widely used [3, 4, 6, 23, 26].

For hierarchical methods, Lobatto's functions are used largely and, for cable elements, it was chosen the function:

$$l_2(\xi) = \frac{1}{2}\sqrt{\frac{3}{2}}\,(\xi^2 - 1\,) \tag{12}$$

Also, for the finite element drawn in Figure 3, two sets of enrichment functions are adopted here:

$$\boldsymbol{HY1^T} = \begin{bmatrix} HY1_1 \\ HY1_2 \end{bmatrix} = \begin{bmatrix} -\cosh(\xi + 1) - \xi \\ -\cosh(\xi - 1) + \xi \end{bmatrix} \tag{13}$$

$$\boldsymbol{HY2^T} = \begin{bmatrix} HY1_1 \\ HY1_2 \end{bmatrix} = \begin{bmatrix} -\cosh(\xi . k + k) + \xi^2 \\ -\cosh(\xi . k - k) + \xi^2 \end{bmatrix} \tag{14}$$

where $k = \mu/T_x$ is constant. Figure 4 shows the enrichment functions l_2, $HY1$ and $HY2$.

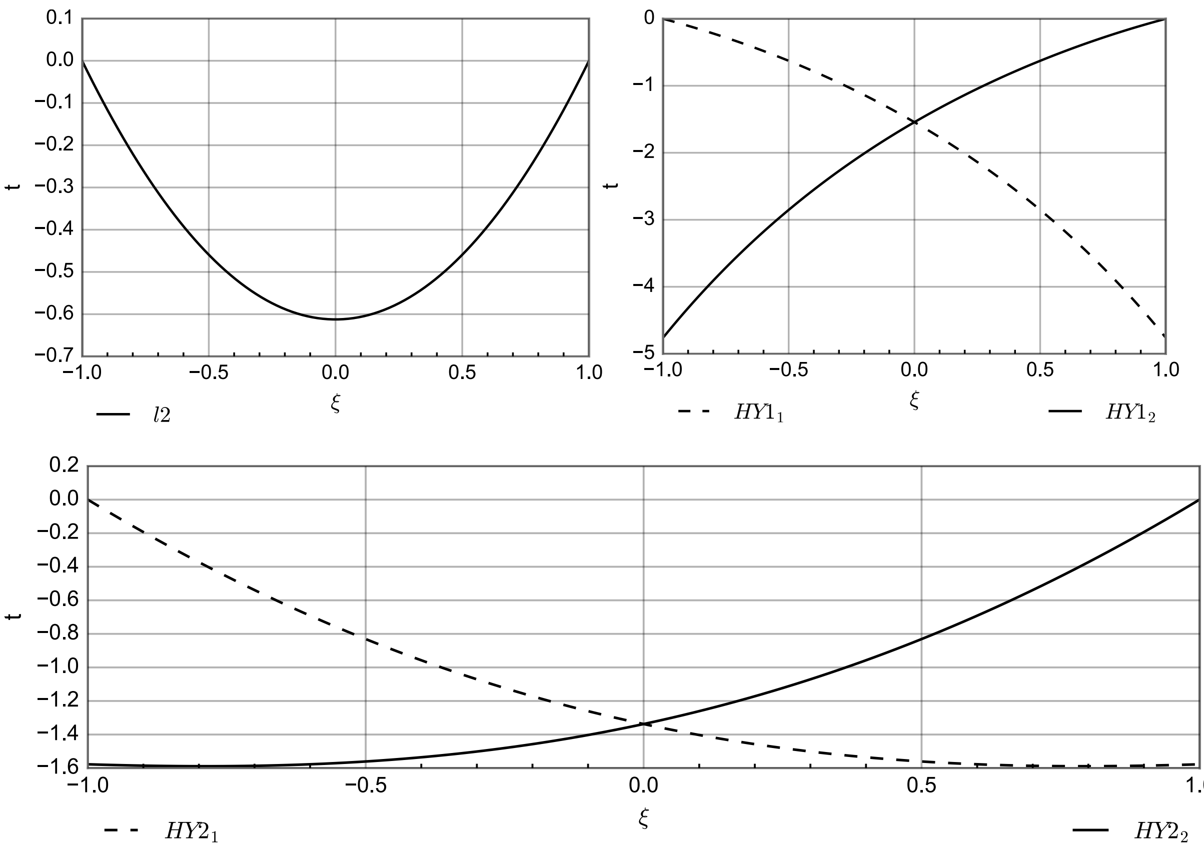

Figure 4 – Enrichment functions l_2, $HY1$ and $HY2$, where $k = 0{,}8$

The interpolations of GFEM can be divided in two parts, the conventional functions of FEM plus the enrichment functions, as indicated in Equation (15):

$$y^e(\xi) = y^e_{FEM} + y^e_{ENRICH} \tag{15}$$

$$y^e_{FEM}(\xi) = \sum_{i=1}^{n} N_i(\xi) \times y_i \tag{16}$$

$$y^e_{ENRICH}(\xi) = \sum_{i=1}^{2} N_i(\xi) \times \left[\sum_{j=1}^{n_l} \gamma_j(\xi) \times a_{ij}\right] \tag{17}$$

$$\boldsymbol{N} = [N_1 \quad N_2] = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right] \tag{18}$$

where n_i is the number of enrichment levels, $N_i(\xi)$ are partition of unit linear functions, given by Equation (18) and a_{ij} are the field degrees of freedom related to the enrichment functions γ_j, which have no physical meaning but are necessary to interpolation process. Functions determined by Equation (17) are shown in Figure 5 for $l2^{enr}$, $HY1^{enr}$ and $HY2^{enr}$ functions.

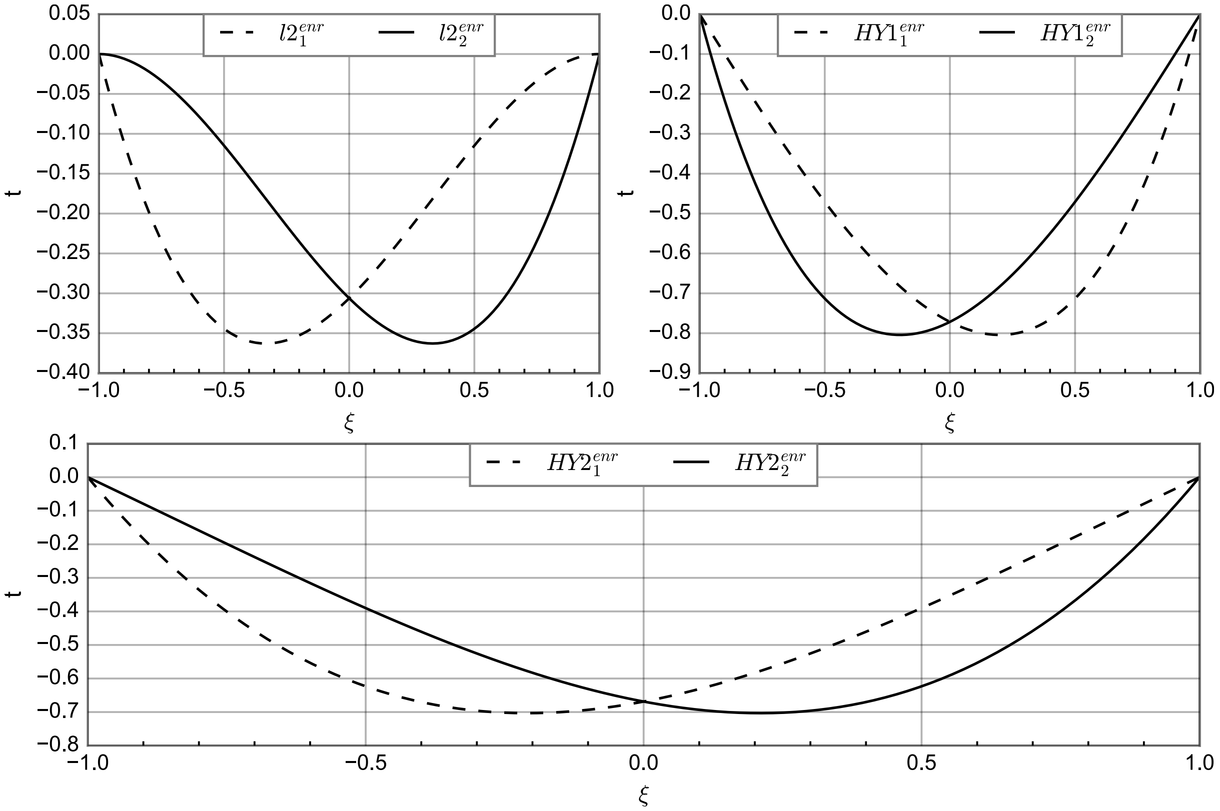

Figure 5 – Enriched functions $l2^{enr}$, $HY1^{enr}$ and $HY2^{enr}$, where $k = 0{,}8$

The cable GFEM element can be built by the following vectors:

$$\boldsymbol{\Phi} = [N_1 \quad N_2 \quad R_1 \quad R_2] \tag{19}$$

$$\boldsymbol{\Phi}' = [N'_1 \quad N'_2 \quad R'_1 \quad R'_2] \tag{20}$$

where R_1 and R_2 are the enriched functions $l2_1^{enr}$ and $l2_2^{enr}$ or $HY1_1^{enr}$ and $HY1_2^{enr}$ or $HY2_1^{enr}$ and $HY2_2^{enr}$. So, the stiffness matrix and the load vector can be easily built by:

$$\boldsymbol{K^e} = T_x \int_{-1}^{1} \boldsymbol{\Phi}'^T . \boldsymbol{\Phi}' . \frac{2}{L_e} . d\xi \tag{21}$$

$$\boldsymbol{f^e} = -\int_{-1}^{1} [q + g(x(\xi))]. \boldsymbol{\Phi}^T . \frac{L_e}{2} . d\xi \tag{22}$$

2.4. The Condition Number

The numerical stability of GFEM is strongly affected by the level of enrichment adopted. Many works show that the condition number of the stiffness matrix for the enrichment system of GFEM grows up indicating an ill conditioning of the system equation [2, 23]. So, for each new enrichment level, the stiffness matrix numerically approaches more and more of a singular matrix, making difficult the numerical solution or even forbidden it. The condition number can be obtained as:

$$\text{cond}(M) = \frac{\max(\lambda)}{\min(\lambda)} \tag{23}$$

where λ is the eigenvalue of the stiffness matrix.

2.5 Convergence criterium

Due the nonlinearity of the cable problem as can be seen in Equation (1), two different criterions for verification of convergence are adopted. The first one is given by

$$\frac{\sqrt{\sum_{i=1}^{n} ({}^k\boldsymbol{\Psi}_i)^2}}{\sqrt{\sum_{i=1}^{n} ({}^k\text{Img}(y)_i)^2}} \times 100 \leq \ tol \tag{24}$$

where *n* is the total number of values in the image of function ${}^k\text{Img}(y)$ and k is the iteration step. The vector ${}^k\boldsymbol{\Psi}$ is a residue vector obtained as:

$${}^k\boldsymbol{\Psi}(\boldsymbol{y}) = \left|\left|{}^k\text{Img}(y) - {}^{k-1}\text{Img}(y)\right|\right| \neq 0 \tag{25}$$

The other criterium convergence is:

$$\left|y^*({}^kT_x)\right| - (h + f) \leq \ tol_y \tag{26}$$

where $y^*({}^kT_x)$ is the smallest value of the function image in iteration step k given by:

$$y^*({}^kT_x) = \min(\text{Img}(y)) \qquad k = 0, 1, 2 \ldots, n \tag{27}$$

and the *h* and *f* values are initial input data in the program illustrated in the Figure 2.

4. Examples

Two examples are presented here. The first one is a suspended cable subjected to its own weight. The second one is a suspended cable subjected to concentrated forces. Both examples are solved by

FEM, GFEM with many different enrichment levels that are presented in Table 1, and the results are compared with the analytical solution and some solutions found in the literature.

Table 1 – Enrichments used in the present paper

Results	**Enrichment Functions Used**
GFEM 01	l_2 (1st level)
GFEM 02	$HY1$ (1st level)
GFEM 03	$HY2$ (1st level)
GFEM 04	l_2 (1st level); $HY1$ (2nd level)
GFEM 05	l_2 (1st level); $HY2$ (2nd level)
GFEM 06	$HY1$ (1st level); $HY2$ (2nd level)

4.1 Example 01: Suspended cable subjected to its own weight.

For this example, the cable transversal area is $A = 0{,}5\ cm^2$ and its elasticity modulus is $E = 165000\ MPa$. The geometric characteristics of the problem are illustrated in Figure 6. The results are presented in Table 2, where is shown the numerical solutions presented by Pereira Jr [19], and Costa [8]. Both references used conventional finite elements with different refinements. The analytical solution is presented by Hibbeler [25]. The GFEM solutions were obtained using just one element with the enrichment levels shown in Table 1. A solution with 10 elements and one enrichment level is also presented. The results presented in Table 2 show that the GFEM analysis can reach the analytical solution with just one finite element with two levels of enrichments. The approximation is as good as the solutions of Pereira Jr [19] and Costa [8] with 500 elements. It is important to note that the number of interactions is grater for GFEM and the condition number really grows up when the enrichment level is incremented. However, this fact did not impact the final results for this case, since the biggest value of the condition number was 6,1E+11 for the five levels case.

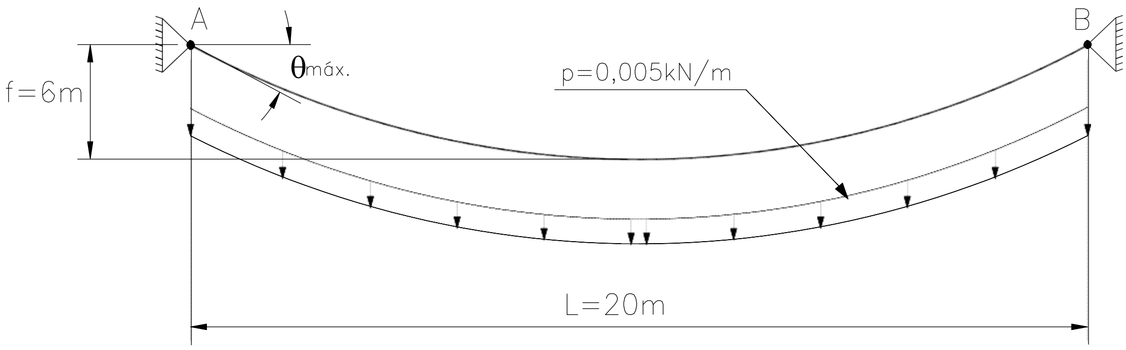

Figure 6 – Suspended cable subjected to its own weight (Source: Pereira Júnior, 2002)

Table 21 – Results for suspended cable subjected to own weight

Results	**S_o (cm)**	**T_{max} (N)**	**T_x (N)**	**θ_{max} (°)**	**Ite.** -	**cond(K_{res})** -
Pereira J. (2002) - 10 elements	2415,47	70,05	46,05	48,89	2	-
Pereira J. (2002) - 500 elements	2418,82	75,81	45,94	52,70	1	-
Costa (2014) - 10 elements	2415,49	70,05	46,05	48,90	3	-
Costa (2014) - 500 elements	2418,84	75,81	45,94	52,70	3	-

Results	**S_o (cm)**	**T_{max} (N)**	**T_x (N)**	**θ_{max} (°)**	**Ite.** -	**cond(K_{res})** -
GFEM 01(1 element)	2408,69	76,11	46,65	52,20	4	1,7E+00
GFEM 02 (1 element)	2418,57	75,95	45,96	52,76	4	9,6E+00
GFEM 03 (1 element)	2408,69	76,11	46,65	52,20	4	1,7E+00
GFEM 04 (1 element)	2418,76	75,95	45,95	52,77	4	1,7E+04
GFEM 05 (1 element)	2418,94	75,94	45,94	52,78	4	6,1E+11
GFEM 06 (1 element)	2418,76	75,95	45,95	52,77	4	1,4E+04
GFEM 02 (10 elements)	2418,80	75,95	45,95	52,77	4	1,3E+02
Hibbeler (2011) - Analytical	2418,82	75,94	45,94	52,77	-	-

*Results GFEM 02, GFEM 05 and GFEM 06 were obtained by $HY2$ with $k = 41{,}67$.

4.2. Example 02: Suspended cable subjected to concentrated loads.

The second example intends to examine the influence of concentrated loads over the cable, since they try to polygonise the cable configuration. The problem was also analyzed by Pereira Jr [19] and Costa [8], that used a conventional finite element method approach. The geometrical and load characteristics of the cable can be seen in Figure 7. The own weighted of the cable is $\mu = 25\ N/m$, the cross transversal area is $A = 2{,}5\ cm^2$ and the Young Modulus is $E = 165000\ MPa$. The reference solution is presented in Hibbeler [25] and it was obtained by single static equilibrium, without consider the elastic strains of the cable and without the self-weight of the cable. So, some little difference is expected between the numerical solutions and the analytical one.

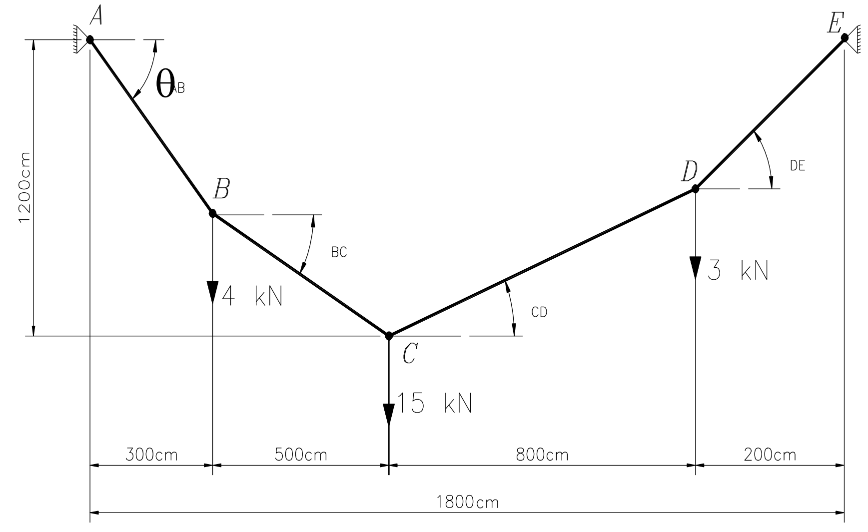

Figure 7 – Example 02 – Suspended cable subjected to concentrated loads
(Source: Pereira Júnior, 2002)

The GFEM solution was obtained with 18 finite elements and the enrichment function was the HY1, Equation (13). As can be seen in Table 3, the GFEM solution is closer to the analytic one. The numerical solution was obtained after 15 iterative steps. The difference observed between GFEM and the others computational solutions is due the nonlinear formulation, that is not implemented yet.

Table 3 – Solutions for the suspended cable subjected to concentrated loads

Variáveis	**Hibbeler [25]**	**Pereira Junior [19]** **Cables - NLFG**	**Costa [8]** **ASTRAS**	**Present** **GFEM**
S_o (cm)	3015,00	3024,07	3024,13	3014,84
f_{max} (cm)	1200,00	1199,95	1199,99	1200,00
θ_{AB} (graus)	62,20	63,33	63,33	62,31
θ_{BC} (graus)	51,60	53,52	53,52	51,79
θ_{CD} (graus)	47,90	45,04	45,04	47,39
θ_{DE} (graus)	57,70	55,94	55,95	57,76
T_x (kN)	6,33	6,54	6,54	6,42
T_{AB} (kN)	13,60	14,56	14,56	13,81
T_{BC} (kN)	10,20	10,99	10,99	10,37
T_{CD} (kN)	9,44	9,25	9,25	9,48
T_{DE} (kN)	11,80	11,67	11,67	12,03
V_A (kN)	12,00	13,05	13,05	12,23
V_B (kN)	10,00	9,71	9,71	10,23

5. Conclusions

This work shows a first application of GFEM for cable structures. Due the characteristics of the cable, whose deformed configuration tends to be approximated by a catenary or a hyperbolic function, the use of convenient enrichment functions can lead the solutions for a best approximation. The consequence is that meshes with just one element and lower number of enrichment levels are sufficient to reach good results, comparable with the conventional FEM with a large number of elements. The examples show that the condition number increase as the enrichment level number is greater, but for the six results presented, its magnitude was not significant for the static analysis. The results here obtained are specific for linear static analysis of a suspended cable. The next steps are to introduce the geometrical nonlinear formulation for incorporate large displacements and, afterword, investigate the dynamic behavior of cable models, mainly their frequencies and vibration modes.

Acknowledgements

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