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FINITE DIFFERENCES FOR PLATES MODAL ANALYSIS

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Abstract. Plates are flat structural members with a thickness much less than the other two dimensions, loaded in the direction perpendicular to the plane containing these two larger lengths. In case the thickness does not exceed 1/10 of the other dimensions, these structures are called thin plates. In this case, it is possible to adopt the so-called classical thin plate theory of plate dynamics, developed by Lagrange / Sophie-Germain, in which Kirchhoff's hypotheses are given as valid. Due to the difficulty of obtaining analytical solutions for the differential equations that govern this structural model, and to the advancement of software and computational hardware, numerical methods have been used in the modeling of this type of structural system. Our objective in this paper is to present modal analysis of aircraft plates using a computer implementation of the Central Finite Differences Method. The numerical results will be compared to solutions available in the literature. The numerical method of finite differences is an approach to obtain the approximation of the solution of differential equations. The basic idea of this method is to transform the resolution of a differential equation into a system of algebraic equations, replacing the derivatives by differences.

Keywords: Plates, Modal analysis, Finite differences

1 Introduction

In this paper, we present a simple Finite Differences algorithm, implement in MATLAB environment, to compute frequencies and modes of free undamped vibrations of thin elastic plates.

The massive industrial and academic use of structural analysis Finite Elements software in the last half-century has unduly obscured research on other sometimes more efficient numerical methods for some particular applications. That is certainly the case of the use of Finite Differences schemes for plate analysis both in statics and dynamics, as presented by, for example, Crandall [1].

The theoretical basis is Sophie-Germain/Lagrange equation valid in Kirchhoff's thin plate hypothesis (Timoshenko [2]). Next, Central Finite Differences approximations are introduced and the undamped free vibrations linear algebraic eigenvalue problem formulated under the assumption that these vibrations are harmonic.

Although none of this is new, the authors feel that their implementation effort is a valid contribution to both academic and commercial applications.

We present several application examples, and comparisons are made with classical results by Leissa [3], using the assumed modes technique.

2 Theoretical background

2.1 Sophie-Germain/Lagrange equation

Consider a plate of small thickness h, made of homogeneous linear elastic material with Young modulus E, Poisson's coefficient ν and density ρ. Let $w = w(x, y, t)$ be the time dependent transverse displacements of the points of it mid surface defined by coordinates x and y contained into that plane. Sophie-Germain/Lagrange equation for free undamped vibrations (Timoshenko [1]), is

$$D\nabla^4 w + \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

conditioned to the appropriate boundary conditions: null displacements and first derivative for clamped conditions; null displacements and second derivative for simply support conditions. In Eq. (1) we use the **nabla-four** operator

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2}$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{3}$$

2.2 Central Finite Differences

Consider the planar domain of a plate uniformly meshed by straight lines h_x apart in the x direction and h_y apart in the y direction. The partial derivatives present in Eq. (2) may be approximated by the following Central Finite Differences expressions, for a certain j,k node of the mesh:

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{j,k} = \frac{w_{j,k+2} - 4w_{j,k+1} + 6w_{j,k} - 4w_{j,k-1} + w_{j,k-2}}{{h_x}^4} \tag{4}$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{j,k} = \frac{w_{j+2,k} - 4w_{j+1,k} + 6w_{j,k} - 4w_{j-1,k} + w_{j-2,k}}{{h_y}^4} \quad (5)$$

$$\left(\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)_{j,k} = \frac{w_{j+1,k-1} - 2w_{j+1,k} + w_{j+1,k+1} - 2w_{j,k-1} + 4w_{j,k} - 2w_{j,k+1} + w_{j-1,k-1} - 2w_{j-1,k} + w_{j-1,k+1}}{{h_x}^2 {h_y}^2}$$

(6)

For clamped boundary conditions, the first partial derivatives will be approximated by

$$\left(\frac{\partial w}{\partial x}\right)_{j,k} = \frac{w_{j,k+1} - w_{j,k-1}}{2h_x} \quad (7)$$

$$\left(\frac{\partial w}{\partial y}\right)_{j,k} = \frac{w_{j+1,k} - w_{j-1,k}}{2h_y} \quad (8)$$

and, for simply supported boundaries, the second partial derivatives will be approximated by

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{j,k} = \frac{w_{j,k+1} - 2w_{j,k} + w_{j,k-1}}{{h_x}^2} \quad (9)$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{j,k} = \frac{w_{j+1,k} - 2w_{j,k} + w_{j-1,k}}{{h_y}^2} \quad (10)$$

2.3 The linear algebraic eigenvalue problem

In Eq. (1), the time dependent $w = w(x, y, t)$ transverse displacements, **in free undamped vibrations**, are considered to be composed of a spatial time independent function, the **vibration mode** $W(x,y)$, times a harmonic function of time only,

$$w(x, y, t) = W(x, y)\, sin\, \omega t \quad (11)$$

where ω are the vibration frequencies. Under this supposition, the accelerations field $\ddot{w} = \ddot{w}(x, y, t)$ can also be put in the form

$$\ddot{w}(x, y, t) = -\omega^2 W(x, y)\, sin\, \omega t \quad (12)$$

The vibration mode $W(x,y)$ can now be discretized using the Finite Differences approximations of Eqs. 4 through 10 and plugged into Eqs. (11) and (12) and then substituted into Eq. (1) to render a linear algebraic eigenvalue problem,

$$(\boldsymbol{A} - \lambda)\boldsymbol{W} = \boldsymbol{0} \quad (13)$$

where $\boldsymbol{W}$ is a discretized vector version of the vibration mode whose components are the normalized modal displacements at each node of the adopted discretization mesh, and

$$\lambda = \frac{\rho \omega^2}{D} \quad (14)$$

Coefficients of matrix ***A*** are derived from Eqs. 4 through 10 and are not displayed in this paper for the sake of concision. They are listed in Brasil [4].

The implementation of this algorithm in MATLAB environment is straight forward using the resident function *eig*.

3 Results of examples

In this section, we present a series of results obtained with our implementation in MATLAB of a Finite Differences algorithm to compute frequencies and modes of free undamped vibrations of plates with varied boundary conditions. The material is aluminum, E = 73 GPa, ρ = 2800 kg/m³, ν = 0,25. Thickness for all examples is 100 mm.

We analyze square plates, 5x5m, and rectangular ones, 5x8m. We also present results for four different meshes: 4, 16, 32 and 64 divisions. As for boundary conditions, six cases are displayed, as follows:

Case 1: clamped in all four edges;
Case 2: clamped in three edges and simply supported in the fourth one;
Case 3: clamped in two opposite edges and simply supported in the two opposite others;
Case 4: clamped in two adjacent edges and simply supported in the other adjacent two;
Case 5: only one face clamped and simply supported in the other three;
Case 6: simply supported in all four edges.

Results are compared to those available in Leissa [3], using the assumed modes technique. It is important to mention that that author himself declares that some of his results are quite poor.

3.1 Square 5x5m plates

Table 1. First frequency for square 5x5m plates (rad/s)

Case	4x4 mesh	16x16 mesh	32x32 mesh	64x64 mesh	Leissa [3]
1	55.429	68.037	68,969	69.211	71.672
2	50.705	60.359	61.047	61.224	63.697
3	46.771	54.970	55.549	55.698	57.015
4	46.771	54.970	51.961	52.062	54.554
5	46.771	45.195	45,448	45.512	46.785
6	46.771	37.887	37.979	38.002	38.010

3.2 Rectangular 5x8m plates

Table 2. First frequency for rectangular 5x8m plates (rad/s)

Case	4x4 mesh	16x16 mesh	32x32 mesh	64x64 mesh	Leissa [3]
1	40.521	49.722	50.383	50.554	52.172
2	39.584	48.053	48.634	48,783	50.379
3	27.703	31.074	31.334	31.401	32.115
4	32.538	36.844	37.119	37.189	38.806
5	26.312	28.375	28.508	28.542	29.240
6	25.098	26.344	26.407	26.423	26.428

3.3 Mode shapes

For the sake of illustration, Figs. 1 and 2 display plots the first vibration modes for the simply supported edges cases for the square 5x5m and the rectangular 5x8m pates, respectively. The plotted mesh is 32x32.

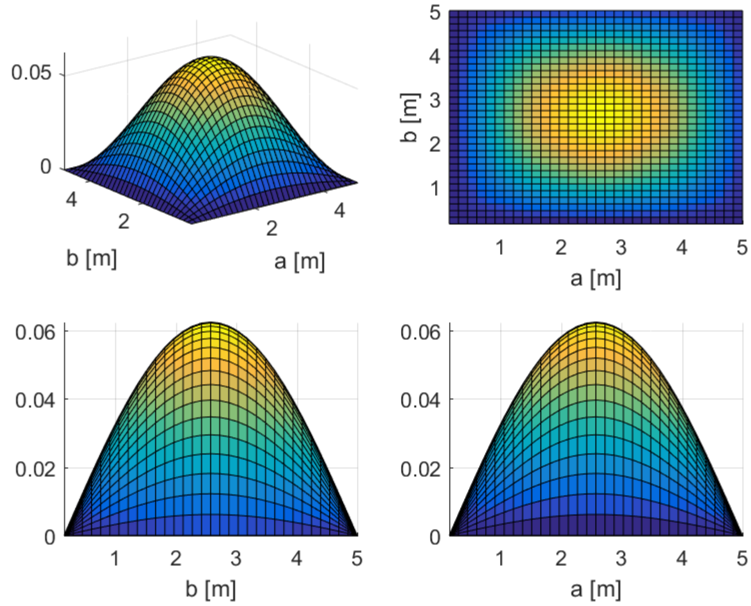

Figure 1. First mode 5x5m plate simply supported, mesh 32x32

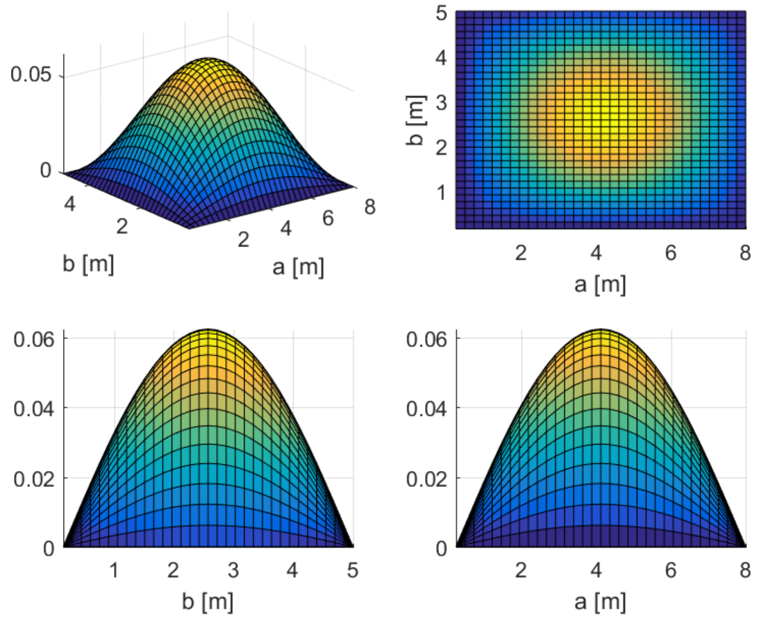

Figure 2. First mode 5x8m plate simply supported, mesh 32x32

4 Conclusions

In this paper we present an implementation of a Central Finite Differences algorithm for numerical computation of frequencies and modes of free undamped vibrations of linear elastic thin plates. Results are compared to values available in the classical literature.

Results in section 3 compare very well with available values in literature with considerable less computational cost than using the FEM.

We consider this implementation effort a valuable contribution to academics and professionals as an alternative to massive use of Finite Elements commercial software.

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