XL CILAMCE
IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING
NOVEMBER 11-14, 2019
Praiamar Natal Hotel & Convention
Natal, RN-BRAZIL

STUDY OF POLYMOMIAL ENRICHMENT FOR ANALYSIS OF STRUCTURES IN DAMAGE PROCESS USING S/GFEM

Guilherme Olivera Ferraz de Paiva
guilherme.paiva@ifpi.edu.br
Instituto Federal de Ciência e Tecnologia do Piauí - IFPI
R. Álvaro Mendes, 94 - Centro, Teresina - PI, 64000-040
Francisco Evangelista Junior
fejr.unb@gmail.com
Universidade de Brasília - UnB
Programa de Pós Graduação em Estruturas e Construção Civil, Prédio SG 12, primeiro andar Campus Darcy Ribeiro, Universidade de Brasília, 70910-900, Brasília, Distrito Federal, Brasil.

Abstract. This work aimed to implement polynomial enrichment functions, according to the strategy of the Stabilized Generalized Finite Element Method (S/GFEM), for application in simulations of failure problems in structures using a bilinear damage model for quasi-brittle materials. The results were verified by comparison with experimental curves drawn from the three-point bending test in central notched beam. The efficiency and accuracy of the polynomial S/GFEM were then tested to improve the prediction of the rupture behavior over a conventional Finite Element analysis. This has become more evident in simulations with coarse meshes, presenting results with precision equivalent to others found in the literature, but with a much smaller number of elements.

Keywords: S/GFEM, Polynomial Enrichment, Continuum Damage Mechanics, Quasi-brittle materials.

1 Introduction

The conventional Finite Element Method (FEM) cannot satisfactorily describe some boundary value problems (BVPs) such as problems dealing with crack propagation, whose description is naturally more complex because it involves the singularity phenomenon in the stress field, high deformations and discontinuity in the unknown fields. More complicated cases require high mesh refinement making analysis prohibitive due to the computational cost involved (Babuška *et al.* [1]).

In this context, the methods without mesh started to gain space. These are numerical methods used in the BVP solution in order to reduce or even eliminate the dependence between the equations that govern the problem and their discretization. And while not a meshless method per se, the Generalized Finite Element Method (GFEM) follows the same principle of seeking to minimize the importance of mesh to the quality of the result (Babuška *et al.* [1]; Belytschko *et al.* [2]; Duarte [3]; Duarte *et al.* [4], Paiva *et al.* [5]).

Recently, Babuška and Banerjee [6] and Babuška and Banerjee [7] have presented a new approach to one-dimensional domains, the so-called Stabilized Generalized Finite Element Method (S/GFEM), which aims to improve the conditioning of the GFEM stiffness matrix.

This work implemented S/GFEM formulations applied in the analysis of structures in the process of damage. Polynomial enrichment functions, according to the method strategy, were used to simulate structural failure problems using a bilinear damage model. The enrichment was employed in order to optimize the results obtained in simulations using coarse meshes.

2 Generalized Finite Element Method

The GFEM is based on the conventional FEM. It was initially proposed by Babuška *et al.* [1] under the designation of the Special Finite Element Method. The strategy of the GFEM consists of the combination of the functions derived from the Partition of Unit (PU) – FEM standard shape function - and linearly independent functions, defined in $\mathcal{I}_j \stackrel{\text{def}}{=} \{L_{j1}(x), L_{j2}(x), \ldots, L_{jq}(x)\}$ with $\{L_{j1}(x) = 1\}$ where $L_{ji}(x)$ are the enrichment functions defined at each node x_j of the domain, and q the total number of enrichment functions relative to node x_j.

These functions can be polynomial or obtained from a priori knowledge of the behavior of the BVP solution to be analyzed. The product between the form functions derived from the PU by the enrichment functions results in the product function or form function ϕ_{ji} of GFEM (Eq. (1)).

$$\{\phi_{ji}(x)\}_{i=1}^{q} = \mathcal{N}_j(x)\,\{L_{ji}(x)\}_{i=1}^{q} \tag{1}$$

As a result of this process, the GFEM form functions ϕ_{ji}(x) are tied to node x_j. This is because they inherit from the PU the property. Maintaining such characteristics is vital to ensure continuity between the initial mesh elements (Strouboulis *et al.* [8]). The approximation of the GFEM is thus obtained from the following linear combination presented in Eq. (2) where $\mathcal{N}_j(x)$ refers to the FEM, and the product $\mathcal{N}_j(x)L_{ji}(x)$, refers to GFEM.

$$\tilde{u}(x) = \sum_{j=1}^{n} \mathcal{N}_j(x).\left\{u_j + \sum_{i=2}^{q} L_{ji}(x).\,b_{ji}\right\} \tag{2}$$

Mathematically u_j and b_{ji} represent, respectively, the degrees of freedom of the structure tied to the node x_j of the cloud w_j; wherein the latter represents the additional degrees of freedom corresponding to each component *i* of the enriched form functions. The monomials used to enrich the domain nodal points can be constructed hierarchically with the Pascal Triangle generally assuming the following format:

$$L(p,q) = \frac{(X - X_j)^p\ (Y - Y_j)^q}{h^{p+q}} \tag{3}$$

where X and Y are the coordinates of the Gauss points in each element; X_j and Y_j, are the coordinates of the nodal points to enrich; p and q are the powers that determine the degree of enrichment. The term h acts as a standardization so that information associated with elements is not introduced into enrichment (Duarte *et al*. [9]). In this work the enrichment function sets were constructed using the extreme terms of the Pascal Triangle. Then, for example, $L_{P=2} = \left\{\frac{x-x_\alpha}{h_\alpha}, \frac{y-y_\alpha}{h_\alpha}, \frac{(x-x_\alpha)^2}{h_\alpha^2}, \frac{(y-y_\alpha)^2}{h_\alpha^2}\right\}$.

2.1 Stabilized GFEM (S/GFEM)

It is recognized that both GFEM and XFEM have excellent convergence properties, but the stiffness matrix associated with these methods can be (and often is) poorly conditioned. First approached in Babuška and Banerjee [6], the S/GFEM arose from the proposition of a simple modification to the existing GFEM structure that ensures the elimination of the problem of poor conditioning.

Babuška and Banerjee [6] showed that the S/GFEM have a good level of convergence. Another advantage is that S/GFEM does not need the so-called ramp functions on transition elements in regions where different types of enrichment functions are applied, as proposed by Fries [10]. In S/GFEM a simple local modification of the enrichments used in GFEM is used to construct the approach spaces $\mathcal{X}_\alpha$, where $\alpha \in I_h^e$, as the Eq. (4).

$$\tilde{L}_{\alpha i}(\boldsymbol{x}) = L_{\alpha i}(\boldsymbol{x}) - I_{\omega\alpha}(L_{\alpha i})(\boldsymbol{x}); \qquad \tilde{\mathcal{X}}_\alpha = span\{\tilde{L}_{\alpha i}\}_{i=1}^{m_\alpha} \tag{4}$$

where: $I_{\omega\alpha}(L_{\alpha i})$ is the linear or bilinear interpolation portion of the enrichment function $L_{\alpha i}$ applied to the node ω_α; $\tilde{L}_{\alpha i}$ is the modified enrichment function of S/GFEM. In this case, the term $L_{j1}(x)$ of $\mathcal{I}_j$ is deleted because it results in a null value.

3 Damage model

The damage model used in this work was proposed by Moreira and Evangelista Jr. [11] and is applied to *quasi*-brittle materials under loading conditions that produce crack propagation mode I or mixed mode. The capacity and limitations of this model are defined by the following hypotheses: the material is considered an elastic medium when evolving damage process and, therefore, are not considered plastic deformations; material damage is the result of extensions along the main stress directions, i.e. local ruptures begin and develop in mode I; finally, the damage is represented by a scalar variable D ($0 \leq D \leq 1$), which means assuming an isotropic damage condition for the material (Moreira [12]; Moreira and Evangelista Jr. [11]; Paiva [13]; Paiva *et al*. [5]). In order for the damage model to be thermodynamically compatible, a control variable, a damage initiation surface, and a damage evolution law must be described (Lemaitre and Chaboche [14]).

The control variable adopted was the equivalent strain (ε_{eq}^{MA}) pesented by Mazars [15] since it is suggested by the literature as a simple and efficient measure (Mazars [15]; Mazars and Pijaudier-Cabot [16]; Schlangen [17]; Geers [18]; Simone [19]; Proença and Torres [20]; Hofstetter and Meschke [21]). The Eq. (5) presents the relationship between the control variable and the strain tensor.

$$\varepsilon_{eq}^{MA} = \sqrt{\sum_{i=1}^{3}\left(\langle\varepsilon_i\rangle_+\right)^2} \tag{5}$$

where $\langle\varepsilon_i\rangle_+ = (\varepsilon_i + |\varepsilon_i|)/2$ and ε_i the main deformation in the direction i.

The Fig. 1 shows the idealization model of the fracture process, where the softening law is inserted as the material reaches its mechanical tensile strength (Camacho and Ortiz [22]; Ortiz and Pandolfi [23]). The fracture process zone is described by a stress-strain relationship (σ - ε). The micro cracks grow and coalesce after the concrete tensile strength (f_t) is reached. However, the paste-aggregate interaction remains imposing some resistance to crack opening. This step is responsible for dissipating the initial fracture energy (G_f), which defines the first slope of the model curve, whose coordinates are defined by the deformation ε_k and by the kink point ψ.

The total fracture energy (G_F) and the f_t, together define the maximum load of the structure. When the crack opening displacement reaches a certain magnitude (ε_f) a force-free surface arises that corresponds to a macro crack (Planas and Elices [24]; Bazant [25]; Elices *et al.* [26]).

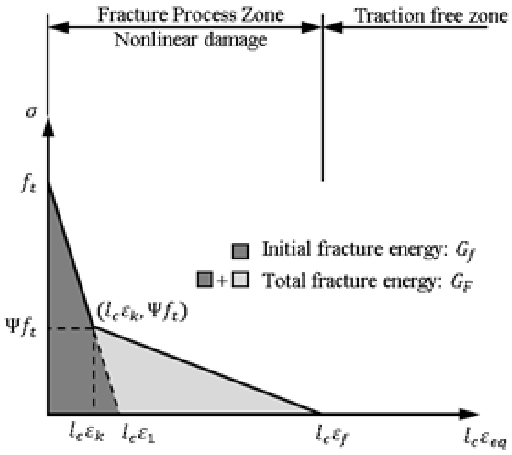

Figure 1. Softening behavior – Constitutive relation force-displacement and equivalence of strains as a function of the characteristic length $\boldsymbol{l_c}$ (adapted from Evangelista Jr *et al.* [27]).

The criterion of onset and evolution of damage is directly related to equivalent deformation. Damage onset occurs when the equivalent strain reaches a critical strain (ε_{d0}) corresponding to the maximum stress of a uniaxial tensile specimen. The law of damage evolution is related to the fracture energy G_F across a characteristic length l_c (Oliver [28]).

3.1 Damage and S/GFEM

Certain phenomena, such as crack propagation, can only be satisfactorily described by continually modifying and updating the mesh. Special methods of analysis, such as S/GFEM, were developed to overcome this and other problems. As in Moteiro *et al.* [29], where the polynomial GFEM was employed for nonlinear analyzes using a constitutive model of elastoplastic damage.

Continuous Damage Mechanics (CDM) is efficient for describing material behavior when analyzing nonlinear physical behavior structures. It is possible to combine this feature of CDM with another great quality of S/GFEM, the flexibility of application, so that it is possible to refine the response only in regions where there is a real need. Polynomial shape functions are able to reproduce with good quality the displacement field for various damage levels (Barros *et al.* [30]).

4 Numerical simulations

The Three Point Bending (TPB) was analyzed. The numerical simulations results for conventional cementitious materials were compared with experimental results from Roesler *et al.* [31] and Gaedicke and Roesler [32]. The Fig. 2 illustrates the geometry used for this test.

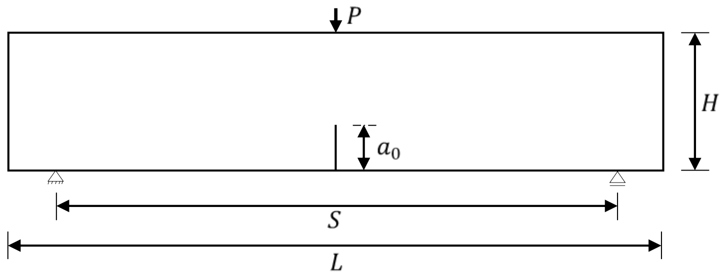

Figure 2. Model geometry, loading, and boundary conditions.

The geometric data of the tested beam are: H=150mm, S=600mm, L=700mm, a_0=50mm, e=80mm. The table 1 shows the fracture and material parameters, respectively. Mazars equivalent strain (ε_{eq}^{MA}). The notch has a width of 2.0 mm. The simulations were performed using controlled displacement.

Table 1. Fracture and material parameters of the TPB test

Fracture parameters			Material parameters			
G_F(N/m)	G_f(N/m)	ψ	E (MPa)	f_c (Mpa)	f_t (Mpa)	υ
164,0	56,7	0,25	32000	58,3	4,15	0,20

Three finite element meshes (all with linear triangular elements of three nodes), two with 2mm rectangular notch (Fig. 3 (a) and 3 (b)) and one with "V" notch (Fig. 3(c)), both with mouth opening of two millimeters. For comparative purposes, we sought to maintain the correspondence regarding the quantity of elements (total quantity and especially in the ligament region - region between the notch tip and the displacement application point) between the last two.

Nodes *Pn* indicate where the degree of polynomial enrichment has varied, with the remaining nodes constantly being enriched with *P1* (where *P1* designates first degree polynomial enrichment and so on) and *P0* indicates that the node has not undergone any kind of enrichment, therefore corresponding to the conventional FEM.

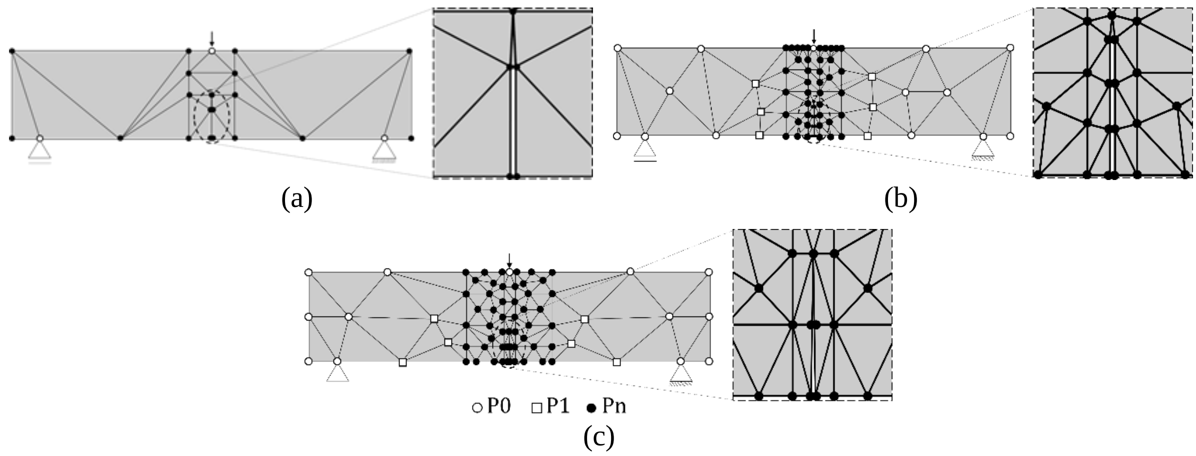

Figure 3. Finite element meshes for TPB: (a) 25 elementes; (b) 101 elementes; (c) 106 elements.

In the meshes of Fig. 3(a) and Fig. 3(b) there is an almost acicular element whose base edge measuring 2 mm describes the end of the notch. This does not occur in the mesh of Fig. 3(c) due to the notch geometry difference. Figure 4 presents the experimental and numerical results of the force (*P*) curves as a function of Crack Mouth Opening Displacement (*CMOD*).

It is noticed that there is no quality in the results without enrichment (*P0*). By contrast, it is apparent that enrichment contributes to improving the quality of results. This potential of the polynomial S/GFEM stands out in the simulations with the coarser meshes, given the good estimate of the maximum resisted load (P_{max}) per beam even using very few finite elements in the test.

It is observed that both the P_{max} and the softening region of the *P* - *CMOD* curve are better represented as the polynomial degree of applied enrichment increases in the same mesh. Naturally, the more refined the model, the smaller the degree of polynomial required. For mesh with 25 elements, for example, the degree of enrichment *P3* was required, and for mesh with 106 elements only *P1* was sufficient to obtain results close to the experimental one.

When using "V" notch, the maximum loads tend to be higher than those presented by the respective simulations with rectangular notch. Due to the symmetry of the problem, the node ahead of the probable "damage path" suffers momentary uncertainty as to which side it will move. This contributes to the fact that P_{max} values, in this case, are relatively higher since the damage model depends on the equivalent deformation of the element. Note also that there is mesh objectivity in all cases, i.e., both P_{max} and the softening region achieve convergence (do not remain decreasing indefinitely).

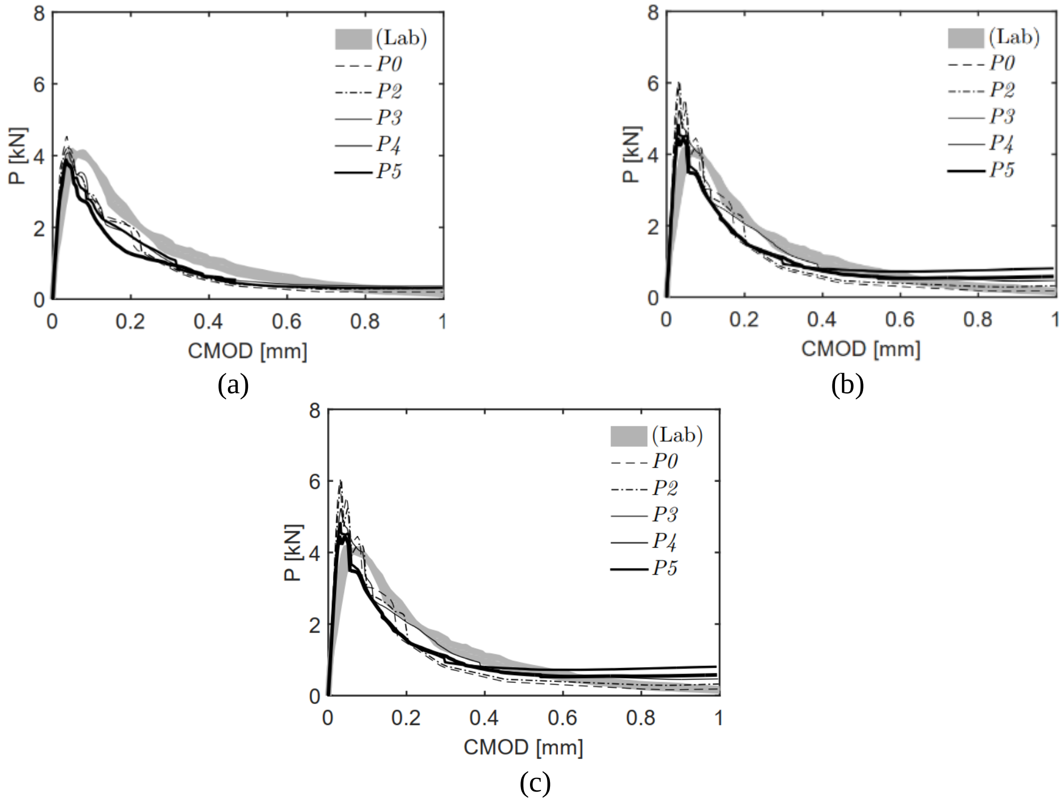

Figure 4. Experimental and numerical *P - CMOD* curves: (a) mesh with 25 elements; (b) mesh with 101 elements; (c) mesh with 106 elements – "V" notch.

By analyzing the distribution of the damage in the fracture zone at the end of the simulation (Fig. 5), it was noted that the damage presented an expected behavior as a function of the experimental result presented by Roesler *et al.* [31]. It is also observed that a better description occurs in the simulations with higher enrichment for the same mesh (damage tends to concentrate in the fracture line). As in the *P – CMOD* curves, the S/GFEM was able to represent well the problem of damage distribution due to the better representation of stress and strain fields as a function of the applied enrichment.

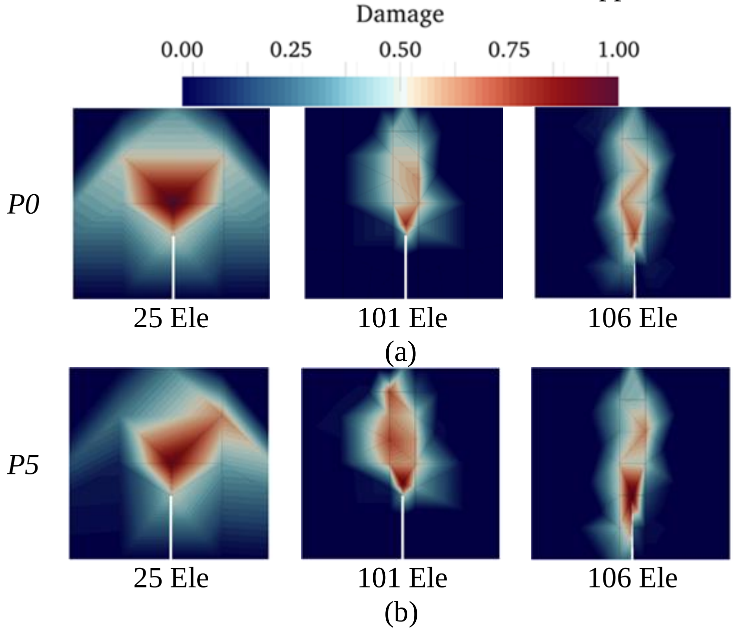

Figure 5. Damage distribution at end of simulations as function of polynomial enrichment degrees.

The Fig. 6 illustrates the convergence study for the mesh with 25 elements, comparing the *P3* simulations using the GFEM and *P3* with S/GFEM strategy. It is noticed that the simulation with S/GFEM with polynomial enrichment presents a larger amount of iterations until reaching convergence in each displacement step, presenting lower computational performance than GFEM.

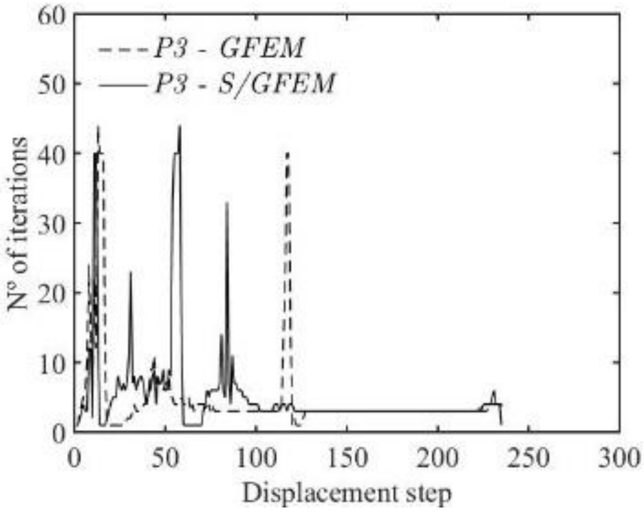

Figure 6. Convergence assessment for the 25 element mesh – comparison between *P3 - GFEM* and *P3 – S/GFEM* simulations.

5 Conclusions

This work implemented S/GFEM formulations applied to the analysis of damaged structures. Polynomial enrichment functions were employed, according to the S/GFEM strategy, to simulate structural failure problems using a bilinear damage model. Results were verified by comparing with experimental results taken from the three point bending test on central notched beams. Enrichment was employed to enable simulations using coarse meshes.

The obtained results proved the efficiency and accuracy of the polynomial S/GFEM to improve the prediction of the rupture behavior even when performing simulations with coarse meshes. In the numerical simulations performed, it was observed good reproduction capacity of the maximum load (P_{max}) as well as the softening behavior observed in the experimental tests, even using a very small number of elements in the mesh.

The S/GFEM presented a larger total amount of iterations than the GFEM. An alternative to improve the computational efficiency of S/GFEM facing the problem analyzed may be the use of the *flat-top* functions. It is also noteworthy that in the enrichment simulations there was a better distribution of damage in the ligament region, mainly due to the better representation of the stress and strain fields, as well as the stress concentration at the notch tip.

Acknowledgements

To the Instituto Federal de Ciência e Tecnologia do Piauí (IFPI) and CNPq for financial support.

References

[1] I. Babuška, G. Caloz and J. E. Osborn. Special Finite Element Methods for a class of second order elliptic problems with rough coefficients. *Journal on Numerical Analysis*, 1994.
[2] T. Belytschko and T. Black. Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering,* 45, 601—620, 1999.

[3] C. A. Duarte and J. T. Oden. An h-p Meshless Method. *Texas Intitute for Computation and Applied Mathematics*, 1995.
[4] C. A. Duarte; T. J. Liszka; W. W. Tworzydlo. Clustered generalized finite element methods for mesh unrefinement, non-matching and invalid meshes. *International Journal for Numerical Methods in Engineering*, 2006.
[5] G. O. F. de Paiva; J. F. A. Moreira; G. S. Alves; F. Evangelista Jr. Estudo do enriquecimento polinomial para análise de estruturas em processo de danificação. *XXXVII Iberian Latin-American Congress on Computational Methods in Engineering - CILAMCE*, 2016.
[6] I. Babuška and U. Banerjee. Stable generalized finite element method (SGFEM). *Technical Report ICES REPORT 11-07, The Institute for Computational Engineering and Sciences, The University of Texas at Austin*, 2011.
[7] I. Babuška and U. Banerjee. Stable generalized finite element method (SGFEM). *Computer Methods in Applied Mechanics and Engineering, 201–204, pp. 91–111*, 2012.
[8] T. Strouboulis, K. Copps and I. Babuška. The Generalized Finite Element Method: an example of its implementation and illustration of its performance. *International Journal for Numerical Methods in Engineering*, 47, pp. 1401–1417, 2000.
[9] C. A. Duarte; O. N. Hamzeh; T. J. Liszka and W. W. Tworzydlo. A generalized finite element method for the simulation of three-dimensional crack propagation. *Computer Methods in Applied Mechanics and Engineering*, 190, pp. 2227–2262, 2001.
[10] T. P. Fries. A corrected XFEM approximation without problems in blending elements. *International Journal for Numerical Methods in Engineering*, 75, 503—532, 2008.
[11] J. F. A. Moreira e F. Evangelista Jr. Modelo bidimensional contínuo-descontínuo de falha para materiais quase-frágeis. *XXXVIII Iberian Latin American Congress on Computacional Methods in Engineering - CILAMCE*, 2016.
[12] J. F. A. Moreira. Modelo bidimensional contínuo-descontínuo de falha para materiais quasi-frágeis em modo I e modo misto. *Master Thesis. Universidade de Brasília, UnB, p. 112*, 2016.
[13] G. O. F. de Paiva. Estudo e aplicação do enriquecimento polinomial para a análise de estruturas em processo de danificação. *Master Thesis. Universidade de Brasília, UnB, p. 66*, 2017.
[14] J. Lemaitre and J. L. Chaboche. Mechanics of solid materials. *Cambrige University Press*, p. 582. ISBN: 0521328535, 1994.
[15] J. Mazars. Application de la mécanique de l'éndommagement au comportementnon lineaire e à la rupture du béton de structure. *Thèse de doctorat. Université* de Paris VI, 1984.
[16] J. Mazars and G. Pijaudier-Cabot. Continuum damage theory - application to concrete.*Journal of Engineering Mechanics*, 115, pp. 345–365, 1989.
[17] E. Schlangen. Experimental and numerical analysis of fracture processes in concrete. Doctoral thesis. *Delft University of Technology*, p. 117, 1993.
[18] M. G. D. Geers. Experimental Analysis and Computational Modelling of Damage and Fracture. *PhD thesis, Eindhoven University of Technology*, 1997.
[19] A. Simone, G. N. Wells and L. J. Sluys. From continuous to discontinuous failure in a gradient-enhanced continuum damage model. *Computer Methods in Applied Mechanics and Engineering*, 192, pp. 4581–4607, 2003.
[20] S. P. B. Proença and I. Torres. Generalized Finite Element method for Nonlinear Three dimensional Analysis of Solids. *International Journal of Computational Methods*, 5, pp. 37–62, 2008.
[21] G. Hofstetter and G. Meschke. Numerical modeling of concrete cracking. *International Centre for Mechanical Sciences*, p. 532. ISBN: 9873709108963, 2011.
[22] G. T. Camacho and M. Ortiz. Computational modeling of impact damage in brittle materials. *International Journal of Solids and Structures*, 33, pp. 2899–2938, 1996.
[23] M. Ortiz and A. Pandolfi. Finite-deformation irreversible cohesive elements for threedimensional crack-propagation analysis. *International Journal for Numerical Methods in Engineerin*, 44, pp. 1267–1282, 1999.
[24] J. Planas and M. Elices. Asymptotic analysis of a cohesive crack: 1. Theoretical background. *International Journal of Fracture*, 55, 153—177, 1992.
[25] Z. P. Bazant and M. Jirásek. Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics*, 128, 1119—1149, 2002.

[26] M. Elices; G.V. Guinea; J. Gómez; J. Planas. The cohesive zone model: advantage, limitations and challenges. *Engineering Fracture Mechanics*, 69, pp. 137–163. 2002.
[27] F. Evangelista Jr, J. R. Roesler and S. P. B. Proença. Three-dimensional cohesive zone model for fracture of cementitious materials based on the thermodynamics of irreversible processes. *Engineering Fracture Mechanics*, 97, pp. 261–280, 2013.
[28] J. Oliver. A consistent characteristic length for smeared cracking models. *International Journal of Numerical Methods in Engineering*, 28, pp. 461–474, 1989.
[29] A. B. Moteiro, A. R. V. Wolenski, F. B. Barros, R. L. da S. Pitangueira and S. S. Penna. GFEM nonlinear analysis using an elastoplastic damage constitutive model. *XXXVIII Iberian Latin American Congress on Computacional Methods in Engineering*, 2016.
[30] F. B. Barros, S. P. B. Proença and C. S. Barcellos. Generalized finite element method in structural nonlinear analysis - a p-adaptative strategy. *Computational Mechanics*, 33, pp. 95–107, 2004.
[31] J. R. Roesler, G. H. Paulino, K. Park and C. Gaedicke. Concrete Fracture Prediction Using Bilinear Softening. *Cement and Concrete Composites*, 29, pp. 300–312, 2007.
[32] C. Gaedicke and J. Roesler. Fracture-Based Method to Determine Flexural Capacity of Concrete Beams on Soil. *Road Materials and Pavement Design*, 11, pp. 361–385, 2010.
[33] F. M. Sato. Experimentos numéricos com versões estáveis do Método dos Elementos Finitos Generalizados. Master Thesis. *Escola de Engenharia de São Carlos, Universidade de São Paulo, São Carlos*, p. 97, 2017.
[34] C. S. Ramos and S. P. B. Proença. On the employment of flat-top partition of unity on GFEM. *XIII Simpósio de Mecânica Computacional – SIMMEC*, 2018.