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# MODEL UPDATING ANALYSIS BASED ON BAYESIAN INFERENCE

**Michael L. M. de Souza**
**Ney Roitman**
*michael.souza@coc.ufrj.br*
*roitman@coc.ufrj.br*
*Av Athos da Silveira Ramos, 149, 21941-909, Rio de Janeiro/RJ - Brazil*
*Department of Civil Engineering COPPE/UFRJ , Brazil*
**Daniel A. Castello**
**Thiago G. Ritto**
*castello@mecanica.coppe.ufrj.br*
*tritto@mecanica.coppe.ufrj.br*
*Av Athos da Silveira Ramos, 149, 21941-909, Rio de Janeiro/RJ - Brazil*
*Department of Mechanical Engineering COPPE/UFRJ*

**Abstract.** The aim of this work is to estimate unknown system parameters based on observed dynamic data, e.g. natural frequencies and damping rates. These dynamical inputs are extracted from experimental modal analyses on a simply supported aluminum beam. Measurements come from three accelerometers and the input excitation is provided by an impact hammer. The experimental setup is submitted into a process of assembling and disassembling the beam for the purpose of increase the variability of modal data. The Young's modulus and the coefficients of the proportional damping model are considered the updating variables in this study. The exploration of the posterior density function (pdf) of these unknown model parameters is performed by a novel Markov Chain Monte Carlo method (MCMC) named Delayed Rejection Adaptive Metropolis (DRAM). In other words the Bayesian paradigm for inverse problems is adopted to tackles the structural identification. The impact of two likelihood functions at posterior parameter distributions is analyzed.

**Keywords:** Model updating, Bayesian framework

# 1 Introduction

Numerical models are used to simulate the behaviour of physical systems in all areas of science and engineering. There are several applications in design, decision making and predictive analysis. A key-aspect for these analyses is the reliability of the predictions provided by the adopted model, Mottershead and Friswell [1].

It is not easy to ensure the compliance with the real system due to the great number of unknowns and uncertainties related to material and physical properties, boundary conditions, load conditions, etc. Simplifying hypotheses are often made which may detract the quality and accuracy of the predictions provided by the computational model.

These issues had led to the development of model updating techniques Simoen et al. [2], which aims to calibrate unknown system parameters, based on observed data of the real structure. In a structural mechanics context, measurements of the system of interest (accelerations, natural frequencies, damping ratios, modal shapes, etc.) provide the experimental information required to perform model updating analyses.

Unfortunately, measurements are always subject to some level of variability which diminishes the confidence on models inasmuch as these are calibrated based on measurements. Therefore, it is imperative to determine the reliability of predictions obtained from numerical models.

In this work the Bayesian inference for inverse problems is adopted, Tarantola [3], Kaipio and Somersalo [4], Aster et al. [5]. The unknown parameters are modeled as random variables and the uncertainties inherent to the system are encoded in their probability density distributions (pdf's), Smith [6].

There are two hypotheses made about the covariance, one formulates it as a diagonal matrix and the other considers the covariance as a full matrix.

# 2 Theoretical approach

The model updating analysis and uncertainty evaluation on a simply supported beam are performed based on a finite element (FE) computational model. Some pre-defined functions of CALFEM toolbox (Austrell et al, 2004) are used to construct the FE mesh, assemble global matrices, solve eigenvalue problems, etc.

In a dynamic structure context the physical problem is governed by the well known dynamics's ordinary differential equation, Clough and Penzien [7] and Géradin and Rixen [8].

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{f}$ are respectively the global mass, damping, stiffness and external excitation matrices, and as $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ are the acceleration, velocity and displacement vectors.

## 2.1 Bayesian inference

In the present work the Bayesian inference is performed under the following premises, Gelman et al. [9]:

There are three updating variables which are the Young's modulus ($E$) and the coefficients of Rayleigh proportional damping model, e.i., $\alpha$ and $\beta$ $\boldsymbol{\theta}$

$$\boldsymbol{\theta} = \{E \quad \alpha \quad \beta\}^T \tag{2}$$

The misfit between the computational model $\mathbf{A}(\boldsymbol{\theta})$ and the observed data $\mathbf{y}$ is generated by an additive Gaussian noise $\mathbf{e}$. Thus:

$$\mathbf{y} = \mathbf{A}(\boldsymbol{\theta}) + \mathbf{e} \tag{3}$$

For additive noise, the likelihood probability function $\pi$ casts as:

$$\pi(\mathbf{y}|\boldsymbol{\theta}) = \pi_e(\mathbf{y} - \mathbf{A}(\boldsymbol{\theta})) \tag{4}$$

Bayes rule, Kaipio and Somersalo [4], Bayes [10]:

$$\pi(\boldsymbol{\theta}|\mathbf{y}) = \frac{\pi(\mathbf{y}|\boldsymbol{\theta})\,\pi(\boldsymbol{\theta})}{\pi(\mathbf{y})} \tag{5}$$

A key aspect to build the prior statistical model is that every random variable, e.g. the Young's modulus $E$, and the damping coefficients $\alpha$ and $\beta$, was constructed under the hypothesis that they are mutually independent. Therefore, the prior joint density can be computed by the multiplication of the marginal densities. Moreover, the marginal prior for the variable $r$ is defined to be within the set $D_r$ as follows:

$$\begin{aligned} D_E &= \{E \in \mathbb{R} \,|\quad 6.5 < E < 9\} \qquad [10^{10} kPa] \\ D_\alpha &= \{\alpha \in \mathbb{R} \,|\quad 1 < \alpha < 5\} \qquad [10^{-1}\, s^{-1}] \\ D_\beta &= \{\beta \in \mathbb{R} \,|\quad 0 < \beta < 20\} \qquad [10^{-6}\, s] \end{aligned}$$

### 2.2 Markov chain monte carlo

The Markov Chain Monte Carlo methods, named MCMC, are any mathematical procedure adopted which produces samples $\{\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(N)}\}$ whose stationary distribution is $\bar{\pi}(\boldsymbol{\theta})$, Gamerman and Lopes [11], Robert and Casella [12]. The Metropolis-Hastings and the Gibbs Sampler are two examples of these methods, Metropolis and Ulam [13], Metropolis et al. [14], Hastings [15], Geman and Geman [16].

In this work the Delayed Rejection Adaptive Metropolis (DRAM) is adopted to estimate the Markov chain, Haario et al. [17]. This algorithm combines the Adaptative Metropolis (AD) and the Delayed Rejection (DR) in order to improve its efficiency in accepting/rejecting procedure.

In a classic Metropolis Hastings approach a new candidate $\boldsymbol{\theta}^{(c)}$ is generated from a proposal distribution $q_1$ and its acceptance depends on the MH-acceptance ratio $\alpha_{\mathrm{MH}}^{(1)}$ shown in Eq. (6).

$$\alpha_{\mathrm{MH}}^{(1)}(\boldsymbol{\theta}^{(c)}|\boldsymbol{\theta}^{(t-1)}) = min\left\{1\,,\ \frac{\bar{\pi}(\boldsymbol{\theta}^{(c)})q_1(\boldsymbol{\theta}^{(c)}|\boldsymbol{\theta}^{(t-1)})}{\bar{\pi}(\boldsymbol{\theta}^{(t-1)})q_1(\boldsymbol{\theta}^{(t-1)}|\boldsymbol{\theta}^{(c)})}\right\} = min\left\{1\,,\ \frac{N_1}{D_1}\right\} \tag{6}$$

Upon rejection, instead of keeping the chain at the state $(t-1)$, a second candidate $\boldsymbol{\theta}^{(c,2)}$ is proposed. A second proposal, $q_2$, depends not only on the current position $\boldsymbol{\theta}^{(t-1)}$ of the chain but also on what one have just proposed and rejected $\boldsymbol{\theta}^{(c)}$. The probability of acceptance depends on the updated MH-acceptance ratio $\alpha_{\mathrm{MH}}^{(2)}$ which is ruled by Eq. (7).

$$\begin{aligned}\alpha_{\mathrm{MH}}^{(2)}(\boldsymbol{\theta}^{(c,2)}|\boldsymbol{\theta}^{(c)},\boldsymbol{\theta}^{(t-1)}) =& min\left\{1\,,\ \frac{\bar{\pi}(\boldsymbol{\theta}^{(c,2)})q_1(\boldsymbol{\theta}^{(c)}|\boldsymbol{\theta}^{(t-1)})q_2(\boldsymbol{\theta}^{(c,2)}|\boldsymbol{\theta}^{(c)},\boldsymbol{\theta}^{(t-1)})[1-\alpha_1(\boldsymbol{\theta}^{(c)}|\boldsymbol{\theta}^{(t-1)})]}{\bar{\pi}(\boldsymbol{\theta}^{(t-1)})q_1(\boldsymbol{\theta}^{(t-1)}|\boldsymbol{\theta}^{(c)})q_2(\boldsymbol{\theta}^{(t-1)}|\boldsymbol{\theta}^{(c,2)},\boldsymbol{\theta}^{(c)})[1-\alpha_1(\boldsymbol{\theta}^{(t-1)}|\boldsymbol{\theta}^{(c)})]}\right\}\\ =& min\left\{1\,,\ \frac{N_2}{D_2}\right\}\end{aligned} \tag{7}$$

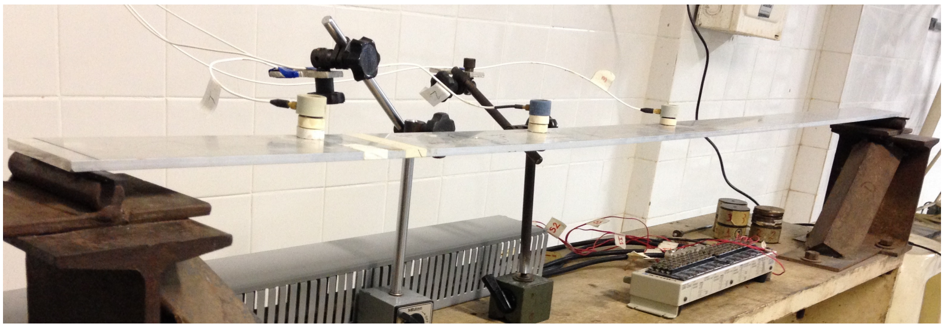
As for the Adaptive Metropolis, the basic ideia of AM is to calibrate the covariance matrix of the proposal distribution, e.g., the Normal distribution, using the sample path of the MCMC, i.e., the past states of the chain. The process starts from an initial covariance matrix $\boldsymbol{\Sigma}^{(0)}$ and the covariance of the proposal pdf is updated at arbitrary intervals according to the rule given by Eq. (8).

$$\boldsymbol{\Sigma}^{(t)} = \left(Cov(\boldsymbol{\theta}^{(t-k)}, \cdots, \boldsymbol{\theta}^{(t-1)}, \boldsymbol{\theta}^{(t)}) + \mathbf{I}_n\eta\right) s \tag{8}$$

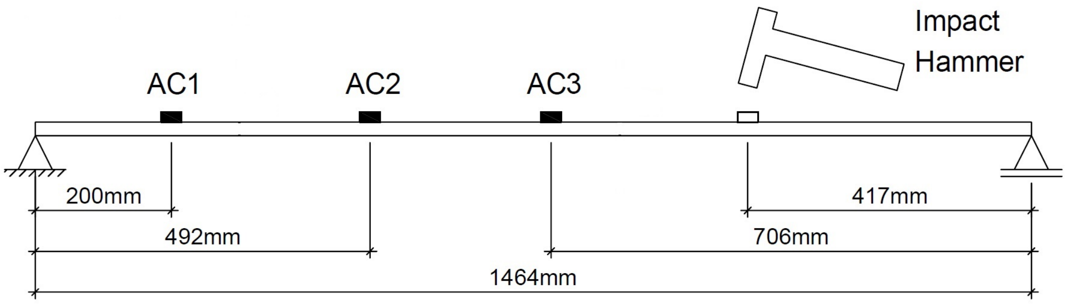
where $\mathbf{I}_n$ represents the $n$-dimensional identity matrix, $\eta$ is a small number used to prevent that the updated covariance matrix $\boldsymbol{\Sigma}^{(t)}$ becomes singular and $s$ is a scaling factor.

## 3 Experimental procedure

A simply supported aluminium beam is taken to perform model updating analyses. The beam's physical properties are: 1464 mm length; rectangular cross section with 76,2 mm of height and 6 mm width and 2700 kg/m$^3$ mass density. A SIMO analysis (single input multiple output) is adopted to perform the modal test. The excitation is provided with non correlated impact of the hammer. Each analysis considers twenty impacts which are performed with thirty seconds mean gap. Measuring frequency range is set as 0-250 Hz and the signal is low-pass filtered with a cut-off frequency of 500 Hz as well as data acquisition frequency set to 1000 Hz. Fig. 1 shows an illustration of the accelerometer's positions, and the impact point of the hammer.



(a) Aluminium beam



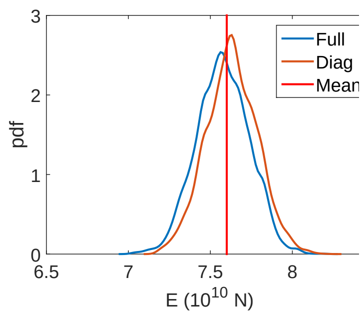

(b) Ilustrative sketch of accelerometers AC1, AC2 and AC3, and impact hammer position.

Figure 1. Experimental set up.

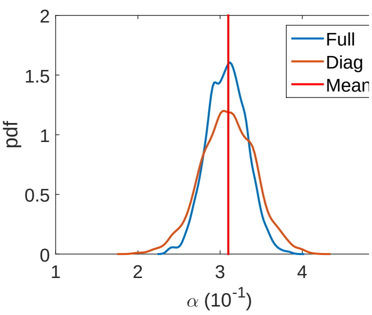
## 4 Results

The results and assessments of the proposed approach using measured data are presented in this section. The experimental data used for Bayesian model updating comprise the natural frequencies and damping rates computed from accelerations measured by the accelerometers AC1, AC2, and AC3 when an impact force is applied to the structure. The dynamic analysis also provided the FRFs for all accelerometer's positions. Computational models were validated by comparisons between the experimental FRFs and their computed counterparts at the accelerometer's position AC1.

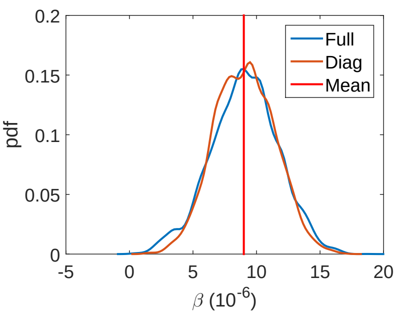
Mean values for unknown vector parameter is $\boldsymbol{\mu}_\theta = [7.6 \times 10^{10} N \quad 3.1 \times 10^{-1} \quad 9 \times 10^{-6}]^T$. Fig. 2, presents the comparison between diagonal and full covariance matrix and their respective mean values. Fig. 3 details the scatter plot, 4 the posteriors distributions of the unknown vector and Fig. 5 compares the experimental and computational FRFs for both covariance hypotheses.




(a) PDF for $E$



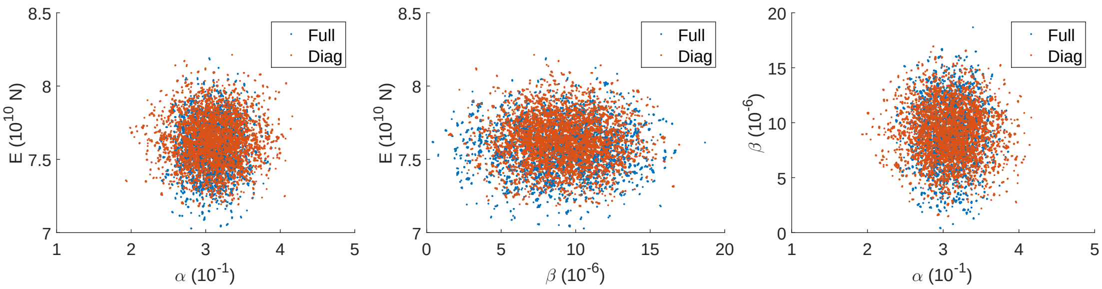

(b) PDF for coef. $\alpha$




(c) PDF for coef. $\beta$

Figure 2. Posterior marginal distributions.

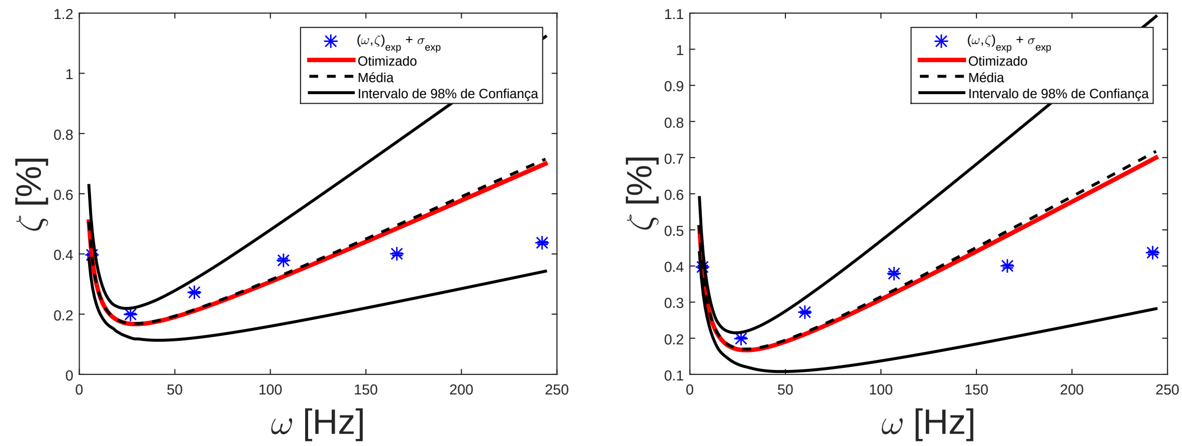
There is no effective difference between the posterior distributions of diagonal and full covariance matrix hypotheses. Even though $\pi(\alpha|\mathbf{y})$ showed some variability in MAP value (posterior's maximum), all the density functions (pdfs) provided, in a simplified perspective, almost the same probability assignment over each random variable domain.




(a) $\alpha \times E$ (b) $\beta \times E$ (c) $\alpha \times \beta$

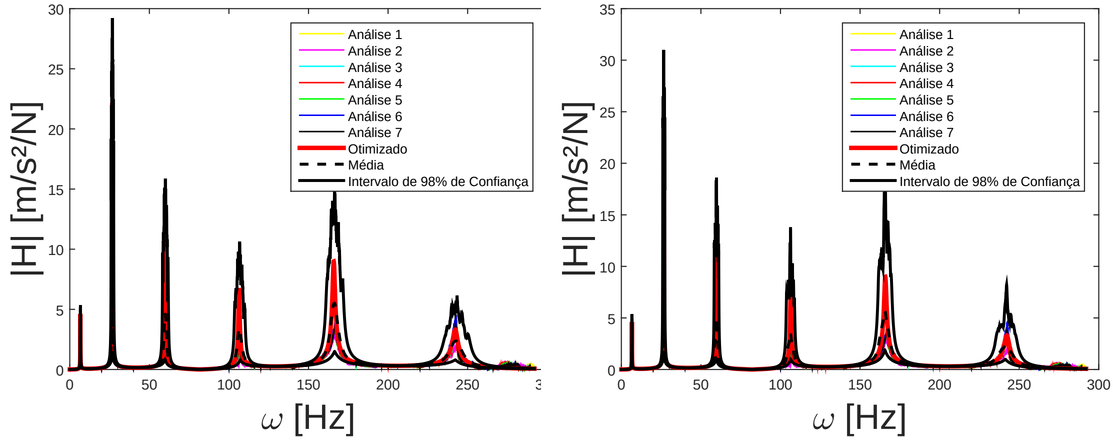
Figure 3. Scatter plots.

In a conceptual analysis, Fig. 3 suggest that both hypotheses produce similar deviation and uncorrelated samples. This result endorses the good agreement of response between these basic assumptions.




(a) Diagonal hypothesis (b) Fully hypothesis

Figure 4. Damping curves with diagonal and fully covariance matrix hypotheses.




(a) Diagonal hypothesis

(b) Fully hypothesis

Figure 5. Frequency response functions for both hypotheses at accelerometer AC1.

Figure 5 illustrates the capability of the computational model simulate the behavior of the real system, independently of the covariance hypothesis. The experimental data are within 98 % credibility interval of the theoretical model.

## 5 Concluding remarks

The results show good agreement between both hypotheses, i.e. diagonal and fully covariance matrix. Moreover, both computational results perform a 98 % credibility interval that contains the measured FRF, Fig. 5. In other words, the diagonal covariance matrix could simulate the behavior of the real structure in this study.

These results show that diagonal values kept the core information of the variance of response data and can be used instead of the fully matrix. The diminishing of the computational effort and partial knowledge prior information of the response data are some of the advantages of the diagonal covariance basic assumption. It is worthy to mention that these conclusions are related to the present work and should not be adopted without rigorous applicability verification.

Based on these model updating results the authors applied the present Bayesian framework to infer the position and magnitude of punctual masses along this structure. Several lumped masses were fixed along the structure's span in order to simulate structural anomalies, i.e. damages. Further details from the damage identification paper may be found in de Souza et al. [18].

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